An invitation to sphere packings
球充填への招待

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Introduction

This is (mostly) a review talk, also mentioning recent breakthrough results by Mikhail Ganzhinov (PhD student at Aalto University).

The topic is the kissing number problem. The goal is to raise interest, and get some progress.

Problem

In $\mathbb{R}^d$ how many pairwise non-overlapping unit spheres can be arranged surrounding a central unit sphere?

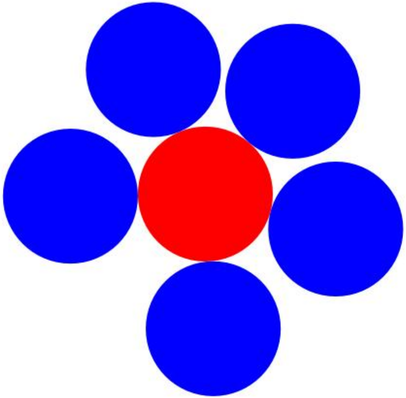
The answer is known in a few cases ($d = 1, 2, 3, 4, 8$, and 24), but mostly we have no clue whatsoever. This maximum number is denoted by $\tau(d)$.

To gain some intuition of higher dimensional geometry, the reader is warmly advised to consult the (highly accessible, eye-opening) slides of

Henry Cohn: "*What is the densest sphere packing in a million dimensions?*"

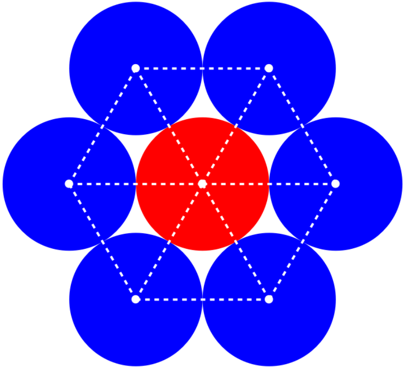
https://www.ams.org/publicoutreach/students/mathgame/cohn-slides-2014-web.pdf

The geometric picture in dimension 2: 5 disks



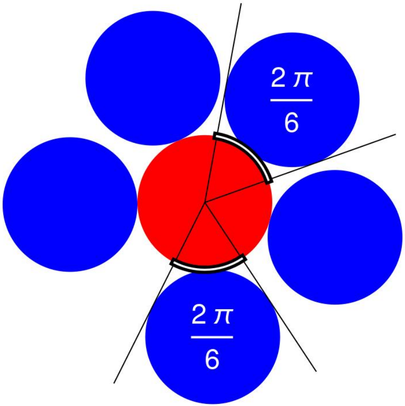
It is easy to see that that $\tau(2) \geq 5$. However, there is a lot of space left around the central unit disk. In fact, there is so much space left, that...

The geometric picture in dimension 2: 6 disks



...that there is enough room for 6 disks, arranged along the hexagonal lattice. Hence $\tau(2) \geq 6$. While it is 'obvious' (?!) that there cannot be room for further disks, for a formal proof...

The geometric picture in dimension 2: n disks



...one should realize that each blue disk 'blocks' $\frac{2\pi}{6}$ perimeter out of the total 2π perimeter of the central unit disk. Since the disks are non-overlapping, neither are the perimeter-segments (they might meet at the endpoints). Therefore, $n \cdot \frac{2\pi}{6} \leq 2\pi$, $n \leq 6$, $\tau(2) = 6$.

The algebraic model

The geometric intuition completely fails for $d \geq 3$: whether or not $\tau(3) = 12$ or $\tau(3) = 13$ was a famous debate between Newton and Gregory. Therefore, we recall the algebraic model.

Definition

A kissing arrangement is a set of n unit vectors $\mathcal{K} \subset \mathbb{R}^d$, such that

$$\langle x, y\rangle \leq \frac{1}{2} \qquad \text{for every } x, y \in \mathcal{K}.$$

The geometric interpretation of this model is what we have discussed so far up to a rescaling by a factor of 2. I will refer to these inner products as angles. When $\langle x, y\rangle = \frac{1}{2}$, then the unit spheres, centered at $2x$ and $2y$, are touching each other (and the unit sphere at the origin).

A set of n vectors in $\mathbb{R}^d$ is conveniently modeled by an $n \times d$ matrix F. The Gram matrix of the kissing arrangement is the positive semidefinite matrix $G = FF^T$ of rank (at most) d. All entries on the main diagonal of G are 1, and all off-diagonal entries are at most $1/2$.

Since we don't understand how higher dimensional Euclidean spaces look like, we focus on this algebraic model.

Further examples

We call the unit vectors $u, v \in \mathbb{R}^d$ compatible, if $\langle u, v \rangle \leq \frac{1}{2}$. In $\mathbb{R}^1$ we have altogether two unit vectors: $v_1 = [1]$ and $v_2 = [-1]$. They are compatible, thus $\tau(1) = 2$. The respective matrices are:

$$F = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \qquad G = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

One immediately observes that vertices of the cross-polytope $\mathcal{X}_d := \{\pm e_i \colon i \in \{1, 2, \ldots, d\}\} \subset \mathbb{R}^d$ is a kissing arrangement. Hence we have $\tau(d) \geq 2d$.

For $d \geq 2$, we can do slightly better by considering all permutations of vectors of shape $[\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}, 0, \ldots, 0] \in \mathbb{R}^d$.

$$\mathcal{D}_d := \{\sigma([\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, \ldots, 0]) \colon \sigma \in S_d\} \cup \{\sigma([\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0, \ldots, 0]) \colon \sigma \in S_d\}$$
$$\cup \{\sigma([-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0, \ldots, 0]) \colon \sigma \in S_d\},$$

where S_d is the symmetric group. Therefore $\tau(d) \geq 2d(d-1)$ for $d \geq 2$.

And one more example: $\mathcal{D}_3$

In $\mathbb{R}^3$ we have $\tau(3) = 12$, with infinitely many pairwise non-isometric examples saturating this value. Again, for an example let's have a look at $\mathcal{D}_3$:

$$F = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \\ -1 & -1 & 0 \\ -1 & 0 & 1 \\ -1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & -1 & -1 \end{bmatrix}, G = \frac{1}{2} \begin{bmatrix} 2 & 0 & 1 & 1 & 0 & -2 & -1 & -1 & 1 & 1 & -1 & -1 \\ 0 & 2 & 1 & 1 & -2 & 0 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & 2 & 0 & -1 & -1 & 0 & -2 & 1 & -1 & 1 & -1 \\ 1 & 1 & 0 & 2 & -1 & -1 & -2 & 0 & -1 & 1 & -1 & 1 \\ 0 & -2 & -1 & -1 & 2 & 0 & 1 & 1 & 1 & 1 & -1 & -1 \\ -2 & 0 & -1 & -1 & 0 & 2 & 1 & 1 & -1 & -1 & 1 & 1 \\ -1 & -1 & 0 & -2 & 1 & 1 & 2 & 0 & 1 & -1 & 1 & -1 \\ -1 & -1 & -2 & 0 & 1 & 1 & 0 & 2 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 2 & 0 & 0 & -2 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 0 & 2 & -2 & 0 \\ -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 0 & -2 & 2 & 0 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -2 & 0 & 0 & 2 \end{bmatrix}.$$

These matrices are the models of (optimal) kissing arrangements.

The best known bounds on $\tau(d)$

The highlighted lower bounds 510, 592 and 1932 are recent results of Mikhail Ganzhinov.

d	LBD	UBD	d	LBD	UBD	d	LBD	UBD
1	2		9	306	363	17	5346	11014
2	6		10	510	553	18	7398	16469
3	12		11	592	869	19	10668	24575
4	24		12	840	1356	20	17400	36402
5	40	44	13	1154	2066	21	27720	53878
6	72	78	14	1932	3177	22	49896	81376
7	126	134	15	2564	4858	23	93150	123328
8	240		16	4320	7332	24	196560	

This table shows that we know very little: the gap between the lower and upper bounds is significant. There are two main construction methods:

- Find a 'large' configuration, and then look at lower-dimensional cross-sections of it. (That is, select vectors living in a smaller dimensional subspace). This is the method of 'slicing'.
- Constructions from binary codes: Find large set of $\{0,1\}$-vectors, which are 'almost' orthogonal, and then 'transform' these to unit vectors whose inner product is at most $1/2$.

Examples in $\mathbb{R}^5$

The smallest open case where the exact value of $\tau(d)$ is not known is $d = 5$. The vectors of $\mathcal{D}_5$ give the obvious lower bound of 40. There is, however, another nonisometric example due to John Leech from 1967. We call this $\mathcal{L}_5$:

$$\mathcal{L}_5 := (\mathcal{D}_5 \setminus \{v \in \mathcal{D}_5 \colon v_5 = -\frac{1}{\sqrt{2}}\}) \cup \{vH \colon v \in \mathcal{D}_5, v_5 = -\frac{1}{\sqrt{2}}\},$$

where H is the orthogonal transformation via an augmented (and rescaled) Hadamard matrix:

$$H = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & 1 & 0 \\ 1 & -1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 1 & 0 \\ 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

Basically, what you do is you move around a set of 8 vectors. More concretely, you exchange

$$[\frac{1}{\sqrt{2}}, 0, 0, 0, -\frac{1}{\sqrt{2}}] \quad \rightsquigarrow \quad [-\frac{1}{\sqrt{8}}, \frac{1}{\sqrt{8}}, \frac{1}{\sqrt{8}}, \frac{1}{\sqrt{8}}, -\frac{1}{\sqrt{2}}], \text{ etc.}$$

The profile of an arrangement

The profile $\pi(\mathcal{K})$ of an arrangement $\mathcal{K}$ is the the multiset of the entries of the Gram matrix G. Clearly, this multiset is an invariant up to isometry. We have:

$$\pi(\mathcal{D}_5) = \{[-1]^{40}, [-1/2]^{480}, [0]^{560}, [1/2]^{480}, [1]^{40}\},$$

and further,

$$\pi(\mathcal{L}_5) = \{[-1]^{24}, [-3/4]^{64}, [-1/2]^{384}, [0]^{544}, [1/2]^{480}, [1]^{40}\}.$$

Thus, these two examples are not isometric. Further, while the vectors of $\mathcal{D}_5$ form an antipodal configuration (that is: $x \in \mathcal{D}_5$ if and only if $-x \in \mathcal{D}_5$), $\mathcal{L}_5$ is not antipodal. Indeed, if there are two vectors $x, y \in \mathcal{K}$, such that $-1 < \langle x, y \rangle < -1/2$, then x and $-y$ are not compatible. Thus $\mathcal{K}$ cannot be antipodal.

According to the website of Henry Cohn, these are the only known optimal examples in $\mathbb{R}^5$.

`https://cohn.mit.edu/5d-kissing`

Well... not anymore.

A new result

Theorem[F. Sz., 2023+]

The set of 40 unit vectors $\mathcal{Q}_5 := (\mathcal{D}_5 \setminus \mathcal{X}) \cup \mathcal{Y}$, where the sets $\mathcal{X}$ and $\mathcal{Y}$ are the rows of the following matrices

$$X = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & -1 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 & -1 \end{bmatrix}, \qquad Y = \frac{1}{5\sqrt{2}} \begin{bmatrix} -1 & 1 & -4 & -4 & -4 \\ -1 & -4 & 1 & -4 & -4 \\ -1 & -4 & -4 & 1 & -4 \\ -1 & -4 & -4 & -4 & 1 \\ 4 & 1 & 1 & -4 & -4 \\ 4 & 1 & -4 & 1 & -4 \\ 4 & 1 & -4 & -4 & 1 \\ 4 & -4 & 1 & 1 & -4 \\ 4 & -4 & 1 & -4 & 1 \\ 4 & -4 & -4 & 1 & 1 \end{bmatrix},$$

forms a new, previously unknown kissing arrangement in $\mathbb{R}^5$ with profile

$$\pi(\mathcal{Q}_5) = \{[-1]^{20}, [-4/5]^{60}, [-1/2]^{360}, [-3/10]^{120}, [0]^{500}, [1/5]^{20}, [1/2]^{480}, [1]^{40}\}.$$

The appearance of the angles $1/5$, $-4/5$, and especially $-3/10$ (?!) is rather unexpected.

The structure of $\mathcal{Q}_5$

The Gram matrices satisfy $XX^T = YY^T$, therefore there is an orthogonal transformation

$$U = \frac{1}{5}\begin{bmatrix} -3 & -2 & -2 & -2 & -2 \\ -2 & -3 & 2 & 2 & 2 \\ -2 & 2 & -3 & 2 & 2 \\ -2 & 2 & 2 & -3 & 2 \\ -2 & 2 & 2 & 2 & -3 \end{bmatrix}$$

such that $XU = Y$. But it is not so clear, what X is in the first place. It is a 10 point 2-distance set, in dimension 5 (there are several such sets). So

$$\mathcal{Q}_5 = (\mathcal{D}_5 \setminus \mathcal{X}) \cup \{vU \colon v \in \mathcal{X}\}.$$

Because of the angles in the Gram matrix of $\mathcal{Q}_5$, it is not a subset of any of the largest known kissing arrangements up to dimension 24.

How we discovered this new arrangement (1)

Basic approach:

- Pick a basis B, containing d pairwise compatible vectors.
- Fix a (finite!) angle set $\mathcal{A}$ with $\max \mathcal{A} \leq 1/2$.
- Generate the multiangular cloud $\mathcal{C}_{\mathcal{A},B} \subset \mathbb{R}^d$ around the basis B w.r.t. the angle set $\mathcal{A}$, that is: Find all those (i) unit vectors v, which are (ii) in the linear span of B, such that (iii) they are $\mathcal{A}$-compatible with the basis vectors. (That is: $\langle B_i, v\rangle \in \mathcal{A}$).
- Search for a maximal subset of pairwise compatible vectors in $\mathcal{C}_{\mathcal{A},B}$.

From (ii) we see that there is a coordinate vector $c \in \mathbb{R}^d$ such that $v = cB$. From (iii) we see that there is a d-tuple $t \in \mathcal{A}^d$, such that $Bv^T = t^T$. It follows that $t = vB^T = cBB^T$, thus $t(BB^T)^{-1}B = cB = v$. Finally, from (i), we have $t(BB^T)^{-1}BB^T(BB^T)^{-1}t^T = 1$, thus $t(BB^T)^{-1}t^T = 1$. Therefore:

$$\mathcal{C}_{\mathcal{A},B} = \{t(BB^T)^{-1}B \colon t(BB^T)^{-1}t^T = 1, t \in \mathcal{A}^d\}.$$

Unfortunately, this approach is too restrictive from two points of view:

- There are many bases, which one to choose?
- The condition that the vectors v are $\mathcal{A}$-compatible with the basis seems very restrictive.

How we discovered this new arrangement (2)

Modified approach: The key idea is to require $\mathcal{A}$-compatibility to only part of a basis.

- Pick d linearly independent pairwise compatible vectors in $\mathbb{R}^{d+1}$. They are represented by a $d \times (d+1)$ matrix B.
- Fix a (finite!) angle set $\mathcal{A}$ with $\max \mathcal{A} \leq 1/2$.
- Generate the multiangular cloud $\mathcal{C}$ around the independent set B w.r.t the angle set $\mathcal{A}$.
- Search for a maximal subset of pairwise compatible vectors in $\mathcal{C}$.

The orthogonal complement of B is spanned by a single unit vector B_{d+1}. Therefore the cloud can be described as

$$\mathcal{C}_{\mathcal{A},B} = \{t(BB^T)^{-1}B \pm \sqrt{1 - t(BB^T)^{-1}t^T}B_{d+1} \colon t(BB^T)^{-1}t^T \leq 1, t \in \mathcal{A}^d\}.$$

In this case $\mathcal{A}$-compatible vectors are either unit vectors in the span of B, or they are shorter. If that is the case, they can be made unit length along the direction of B_{d+1} (or $-B_{d+1}$). This does not affect $\mathcal{A}$-compatibility.

A case study

- We start by four vectors in $\mathbb{R}^4$ forming a basis B, whose pairwise inner product is $1/2$. We embed this into $\mathbb{R}^5$ in the natural way, and think of B as a 4×5 matrix:

$$B = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad G = BB^T = \frac{1}{2} \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}.$$

- Fix the angle set to be $\mathcal{A} = \{-1, -1/2, 0, 1/2\}$; and
- take the orthocomplement $B_5 = \frac{1}{\sqrt{5}}[1, -1, -1, -1, -1]$; and
- generate the cloud $\mathcal{C}$. This time, $\mathcal{C}$ has 78 vectors.
- Then, we search for maximum cliques in the compatibility graph Γ of $\mathcal{C}$.
- There are exactly 4 maximum cliques of size 36. Therefore we have 4 sets of 40 pairwise nonoverlapping spheres. One is $\mathcal{D}_5$, another is isometric to $\mathcal{D}_5$, another two are isometric to $\mathcal{Q}_5$.

The reason for finding these twice is that there is no prescribed basis (I think...)

GANZHINOV's results in a nutshell

Very recently, MIKHAIL GANZHINOV (Aalto University) constructed, using representation theory, new record-breaking kissing configurations in $\mathbb{R}^{10}$, $\mathbb{R}^{11}$, and $\mathbb{R}^{14}$.

- He found a set $\mathcal{M}$ of 80 complex unit vectors in $\mathbb{C}^5$ with angle set $\mathcal{A} = \{\pm\frac{1}{3}, \pm\frac{i}{\sqrt{3}}, -1\}$ such that $\Re[\langle x, y\rangle] \leq \frac{1}{3}$. The choice of the angle set allowed him to triple his construction and have a 240-element set $\mathcal{M} \cup \omega\mathcal{M} \cup \omega^2\mathcal{M}$ such that $\Re[\langle x, y\rangle] \leq \frac{1}{2}$. Then he lifted this to $\mathbb{R}^{10}$, and found an additional set of 270 pairwise compatible vectors. Altogether, this is an arrangement of $240 + 270 = 510$ vectors, with the property of having 80 vectors with $\langle x, y\rangle \leq \frac{1}{3}$. The angle set (after lifting) becomes $\mathcal{A} = \{0, \pm\frac{1}{6}, \pm\frac{1}{4} \pm \frac{1}{3}, \pm\frac{1}{2}, -1\}$.
- Then, he continues modifying this set of 510 vectors. By another lifting it is possible to "double" (any one of) the 80 vectors in it (with $\langle x, y\rangle \leq \frac{1}{3}$) and include two more basis vectors to obtain $510 + 80 + 2 = 592$ vectors in $\mathbb{R}^{11}$. Here irrational angles show up.
- Finally, he comes up with a very elegant gluing construction, where (highly structured) kissing arrangements in $\mathbb{R}^k$ and $\mathbb{R}^\ell$ can be combined to obtain kissing arrangements in $\mathbb{R}^{k+\ell}$. Applying this to $k = \ell = 7$, and to the E_7 kissing arrangement yielded 1932 spheres.

M. GANZHINOV: "*Highly symmetric lines*", preprint, `https://arxiv.org/abs/2207.08266`

Future work and open problems

Open problems:

- The arrangements $\mathcal{D}_5$ and $\mathcal{L}_5$ are known to be rigid. How about $\mathcal{Q}_5$?
- Are there higher dimensional (near-)optimal kissing arrangements, containing an isometric copy of $\mathcal{Q}_5$?

Thank you

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Preprint is on arXiv: https://arxiv.org/abs/2301.08272 (submitted for publication).
F. Szöllősi: "*A note on five dimensional kissing arrangements*" (2023+).

```
Q5={{5,5,0,0,0},{5,0,5,0,0},{5,0,0,5,0},{5,0,0,0,5},{0,0,0,-5,5},
{0,0,0,5,-5},{0,0,0,5,5},{0,0,-5,0,5},{0,0,-5,5,0},{0,0,5,0,-5},
{0,0,5,0,5},{0,0,5,-5,0},{0,0,5,5,0},{0,-5,0,0,5},{0,-5,0,5,0},
{0,-5,5,0,0},{0,5,0,0,-5},{0,5,0,0,5},{0,5,0,-5,0},{0,5,0,5,0},
{0,5,-5,0,0},{0,5,5,0,0},{-5,0,0,0,-5},{-5,0,0,0,5},{-5,0,0,-5,0},
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{-1,1,-4,-4,-4},{-1,-4,1,-4,-4},{-1,-4,-4,1,-4},{-1,-4,-4,-4,1},
{4,1,1,-4,-4},{4,1,-4,1,-4},{4,1,-4,-4,1},{4,-4,1,1,-4},
{4,-4,1,-4,1},{4,-4,-4,1,1}};
```