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CIRCLES AND HELICES IN A COMPLEX PROJECTIVE SPACE

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Abstract. This expository paper consists of two parts. One is to investigate lengths of closed helices in Euclidean sphere by using the information on lengths of closed circles in a complex projective space. The other is to find examples of closed helices in a complex projective plane. This paper bridges between curve theory and submanifold geometry.

1. Helices

We first review the notion of Frenet curves in an $n(\geqq 2)$-dimensional Riemannian manifold M with Riemanian metric $\langle\, ,\, \rangle$. A real smooth curve $\gamma = \gamma(s)$ by its arclength s is called a *Frenet curve of proper order* d if there exist a field of orthonormal frames $\{V_1, \dots, V_d\}$ along γ and positive smooth functions $\kappa_1, \dots, \kappa_{d-1}$ which satisfy the system of ordinary differential equations;

$$(1.1) \qquad \nabla_{\dot\gamma} V_j(s) = -\kappa_{j-1}(s)V_{j-1}(s) + \kappa_j(s)V_{j+1}(s), \quad j = 1, \dots, d,$$

where $V_1 = \dot\gamma,\ V_0 = V_{d+1} = 0$ and $\nabla_{\dot\gamma}$ is the covariant differentiation with respect to the Riemannian metric $\langle\, ,\, \rangle$ of M along γ. The functions κ_j $(1 \leqq j \leqq d-1)$ and the field of orthonormal frames $\{V_1, \dots, V_d\}$ are called the *curvatures* and the *Frenet frame* of γ, respectively. We call a curve a *Frenet curve of order* d if it is a Frenet cuve of proper order $r(\leqq d)$. We use the convention in (1.1) that $\kappa_j = 0$ $(r \leqq j \leqq d-1)$ and $V_j = 0$ $(r+1 \leqq j \leqq d)$. Roughly speaking, a Frenet curve can be regarded as a smooth real curve having no inflection points.

A Frenet curve $\gamma = \gamma(s)$ on M is called a *helix* if all of its curvatures $\kappa_1, \dots, \kappa_{d-1}$ are constants. A helix of order 1 is nothing but a geodesic, and a helix of order 2 is called a *circle*. In the following, for a helix γ of proper order d we adopt constants $k_1, \dots, k_{d-1}$ as curvatures of the curve γ instead of $\kappa_1, \dots, \kappa_{d-1}$. We say that two Frenet curves γ_1 and γ_2 are *congruent* if there exist an isometry φ on M and a constant s_0 with $\gamma_2(s) = (\varphi \circ \gamma_1)(s + s_0)$ for each s.

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A real smooth curve $\gamma = \gamma(s)$ in M is said to be *homogeneous* if it is an orbit of some one-parameter subgroup of the full isometry group $\mathrm{I}(M)$ on M, i.e., the curve γ is an integral curve of a Killing vector field on M. Needless to say, every homogeneous curve must be a helix and there exist many helices which are *not* homogeneous in M. However, it is well-known that in a real space form $M^n(c)$ of constant sectional curvature c, which is globally congruent to either a Euclidean sphere $S^n(c)$, a Euclidean space $\mathbb{R}^n$ or a real hyperbolic space $H^n(c)$, the notions of homogeneous curves and helices are mutually equivalent and that two helices are congruent if and only if they have the same order and the same curvatures. This means that helices of order d in a real space form $M^n(c)$ are parametrized by $d-1$ nonnegative numbers.

On the contrary, in a complex $n(\geqq 2)$-dimensional complex projective space $\mathbb{C}P^n(c)$ of constant holomorphic sectional curvature $c(>0)$, the theory of Frenet curves is a bit complicated. To show this, for a Frenet curve $\gamma = \gamma(s)$ of proper order d in $\mathbb{C}P^n(c)$ we recall the notion of holomorphic torsions $\tau_{ij} = \tau_{ij}(s)$ along the curve γ, which is defined by $\tau_{ij} = \langle V_i(s), JV_j(s)\rangle$ $(1 \leqq i < j \leqq d)$, where J is the Kähler structure of the space $\mathbb{C}P^n(c)$ and $\{V_1, V_2, \ldots, V_d\}$ is the Frenet frame of γ. The congruence theorem for Frenet curves in $\mathbb{C}P^n(c)$ is as follows:

Lemma 1 ([17])**.** *Let γ_1 and γ_2 be two Frenet curves of proper orders p and q in $\mathbb{C}P^n(c)$, respectively. We denote by $\kappa_j^{(\ell)}(s)$ and $\tau_{ij}^{(\ell)}(s)$ the curvature functions and holomorphic torsions of γ_ℓ, $\ell = 1, 2$. Then the curves γ_1 and γ_2 are congruent to each other if and only if they satisfy the following conditions*:
(1) $p = q$;
(2) *There exists a constant s_0 with the following properties*;
  i) $\kappa_i^{(2)}(s) = \kappa_i^{(1)}(s + s_0)$ $(i = 1, 2, \ldots, p-1)$ *for every s*,
  ii) *either $\tau_{ij}^{(2)}(0) = \tau_{ij}^{(1)}(s_0)$ for all i, j with $1 \leqq i < j \leqq p$ or $\tau_{ij}^{(2)}(0) = -\tau_{ij}^{(1)}(s_0)$ for all i, j with $1 \leqq i < j \leqq p$.*

In Condition (2)ii) of Lemma 1, the former holds if γ_1, γ_2 are congruent with respect to some holomorphic isometry and the latter holds if they are congruent with respect to some anti-holomorphic isometry. As an immediate consequence of Lemma 1 we get the following which is a congruence theorem for helices in $\mathbb{C}P^n(c)$:

Lemma 2 ([17])**.** *Let γ_1 and γ_2 be two helices of proper orders p and q in $\mathbb{C}P^n(c)$, respectively. We denote by $k_j^{(\ell)}$ and $\tau_{ij}^{(\ell)}(s)$ the constant curvatures and holomorphic torsions of γ_ℓ, $\ell = 1, 2$. Then the curves γ_1 and γ_2 are congruent to each other if and only if they satisfy the following conditions*:
(1) $p = q$;
(2) *There exists a constant s_0 with the following property* ii);
  i) $k_i^{(2)} = k_i^{(1)}$ $(i = 1, 2, \ldots, p-1)$,
  ii) *either $\tau_{ij}^{(2)}(0) = \tau_{ij}^{(1)}(s_0)$ for all i, j with $1 \leqq i < j \leqq p$ or $\tau_{ij}^{(2)}(0) = -\tau_{ij}^{(1)}(s_0)$ for all i, j with $1 \leqq i < j \leqq p$.*

The following gives a necessary and sufficient condition for a Frenet curve to be homogeneous in $\mathbb{C}P^n(c)$:

Lemma 3 ([17]). *In $\mathbb{C}P^n(c)$, a Frenet curve $\gamma = \gamma(s)$ of proper order d is homogeneous if and only if all of its curvatures κ_i $(1 \leqq i \leqq d-1)$ and holomorphic torsions τ_{ij} $(1 \leqq i < j \leqq d)$ are constant functions along the curve γ.*

Hence in view of Lemmas 2 and 3 we have the following which is a congruence theorem for homogeneous curves:

Lemma 4 ([17]). *Let γ_1 and γ_2 be two homogeneous curves of proper orders p and q in $\mathbb{C}P^n(c)$, respectively. We denote by $k_j^{(\ell)}$ and $\tau_{ij}^{(\ell)}$ the constant curvatures and constant holomorphic torsions of γ_ℓ, $\ell = 1, 2$. Then the curves γ_1 and γ_2 are congruent to each other if and only if they satisfy the following conditions*:
(1) $p = q$;
(2) i) $k_i^{(2)} = k_i^{(1)}$ $(i = 1, 2, \dots, p-1)$,
ii) *either* $\tau_{ij}^{(2)} = \tau_{ij}^{(1)}$ *for all* i, j *with* $1 \leqq i < j \leqq p$ *or* $\tau_{ij}^{(2)} = -\tau_{ij}^{(1)}$ *for all* i, j *with* $1 \leqq i < j \leqq p$.

At the end of this section we recall the classification theorems of homogeneous curves of proper orders 3 and 4 in a complex projective plane $\mathbb{C}P^2(c)$.

Lemma 5 ([15]). *On $\mathbb{C}P^2(c)$ the holomorphic torsions of each homogeneous curve γ of proper order 3 are expressed in terms of its curvatures k_1, k_2 as follows*:

$$\tau_{12} = \frac{k_1}{\sqrt{k_1^2 + k_2^2}},\ \tau_{13} = 0,\ \tau_{23} = \frac{k_2}{\sqrt{k_1^2 + k_2^2}}$$

or

$$\tau_{12} = -\frac{k_1}{\sqrt{k_1^2 + k_2^2}},\ \tau_{13} = 0,\ \tau_{23} = -\frac{k_2}{\sqrt{k_1^2 + k_2^2}}.$$

Conversely for given positive constants k_1 and k_2, there exists a unique homogeneous curve of proper order 3 with curvatures k_1 and k_2 up to isometries of $\mathbb{C}P^2(c)$.

Lemma 6 ([15]). *On $\mathbb{C}P^2(c)$, the holomorphic torsions of every homogeneous curve γ of proper order 4 are expressed in terms of its curvatures k_1, k_2, k_3 as follows*:

$$\begin{cases} \tau_{12} = \tau_{34} = \tau, \tau_{23} = \tau_{14} = k_2\tau/(k_1 + k_3), \tau_{13} = \tau_{24} = 0, \\ \tau = \pm(k_1 + k_3)/\sqrt{k_2^2 + (k_1 + k_3)^2}\ , \end{cases} \tag{1.2}$$

$$\begin{cases} \tau_{12} = -\tau_{34} = \tau, \tau_{23} = -\tau_{14} = k_2\tau/(k_1 - k_3), \tau_{13} = \tau_{24} = 0, \\ \tau = \pm(k_1 - k_3)/\sqrt{k_2^2 + (k_1 - k_3)^2} \text{ if } k_1 \neq k_3, \end{cases} \tag{1.3}$$

$$\tau_{12} = \tau_{34} = \tau_{13} = \tau_{24} = 0,\ \tau_{23} = -\tau_{14} = \pm 1 \text{ if } k_1 = k_3. \tag{1.4}$$

Conversely, for any given positive constants k_1, k_2, k_3, there exists two homogeneous curves of proper order 4 on $\mathbb{C}P^2(c)$, which are given by either (1.2), (1.3) *or* (1.2), (1.4), *having curvatures k_1, k_2 and k_3 with respect to $\mathrm{I}(\mathbb{C}P^2(c))(= \mathrm{SU}(3))$.*

2. Circles

It is well-known that all geodesics in $\mathbb{C}P^n(c)$ are congruent one another. We next consider a congruence theorem for circles of positive curvature. We take a circle $\gamma = \gamma(s)$ of curvature $k(> 0)$ in $\mathbb{C}P^n(c)$, so that it satisfies $\nabla_{\dot\gamma}\dot\gamma = kV_2(s)$ and $\nabla_{\dot\gamma}V_2(s) = -k\dot\gamma$. Then we see easily that the only holomorphic torsion $\tau := \tau_{12}(s) = \langle\dot\gamma, JV_2(s)\rangle$ is automatically constant. In fact,

$$\dot\gamma\langle\dot\gamma, JV_2\rangle = \langle\nabla_{\dot\gamma}\dot\gamma, JV_2\rangle + \langle\dot\gamma, J\nabla_{\dot\gamma}V_2\rangle = k\langle V_2, JV_2\rangle - k\langle\dot\gamma, J\dot\gamma\rangle = 0.$$

Hence in consideration of Lemmas 3 and 4 we obtain the following fundamental lemma on circles of positive curvature in $\mathbb{C}P^n(c)$:

Lemma 7 ([17, 5])**.** (1) *Every circle of positive curvature in $\mathbb{C}P^n(c)$ is homogeneous.*
(2) *Let γ_1 and γ_2 be two circles of positive curvature in $\mathbb{C}P^n(c)$. We denote by $k^{(\ell)}$ and $\tau^{(\ell)}$ the curvatures and holomorphic torsions of γ_ℓ, $\ell = 1, 2$. Then the curves γ_1 and γ_2 are congruent to each other if and only if they satisfy either $k^{(1)} = k^{(2)}$, $\tau^{(1)} = \tau^{(2)}$ or $k^{(1)} = k^{(2)}$, $\tau^{(1)} = -\tau^{(2)}$.*

In order to study circles in $\mathbb{C}P^n(c)$ in connection with submanifold geometry (see [20]), we construct a quotient of a 2-dimensional flat torus in a complex projective plane $\mathbb{C}P^2(4)$. Using a unit circle $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$, we define an automorphism φ of $S^1 \times S^1$ by $\varphi(z_1, z_2) = (-z_1, -z_2)$. On a quotient of a 2-dimensional flat torus $N = (S^1 \times S^1)/\varphi$ we define a Riemannian metric $\langle\ ,\ \rangle$ by

$$\langle(A, \xi), (B, \eta)\rangle = (2/9)\langle A, B\rangle_{S^1} + (2/3)\langle\xi, \eta\rangle_{S^1}.$$

By using the Hopf fibration $\varpi : S^5(1) \to \mathbb{C}P^2(4)$ we define an isometric parallel embedding $f : N \to \mathbb{C}P^2(4)$ by

$$f(e^{i\theta}, (a_1, a_2)) = \varpi\Big(\frac{1}{3}(e^{-2i\theta/3} + 2a_1e^{i\theta/3}), \frac{\sqrt{2}}{3}(e^{-2i\theta/3} - a_1e^{i\theta/3}), \frac{2}{\sqrt{6}}ia_2e^{i\theta/3}\Big),$$

where $a_1^2 + a_2^2 = 1$.

Remark 1. We here give another construction of the above embedding $f : N \to \mathbb{C}P^2(4)$. We set

$$T^3 = S^1\big(1/\sqrt{3}\,\big) \times S^1\big(1/\sqrt{3}\,\big) \times S^1\big(1/\sqrt{3}\,\big) \to S^5(1),$$

which is a parallel submanifold of a unit sphere $S^5(1)$. This, together with the Hopf fibration $\varpi : S^5(1) \to \mathbb{C}P^2(4)$, gives a totally real minimal (isotropic) flat surface $\varpi(T^3)$ with parallel second fundamental form of $\mathbb{C}P^2(4)$. Note that $\varpi(T^3)$ is congruent to the above surface N and the embedding of $\varpi(T^3)$ into the ambient space $\mathbb{C}P^2(4)$ is equivalent to f.

We obtain the following theorem which shows that every geodesic N is mapped to a circle of the same curvature $1/\sqrt{2}$ in $\mathbb{C}P^n(4)$ through this embedding f.

Theorem A ([5])**.** *For a unit vector $X = (\alpha u, v) \in T_xN$ ($|\alpha| \leqq 1$, $\|u\| = 1$) at a point x, we denote by γ_X the geodesic with initial vector X on N. Then the curve $f \circ \gamma_X$ on $\mathbb{C}P^2(4)$ satisfies the following properties*:

(1) *The curve* $f \circ \gamma_X$ *is a circle of curvature* $1/\sqrt{2}$ *and of holomorphic torsion* $4\alpha^3 - 3\alpha$ *and it is a simple curve*;
(2) *It is closed if and only if* $\alpha = 0$ *or* $\sqrt{(1-\alpha^2)/(3\alpha^2)}$ *is rational*;
(3) *When* $\alpha = 0$, *it is a closed curve with length* $2\sqrt{6}\,\pi/3$, *and when* $\alpha = \pm 1$, *it is also a closed curve with length* $2\sqrt{2}\,\pi/3$;
(4) *When* $\alpha \neq 0, \pm 1$ *and* $\sqrt{(1-\alpha^2)/(3\alpha^2)}$ *is rational, we denote it by* p/q *as an irreducible fraction.*

- i) *When* pq *is even, the length of the closed circle* $f \circ \gamma_X$ *is the least common multiple of* $2\sqrt{2}\,\pi/(3|\alpha|)$ *and* $2\sqrt{2}\,\pi/\sqrt{3(1-\alpha^2)}$.
- ii) *When* pq *is odd, the length of the closed circle* $f \circ \gamma_X$ *is the least common multiple of* $\sqrt{2}\,\pi/(3|\alpha|)$ *and* $\sqrt{2}\,\pi/\sqrt{3(1-\alpha^2)}$.

By Lemma 7(2) the congruency of circles in $\mathbb{C}P^n(c)$ is determined by their curvatures and holomorphic torsions. Since the function $4\alpha^3 - 3\alpha$ ($|\alpha| \leqq 1$) takes an arbitrary value on the interval $[-1, 1]$, every circle of curvature $1/\sqrt{2}$ in $\mathbb{C}P^2(4)$ is of the form $f \circ \gamma_X$ up to an isometry of $\mathbb{C}P^2(4)$. If we change the metric $\langle\ ,\ \rangle$ of a Riemannian manifold M homothetically to $\lambda^2\langle\ ,\ \rangle$ for some positive constant λ, the curve $\sigma(s) = \gamma(s/\lambda)$, which is given by a circle of curvature k in $(M, \langle\ ,\ \rangle)$, is a circle of curvature k/λ in $(M, \lambda^2\langle\ ,\ \rangle)$. Since $\mathbb{C}P^2(c)$ is contained as a totally geodesic submanifold in $\mathbb{C}P^n(c)$, all circles of curvature $\sqrt{2c}/4$ in $\mathbb{C}P^n(c)$ are obtained in this way from geodesics on some flat torus in $\mathbb{C}P^2(c)$.

We cannot study circles of other curvatures k, i.e. $k \neq \sqrt{2c}/4$, in $\mathbb{C}P^n(c)$ from this point of view. However, by making use of the Hopf fibration $\varpi : S^{2n+1}(c/4) \to \mathbb{C}P^n(c)$ we can investigate all circles in $\mathbb{C}P^n(c)$ through their horizontal lifts on the sphere $S^{2n+1}(c/4)$ and establish the following:

Theorem B ([5])**.** *Every circle* γ *of curvature* k *and holomorphic torsion* τ *in* $\mathbb{C}P^n(c)$ *is simple and lies on a totally geodesic Kähler submanifold* $\mathbb{C}P^2(c)$. *Moreover, it has the following properties*:
(1) *When* $\tau = \pm 1$, *this circle* γ *is a closed curve with length* $2\pi/\sqrt{k^2 + c}$ *and lies on a totally geodesic holomorphic line* $\mathbb{C}P^1(c)$ *in* $\mathbb{C}P^2(c)$;
(2) *When* $\tau = 0$, *this circle* γ *is a closed curve with length* $4\pi/\sqrt{4k^2 + c}$ *and lies on a totally real totally geodesic* $\mathbb{R}P^2(c/4)$ *in* $\mathbb{C}P^2(c)$;
(3) *When* $\tau \neq 0, \pm 1$, *we denote by* a, b, d ($a < b < d$) *the nonzero solutions to the cubic equation* $c\lambda^3 - (4k^2 + c)\lambda + 2\sqrt{c}\,k\tau = 0$;

- i) *If* (*one of hence*) *all of the three ratios* $a/b, b/d$ *and* d/a *are rational, then* γ *is a closed curve. Its length is the least common multiple of* $4\pi/(\sqrt{c}\,(b-a))$ *and* $4\pi/(\sqrt{c}\,(d-a))$.
- ii) *If each of the three ratios* $a/b, b/d$ *and* d/a *is irrational, then* γ *is an open curve.*

By virtue of Theorem B we see that every circle of holomorphic torsion either $\tau = 0$ or ± 1 is always closed. Circles of holomorphic torsion ± 1, namely they lie on a holomorphic line $\mathbb{C}P^1(c)$ in $\mathbb{C}P^n(c)$, are said to be *Kähler circles*, and circles of null holomorphic torsion, that is, they lie on a totally real totally geodesic surface $\mathbb{R}P^2(c/4)$ in $\mathbb{C}P^n(c)$, are said to be *totally real circles*.

We next study the distribution of length spectrum of circles in $\mathbb{C}P^n(c)$. In general, it is usual that the length spectrum means the set of lengths of closed geodesics. In this section we shall investigate the lengths of closed circles in $\mathbb{C}P^n(c)$. We denote by $\mathrm{Cir}(M)$ the set of congruency classes of circles in a Riemannian manifold M. The length spectrum of circles in M is the map $\mathcal{R} : \mathrm{Cir}(M) \to \mathbb{R} \cup \{\infty\}$ defined by $\mathcal{R}([\gamma]) = \mathrm{length}(\gamma)$, where $[\gamma]$ is the congruency class containing γ. Sometimes we also call the image $\mathrm{RSpec}(M) = \mathcal{R}(\mathrm{Cir}(M)) \cap \mathbb{R}$ in the real line the length spectrum of circles in M. For example $\mathrm{RSpec}(S^n(c)) = (0, 2\pi/\sqrt{c}\,]$, $\mathrm{RSpec}(\mathbb{R}^n) = \mathrm{RSpec}(H^n(c)) = (0, \infty)$. For $\lambda \in \mathrm{RSpec}(M)$ the cardinality $m_c(\lambda)$ of the set $\mathcal{R}^{-1}(\lambda)$ is called the *multiplicity* of the length spectrum $\mathcal{R}$ at λ. When $m_c(\lambda) = 1$, we say that λ is *simple*. When the multiplicity of $\mathcal{R}$ is greater than one at some point λ, this means that there are circles which are not congruent to each other but have the same length λ. So we can say that $\mathcal{R}$ shows some geometry of M. The moduli space $\mathrm{Cir}(M)$ of circles has a natural stratification by their curvatures. We denote by $\mathrm{Cir}_k(M)$ the set of all congrueny classes of circles of curvature k in M and by $\mathcal{R}_k$ the restriction of the map $\mathcal{R}$ in this space. For a Kähler manifold M the moduli space of circles has another stratification by their holomorphic torsions. We denote by $\mathrm{Cir}^\tau(M)$ the set of all congruency classes of circles with holomorphic torsion τ in M and by $\mathcal{R}^\tau$ the restriction of $\mathcal{R}$ onto this space.

Theorem C ([1, 2])**.** *For a complex projective space $\mathbb{C}P^n(c)$ $(n \geqq 2)$ of constant holomorphic sectional curvature c, the length spectrum of circles has the following properties*:

(1) *Both of the sets*

$$\mathrm{RSpec}_k(\mathbb{C}P^n(c)) = \mathcal{R}(\mathrm{Cir}_k(\mathbb{C}P^n(c))) \cap \mathbb{R}$$

and

$$\mathrm{RSpec}^\tau(\mathbb{C}P^n(c)) = \mathcal{R}(\mathrm{Cir}^\tau(\mathbb{C}P^n(c))) \cap \mathbb{R}$$

are unbounded discrete subsets of $\mathbb{R}$ *for each* $k (> 0)$ *and* τ $(0 < \tau < 1)$.

(2) *The length spectrum* $\mathrm{RSpec}(\mathbb{C}P^n(c))$ *of circles coincides with the real half line* $(0, \infty)$.

(3) *For each $k > 0$ the bottom of* $\mathrm{RSpec}_k(\mathbb{C}P^n(c))$ *is* $2\pi/\sqrt{k^2 + c}$, *which is the length of a Kähler circle of curvature k. The second lowest element of* $\mathrm{RSpec}_k(\mathbb{C}P^n(c))$ *is* $4\pi/\sqrt{4k^2 + c}$, *which is the length of a totally real circle of curvature k. They are simple for $\mathcal{R}_k$.*

(4) *The mutiplicity m_c of $\mathcal{R}$ is finite at each point $\lambda \in (0, \infty)$ but not uniformly bounded. It satisfies*

$$\lim_{\lambda \to \infty} \frac{m_c(\lambda)}{\lambda^2 \log \lambda} = \frac{9c}{8\pi^4}.$$

(5) *A postive number λ is simple for $\mathcal{R}$ if and only if it satisfies $2\pi/\sqrt{c} < \lambda \leqq 4\sqrt{5}\,\pi/(3\sqrt{c}\,)$. Here, $2\pi/\sqrt{c}$ is the common length of a geodesic and a totally real*

circle of curvature $\sqrt{3c}/2$ *in* $\mathbb{C}P^n(c)$.
(6) *The multiplicity* $m_c^k(\lambda)$ *of* $\mathcal{R}_k$ $(k>0)$, *the cardinality of the set* $\mathcal{R}_k^{-1}(\lambda)$, *is not uniformly bounded and satisfies* $\lim_{\lambda\to\infty} m_c^k(\lambda)=\infty$. *However, the growth of the multiplicity* $m_c^k(\lambda)$ *with respect to* λ *is not so rapid. It satisfies* $\lim_{\lambda\to\infty}\lambda^{-\delta}m_c^k(\lambda)=0$ *for each positive* δ, *so that the growth order of* $m_c^k(\lambda)$ *is smaller than polynomial growth.*
(7) *Let* $n_c^k(\lambda)$ *denote the number of congruency classes of closed circles of curvature* k *in* M *with length not longer than* λ. *That is, it is the cardinality of the set* $\{[\gamma]\in \mathrm{Cir}_k(M) \mid \mathrm{length}(\gamma)\leqq\lambda\}$. *Then its asymptotic behaviour is of quadratic polynomial growth and satisfies*

$$\lim_{\lambda\to\infty}\frac{n_c^k(\lambda)}{\lambda^2}=\frac{3\sqrt{3}\,(4k^2+c)}{8\pi^4}\tan^{-1}\left(\frac{1}{\sqrt{3}\,\alpha_k}\right),$$

where $\alpha_k\,(\geqq 1)$ *denotes the unique number with*

$$3\sqrt{3}\,ck(4k^2+c)^{-3/2}=(9\alpha_k^2-1)(3\alpha_k^2+1)^{-3/2}.$$

In particular, it satisfies

$$\lim_{\lambda\to\infty}\lambda^{-2}n_c^{\sqrt{2c}/4}(\lambda)=3\sqrt{3}\,c/(32\pi^3).$$

For the length spectrum of circles in a real space form, which is congruent to either $S^n(c)$, $\mathbb{R}^n$ or $H^n(c)$, we can see that $\mathrm{RSpec}_k(M^n(c))=\{2\pi/\sqrt{k^2+c}\,\}$ if $k^2+c>0$, and $\mathrm{RSpec}_k(M^n(c))=\phi$ if $k^2+c\leqq 0$. We hence find that the length spectrum of circles in $\mathbb{C}P^n(c)$ is quite different from that of circles in a real space form. It follows from (2) and (5) in Theorem C that for each positive constant ℓ there exists a closed circle of length ℓ in $\mathbb{C}P^n(c)$ that we can determine its congruency class by ℓ if and only if $2\pi/\sqrt{c}<\ell\leqq 4\sqrt{5}\,\pi/(3\sqrt{c}\,)$. Even if we restrict ourselves to circles of a given curvature, there exist many pairs of closed circles with the same lengths which are not congruent to each other. For example, in $\mathbb{C}P^n(4)$ we consider circles of curvature $1/\sqrt{2}$. If a circle is of holomorphic torsion $5698/(559\sqrt{559}\,)$ and the other is of holomorphic torsion $12502/(559\sqrt{559}\,)$, then they are closed curves with common length $2\sqrt{1118}\,\pi/3$ but not congruent to each other (see Theorem B(3) and Lemma 7(2)).

Remark 2. It is known that $\mathbb{C}P^2(c)$ can be embedded as a totally geodesic submanifold into compact symmetric spaces M of rank one which are $\mathbb{C}P^n(c)$ $(n\geqq 2)$, a quaternionic projective space $\mathbb{H}P^n(c)$ $(n\geqq 2)$ and a Cayley projective plane $\mathbb{C}ayP^2(c)$ of maximal sectional curvature $c(>0)$. Moreover, by virtue of the work ([19]) we know that every circle in these symmetric spaces is congruent to some circle lying on $\mathbb{C}P^2(c)$ up to an isometry of M.

3. Main results

It is well-known that on an $n(\geqq 2)$-dimensional sphere $S^n(c)$ of constant sectional curvature c the length of every circle of curvature $k(\geqq 0)$ is not longer than $2\pi/\sqrt{c}$ which is the length of a great circle. Since all circles can be regarded as helices of order 2, it is natural to pose the following:

Question. When $n \geqq 3$, for every positive constant ℓ, does there exist a closed helix whose length is ℓ on $S^n(c)$?

The purpose of this section is to give the following partial affirmative answer to this problem.

Theorem 1 ([16])**.** *When $n \geqq 6$, for every positive constant ℓ, there exists a closed helix of order* 6 *whose length is ℓ on $S^n(c)$.*

In order to obtain Theorem 1 we consider an SU(n + 1)-equivariant minimal embedding $f_1 : \mathbb{C}P^n(c) \to S^{n(n+2)-1}((n+1)c/(2n))$ with parallel second fundamental form σ_1, which is defined by eigenfunctions associated to the first eigenvalue of the Laplacian Δ on $\mathbb{C}P^n(c)$ (see [12, 25]). We prove the following lemma which clarifies the inner product of the first normal space of the embedding f_1.

Lemma 8. *For any vectors X, Y, Z and W on $\mathbb{C}P^n(c)$ the inner product of the first normal space of the minimal embedding $f_1 : \mathbb{C}P^n(c) \to S^{n(n+2)-1}\big((n+1)c/(2n)\big)$ is expressed as*:

$$\begin{aligned}&\langle \sigma_1(X,Y), \sigma_1(Z,W)\rangle = -(c/(2n))\langle X,Y\rangle\langle Z,W\rangle \\ &\quad + (c/4)(\langle X,W\rangle\langle Y,Z\rangle + \langle X,Z\rangle\langle Y,W\rangle + \langle JX,W\rangle\langle JY,Z\rangle + \langle JX,Z\rangle\langle JY,W\rangle).\end{aligned}$$

Proof. We recall that the scalar curvature ρ of $\mathbb{C}P^n(c)$ is written as: $\rho = n(n+1)c$, and that the scalar curvature ρ of an m-dimensional minimal submanifold M^m of a real space form $\widetilde{M}^{m+p}(\tilde{c})$ of constant sectional curvature $\tilde{c}$ is expressed as: $\rho = m(m-1)\tilde{c} - \|\sigma\|^2$, where $\|\sigma\|$ is the length of the second fundamental form σ of the minimal immersion.

Hence the length $\|\sigma_1\|$ of σ_1 satisfies

$$n(n+1)c = 2n(2n-1)\frac{n+1}{2n}c - \|\sigma_1\|^2,$$

so that

$$\|\sigma_1\|^2 = (n+1)(n-1)c. \tag{3.1}$$

On the other hand, since the embedding f_1 is λ-*isotropic*, the second fundamental form σ_1 satisfies the following symmetric expression (see [23]):

$$\begin{aligned}&\langle \sigma_1(X,Y), \sigma_1(Z,W)\rangle + \langle \sigma_1(X,Z), \sigma_1(Y,W)\rangle + \langle \sigma_1(X,W), \sigma_1(Y,Z)\rangle \\ &\quad = \lambda^2(\langle X,Y\rangle\langle Z,W\rangle + \langle X,Z\rangle\langle Y,W\rangle + \langle X,W\rangle\langle Y,Z\rangle)\end{aligned} \tag{3.2}$$

for all vectors X, Y, Z and W on $\mathbb{C}P^n(c)$. Hence, for orthonormal vectors $e_1, e_2, \ldots, e_{2n}$ on $\mathbb{C}P^n(c)$ we have

$$2\langle \sigma_1(e_i,e_j), \sigma_1(e_i,e_j)\rangle + \langle \sigma_1(e_i,e_i), \sigma_1(e_j,e_j)\rangle = \lambda^2(2\delta_{ij}\delta_{ij} + 1)$$

for $i, j \in \{1, 2, \ldots, 2n\}$, which, together with a fact that trace $\sigma_1 = 0$, implies $2\|\sigma_1\|^2 = \lambda^2(4n + 4n^2)$. Thus from (3.1) we find that $\lambda^2 = (n-1)c/2n$. So, it follows from (3.2) that

$$\begin{aligned}(3.3)\qquad &\langle\sigma_1(X,Y),\sigma_1(Z,W)\rangle+\langle\sigma_1(X,Z),\sigma_1(Y,W)\rangle+\langle\sigma_1(X,W),\sigma_1(Y,Z)\rangle\\ &=\frac{(n-1)c}{2n}(\langle X,Y\rangle\langle Z,W\rangle+\langle X,Z\rangle\langle Y,W\rangle+\langle X,W\rangle\langle Y,Z\rangle)\end{aligned}$$

for all vectors X, Y, Z and W on $\mathbb{C}P^n(c)$.

We here denote by the curvature tensors R and $\widetilde{R}$ of $\mathbb{C}P^n(c)$ and $S^{n(n+2)-1}((n+1)c/2n)$, respectively. Then by Gauss equation we see that

$$\begin{aligned}&\langle\sigma_1(X,Y),\sigma_1(Z,W)\rangle-\langle\sigma_1(Z,Y),\sigma_1(X,W)\rangle\\ &=\langle R(Z,X)Y,W\rangle-\langle\widetilde{R}(Z,X)Y,W\rangle\\ &=(c/4)\{\langle X,Y\rangle\langle Z,W\rangle-\langle Z,Y\rangle\langle X,W\rangle+\langle JX,Y\rangle\langle JZ,W\rangle-\langle JZ,Y\rangle\langle JX,W\rangle\\ &\qquad-2\langle JZ,X\rangle\langle JY,W\rangle\}-(n+1)c/(2n)\{\langle X,Y\rangle\langle Z,W\rangle-\langle Z,Y\rangle\langle X,W\rangle\}.\end{aligned}$$

Here, exchanging W for Z in the above equality, we have

$$\begin{aligned}&\langle\sigma_1(X,Y),\sigma_1(Z,W)\rangle-\langle\sigma_1(W,Y),\sigma_1(X,Z)\rangle\\ &=(c/4)\{\langle X,Y\rangle\langle Z,W\rangle-\langle W,Y\rangle\langle X,Z\rangle+\langle JX,Y\rangle\langle JW,Z\rangle-\langle JW,Y\rangle\langle JX,Z\rangle\\ &\qquad-2\langle JW,X\rangle\langle JY,Z\rangle\}-(n+1)c/(2n)\{\langle X,Y\rangle\langle Z,W\rangle-\langle W,Y\rangle\langle X,Z\rangle\}.\end{aligned}$$

Thus, summing up these two equations and (3.3), we obtain the desirable equality. □

We study geometric properties of the following two-parameters family $\{\gamma_{k_1,k_2}\}$ of order 6 on $S^6(c)$.

Example 1. For constants k_1 and k_2 with $k_1 \geqq \sqrt{c/3}$, $k_2 \geqq 0$, $k_1^2k_2^2 \leqq c(3k_1^2-c)$, a helix γ_{k_1,k_2} of order 6 with the first curvature k_1 and the second curvature k_2 on $S^6(c)$ is defined as follows:
(1) When $k_2=0$, γ_{k_1,k_2} is a small circle of positive curvature $k_1(\geqq \sqrt{c/3}\,)$ on $S^6(c)$.
(2) When $k_1=\sqrt{c/2}$ and $k_2=\sqrt{c}$, γ_{k_1,k_2} is a helix of proper order 3 with the first curvature $\sqrt{c/2}$ and the second curvature $\sqrt{c}$ on $S^6(c)$.
(3) When $k_1 \neq \sqrt{c/2}\,,\sqrt{c/3}$ and $k_2=\sqrt{c(3k_1^2-c)}\,/k_1$, γ_{k_1,k_2} is a helix of proper order 4 with the first curvature k_1, the second curvature k_2 and the third curvature $k_3=|2k_1^2-c|/k_1$ on $S^6(c)$.
(4) When $k_1=\sqrt{c/2}$ and $k_2 \neq \sqrt{c}\,, 0$, γ_{k_1,k_2} is a helix of proper order 5 with the first curvature k_1, the second curvature k_2, the third curvature $k_3=\sqrt{c-k_2^2}$ and the fourth curvature $k_4=\sqrt{c/2}$ on $S^6(c)$.
(5) When $k_1 \neq \sqrt{c/2}\,,\sqrt{c/3}$ and $k_2 \neq \sqrt{c(3k_1^2-c)}\,/k_1$, 0, $\gamma_{k_1k_2}$ is a helix of proper order 6 with the first curvature k_1, the second curvature k_2, the third curvature $k_3=\sqrt{4k_1^2-k_2^2-c}$, the fourth curvature
$k_4=\sqrt{(3ck_1^2-k_1^2k_2^2-c^2)/(4k_1^2-k_2^2-c)}$ and the fifth curvature
$k_5=|2k_1^2-c|/\sqrt{4k_1^2-k_2^2-c}$ on $S^6(c)$.

Remark 3. The moduli space of $\{\gamma_{k_1,k_2}\}$ in Example 1(5) can be regarded as a smooth surface of $\mathbb{R}^3$ which is defined on the following domain:

$$D = \{(k_1, k_2) \in \mathbb{R}^2 |\, 4k_1^2 - k_2^2 > c,\, k_1^2 k_2^2 < c(3k_1^2 - c),\, k_1 \neq \sqrt{c/2}\,, k_1 > \sqrt{c/3}\,, k_2 > 0\}.$$

We shall prove the following which is a key in this section.

Theorem 2 ([16]). *For every positive constant ℓ, there exists a closed helix γ_{k_1,k_2} in Example 1, whose length is ℓ on $S^6(c)$. Moreover, this closed helix exists uniquely in the class of $\{\gamma_{k_1,k_2}\}$ with respect to* SO(7) *which is the full isometry group of the ambient space $S^6(c)$ if and only if ℓ satisfies $(\sqrt{3/c}\,)\pi < \ell \leqq 2(\sqrt{5/(3c)}\,)\pi$.*

Sketch of Proof. Our main tool is the minimal embedding $f_1 : \mathbb{C}P^2(4c/3) \to S^7(c)$. It follows Lemma 8 that the inner product of the first normal space of this isometric embedding f_1 is given by

$$\begin{aligned}\langle \sigma_1(X, Y), \sigma_1(Z, W)\rangle = (c/3)\{&-\langle X, Y\rangle\langle Z, W\rangle + \langle X, W\rangle\langle Y, Z\rangle + \langle X, Z\rangle\langle Y, W\rangle \\ &+ \langle JX, Z\rangle\langle JY, W\rangle + \langle JY, Z\rangle\langle JX, W\rangle\}.\end{aligned}$$

For each circle $\delta_{k,\tau}$ of curvature $k(\geqq 0)$ and holomorphic torsion τ $(0 \leqq \tau \leqq 1)$, using this equality and (1.1) repeatedly, we can see that the curve $f_1 \circ \delta_{k,\tau}$ is a helix of order 6 whose curvatures k_1, k_2, k_3, k_4, k_5 are expressed as:

$$k_1 = \sqrt{\frac{3k^2 + c}{3}}\,,\ k_2 = 3k\sqrt{\frac{c(1-\tau^2)}{3k^2 + c}}\,,\ k_3 = \sqrt{\frac{(6k^2 - c)^2 + 27\tau^2 k^2 c}{3(3k^2 + c)}}\,,$$
$$k_4 = 3k\tau\sqrt{\frac{c(3k^2 + c)}{(6k^2 - c)^2 + 27\tau^2 c k^2}}\,,\ k_5 = |6k^2 - c|\sqrt{\frac{3k^2 + c}{3\{(6k^2 - c)^2 + 27\tau^2 k^2 c\}}}\,.$$

Then we find that for every circle $\delta_{k,\tau}$ on the submanifold $\mathbb{C}P^2(4c/3)$ the curve $f_1 \circ \delta_{k,\tau}$ is a helix of order 6 on $S^7(c)$, so that it lies on a totally geodesic $S^6(c)$ in the ambient sphere $S^7(c)$. Note that our examples $\{\gamma_{k_1,k_2}\}$ are nothing but $\{f_1 \circ \delta_{k,\tau}\}$. Indeed, we here consider the case of (5) in Example 1. It follows from

$$k_1 = \sqrt{\frac{3k^2 + c}{3}}\,, \quad k_2 = 3k\sqrt{\frac{c(1-\tau^2)}{3k^2 + c}}$$

that

$$k = \sqrt{\frac{3k_1^2 - c}{3}} \quad \text{and} \quad \tau = \sqrt{\frac{c(3k_1^2 - c) - k_1^2 k_2^2}{c(3k_1^2 - c)}}\,. \tag{3.4}$$

These, together with the expressions of k_3, k_4 and k_5 in terms of k and τ, yield the desired expressions of k_3, k_4 and k_5 in terms of k_1 and k_2. Hence we can see that every curve γ_{k_1,k_2} in Example 1(5) is a curve $f_1 \circ \delta_{k,\tau}$ satisfying (3.4). Similarly, we can check easily other cases (1), (2), (3) and (4) in Example 1. This, together with Theorem C(2), gives the first half in Theorem 2. Furthermore, the latter half of our statement is an immediate consequence of Theorem C(5). □

Theorem 1 is an immediate consequence of Theorem 2. The following proposition shows that every γ_{k_1,k_2} of (1), (2) and (3) is closed but there exist many *open* $\gamma_{k_1.k_2}$ as well as many closed γ_{k_1,k_2} in other cases (4) and (5). Combining Theorem B with the above discussion, we have the following:

Proposition 9 ([16]). *The closedness of a curve γ_{k_1,k_2} in Example 1 is as follows*:
(1) *When $k_2 = 0$, γ_{k_1,k_2} is a closed simple curve with length $2\pi/\sqrt{k_1^2+c}$.*
(2) *When $k_1 = \sqrt{c/2}$ and $k_2 = \sqrt{c}$, γ_{k_1,k_2} is a closed simple curve with length $2\sqrt{2}\,\pi/\sqrt{c}$.*
(3) *When $k_1 \neq \sqrt{c/2}, \sqrt{c/3}$ and $k_2 = \sqrt{c(3k_1^2-c)}/k_1$, γ_{k_1,k_2} is a closed simple curve with length $2\pi/\sqrt{k_1^2+(c/3)}$.*
(4) *When $k_1 = \sqrt{c/2}$ and $k_2 \neq \sqrt{c}, 0$, using the three distinct real solutions a, b, d $(a < b < d)$ to a cubic equation $2\sqrt{c}\,\lambda^3 - 3\sqrt{c}\,\lambda + \sqrt{2(c-k_2^2)} = 0$, we find the following*:

- (4i) *If one of the three ratios $a/b, b/d, d/a$ is rational, γ_{k_1,k_2} is a simple closed curve whose length is the least common multiple of $2\sqrt{3}\,\pi/(\sqrt{c}\,(b-a))$ and $2\sqrt{3}\,\pi/(\sqrt{c}\,(d-a))$;*
- (4ii) *If each of the three ratios $a/b, b/d, d/a$ is irrational, γ_{k_1,k_2} is a simple open curve.*

(5) *When $k_1 \neq \sqrt{c/2}, \sqrt{c/3}$ and $k_2 \neq \sqrt{c(3k_1^2-c)/k_1}, 0$, using the three distinct real solutions a, b, d $(a < b < d)$ to a cubic equation $c\lambda^3 - 3k_1^2\lambda + \sqrt{3ck_1^2 - k_1^2k_2^2 - c^2} = 0$, we find the following*:

- (5i) *If one of the three ratios $a/b, b/d, d/a$ is rational, γ_{k_1,k_2} is a simple closed curve whose length is the least common multiple of $2\sqrt{3}\,\pi/(\sqrt{c}\,(b-a))$ and $2\sqrt{3}\,\pi/(\sqrt{c}\,(d-a))$;*
- (5ii) *If each of the three ratios $a/b, b/d, d/a$ is irrational, γ_{k_1,k_2} is a simple open curve.*

4. Kähler immersions and homogeneous curves

We here consider a Kähler isometric full immersion of a complex projective space into another complex projective space. It is well-known that such Kähler isometric immersions are nothing but Kähler embeddings defined by

Example 2.

$$f_k^n : \mathbb{C}P^n(c/k) \ni [z_i]_{0\leqq i\leqq n} \mapsto \left[\sqrt{\frac{k!}{k_0!\cdots k_n!}}\, z_0^{k_0}\cdots z_n^{k_n}\right]_{k_0+\cdots+k_n=k} \in \mathbb{C}P^m(c)$$

given by using homogeneous coordinates of complex projective spaces for each positive integer k. Here $m = (n+k)!/(n!k!) - 1$ for given k (cf. [10, 21]). We usually call f_k^n the k-th *Veronese embedding* or *Calabi embedding*, which is an SU(n + 1)-equivariant isometric immersion. In particular, when $k = 1, 2$, they have parallel second fundamental forms. Every Kähler isometric embedding f_k^n has various geometric properties. Among these we point out that for each geodesic

γ on $\mathbb{C}P^n(c/k)$ the curve $f_k^n \circ \gamma$ is a homogeneous curve, so that it is a helix, of proper order k on a totally real totally geodesic submanifold $\mathbb{R}P^k(c/4)$ of the ambient space $\mathbb{C}P^m(c)$. By using this property Pak and Sakamoto ([24]) give a characterization of Veronese embeddings.

Theorem D ([24])**.** *Let $f: M_n \to \mathbb{C}P^m(c)$ be a Kähler isometric full immersion of an n-dimensional Kähler manifold M into a complex projective space. Then the following two conditions are equivalent*:
(1) *The immersion f is locally equivalent to the k-th Veronese embedding, the submanifold M is locally congruent to $\mathbb{C}P^n(c)$ and $m = (n+k)!/(n!k!) - 1$;*
(2) *For each geodesic γ on M, the curve $f \circ \gamma$ lies on a totally real totally geodesic submanifold $\mathbb{R}P^k(c/4)$ of the ambient space $\mathbb{C}P^m(c)$, but it does not lie on a totally geodesic proper submanifold $\mathbb{R}P^d(c/4)$ of $\mathbb{R}P^k(c/4)$.*

We next observe the extrinsic shape of *circles* (, say) γ on the submanifold $\mathbb{C}P^n(c/k)$ through the embedding f_k^n into the ambient space $\mathbb{C}P^m(c)$. We note that all of such circles γ map to homogeneous curves in $\mathbb{C}P^m(c)$, so that in particular they have the constant curvature functions in the sense of Frenet formula (1.1). Though it is not easy to compute curvatures of the curve $f_k^n \circ \gamma$, paying attention to the constancy of its first curvature, we obtain the following characterization of Veronese embeddings.

Theorem 3 ([14])**.** *Let $f: M_n \to \mathbb{C}P^m(c)$ be a Kähler isometric full immersion of an n-dimensional Kähler manifold M_n into $\mathbb{C}P^m(c)$. Then the following conditions are equivalent to each other*:
(1) *The immersion f is locally equivalent to some Veronese embedding f_k^n. That is, the submanifold M_n is locally congruent to $\mathbb{C}P^n(c/k)$ and $f = f_k^n$ with $m = (n+k)!/(n!k!) - 1$;*
(2) *There exists a positive constant κ satisfying that for each circle γ of curvature κ on M_n the first curvature of the curve $f \circ \gamma$ is constant along this curve.*

Proof. (1) $\Longrightarrow$ (2): For each Veronese embedding $f_k^n : \mathbb{C}P^n(c/k) \to \mathbb{C}P^m(c)$ the second fundamental form σ_k satisfies $\|\sigma_k(X,X)\|^2 = c(k-1)/(2k)$ for any unit vector X at each point $x \in \mathbb{C}P^n(c/k)$ (see [22]). Then we find that for each circle γ of curvature κ on $\mathbb{C}P^n(c/k)$ the curve $f_k^n \circ \gamma$ has constant first curvature $\kappa_1 = \sqrt{\kappa^2 + \dfrac{c(k-1)}{2k}}$ in the ambient space $\mathbb{C}P^m(c)$. Hence we have Condition (2).

(2) $\Longrightarrow$ (1): Let $f: M_n \to \mathbb{C}P^m(c)$ be a Kähler isometric full immersion satisfying Condition (2). Then we see easily that M_n is constant (λ-)isotropic in $\mathbb{C}P^m(c)$. On the other hand we denote by R and $\widetilde{R}$ the curvature tensors of M_n and $\mathbb{C}P^m(c)$, respectively. So the Gauss equation is written as:

$$\langle R(X,Y)Z,W\rangle = \langle \widetilde{R}(X,Y)Z,W\rangle + \langle \sigma(Y,Z),\sigma(X,W)\rangle - \langle \sigma(X,Z),\sigma(Y,W)\rangle.$$

Since M_n is a Kähler submanifold of $\mathbb{C}P^m(c)$, from this equation and

$$\widetilde{R}(X,Y)Z = (c/4)(\langle Y,Z\rangle X - \langle X,Z\rangle Y + \langle JY,Z\rangle JX - \langle JX,Z\rangle JY - 2\langle JX,Y\rangle JZ),$$

we find that the holomorphic sectional curvature $K(X, JX)$ of M_n determined by a unit vector X is expressed as:

$$K(X, JX) = \langle R(X, JX)JX, X\rangle = c - 2\|\sigma(X, X)\|^2.$$

Thus we can see that our submanifold M_n is a complex space form. Therefore by virtue of Nakagawa and Ogiue's work ([21]) we obtain Condition (1). □

In the following, we pay particular attention to a special case of Example 2.

Example 3. $f_2^1 : S^2(c/2)(= \mathbb{C}P^1(c/2)) \to \mathbb{C}P^2(c)$, which is defined by $f_2^1(z_0, z_1) = (z_0^2, \sqrt{2}\, z_0z_1, z_1^2)$.

We here study geometric properties of images of small circles in $S^2(c/2)$ under the isometric embedding f_2^1.

Proposition 10 ([3])**.** *For a small circle γ of curvature $k(> 0)$ on $S^2(c/2)$, the curve $f_2^1 \circ \gamma$ is a helix of order 4 in $\mathbb{C}P^2(c)$. More precisely,*

(1) *When $k = \dfrac{\sqrt{c}}{2\sqrt{2}}$, it is a helix of proper order 3 with curvatures $k_1 = \dfrac{\sqrt{3c}}{2\sqrt{2}}$, $k_2 = \dfrac{\sqrt{3c}}{2}$.*

(2) *When $k \neq \dfrac{\sqrt{c}}{2\sqrt{2}}$, it is a helix of proper order 4 with curvatures*

$$k_1 = \sqrt{k^2 + \frac{c}{4}}\,,\ k_2 = \frac{3k\sqrt{c}}{\sqrt{4k^2+c}},\ k_3 = \frac{|8k^2 - c|}{2\sqrt{4k^2+c}}.$$

Proof. Note that the second fundamental form σ of the embedding f_2^1 satisfies

$$\|\sigma(X, X)\| = \frac{\sqrt{c}}{2} \quad \text{for each unit vector } X. \tag{4.1}$$

We denote by $\widetilde{\nabla}$ the Riemannian connection of $\mathbb{C}P^2(c)$. Suppose that

$$\nabla_X X = \kappa Y,\ \nabla_X Y = -\kappa X \quad \text{with } X = V_1 = \dot{\gamma}.$$

Then by the Gauss formula and (4.1) the curve $f_2^1 \circ \gamma$ satisfies

$$\widetilde{\nabla}_{V_1} V_1 = k_1 V_2,\ \text{where } k_1 = \sqrt{k^2 + \frac{c}{4}} \text{ and } V_2 = \frac{1}{k_1}\big(kY + \sigma(X, X)\big).$$

Since $Y = \pm JX$ and f_2^1 is holomorphic, we have $\sigma(X, Y) = \pm J\sigma(X, X)$. As f_2^1 has parallel second fundamental form, we obtain by direct calculations

$$\widetilde{\nabla}_{V_1} V_2 = -k_1V_1 + k_2V_3,\ \text{with } k_2 = \frac{3k\sqrt{c}}{\sqrt{4k^2+c}},\ V_3 = \frac{2}{\sqrt{c}}\sigma(X, Y).$$

Continuing routine calculations, we get the following when $k \neq \dfrac{\sqrt{c}}{2\sqrt{2}}$

$$\widetilde{\nabla}_{V_1} V_3 = -k_2V_2 + k_3V_4, \quad \widetilde{\nabla}_{V_1} V_4 = -k_3V_3,$$

where

$$k_3 = \frac{|8k^2 - c|}{2\sqrt{4k^2+c}}, \quad V_4 = \frac{4}{\sqrt{c(4k^2+c)}}\left(\frac{c}{4}Y - k\cdot\sigma(X,X)\right).$$

When $k = \dfrac{\sqrt{c}}{2\sqrt{2}}$, we find $\widetilde{\nabla}_{V_1}V_3 = -k_2V_2$, so that we get the conclusion. □

Holomorphic torsions of such helices are expressed as:

Proposition 11 ([3])**.** *For a circle γ of curvature $k(>0)$ on $S^2(c/2)$, the helix $f_2^1\circ\gamma$ is closed with length $2\pi/\sqrt{k^2+(c/2)}$, and all of the holomorphic torsions of such a helix are written by*

(1) *When $k > \dfrac{\sqrt{c}}{2\sqrt{2}}$, we have $\tau_{12} = \tau_{34} = \dfrac{\pm k}{\sqrt{k^2+\frac{c}{4}}}$, $\tau_{23} = \tau_{14} = \dfrac{\pm\sqrt{c}}{\sqrt{4k^2+c}}$, $\tau_{13} = \tau_{24} = 0$.*

(2) *When $k = \dfrac{\sqrt{c}}{2\sqrt{2}}$, we have $\tau_{12} = \pm\dfrac{1}{\sqrt{3}}$, $\tau_{23} = \pm\dfrac{\sqrt{2}}{\sqrt{3}}$, $\tau_{13} = 0$.*

(3) *When $k < \dfrac{\sqrt{c}}{2\sqrt{2}}$, we have $\tau_{12} = -\tau_{34} = \dfrac{\pm k}{\sqrt{k^2+\frac{c}{4}}}$, $\tau_{23} = -\tau_{14} = \dfrac{\pm\sqrt{c}}{\sqrt{4k^2+c}}$, $\tau_{13} = \tau_{24} = 0$.*

Proof. It is known that every circle γ of curvature $k(\geqq 0)$ on an n-dimensional Euclidean sphere $S^n(c)$ is a simple closed curve with length $\dfrac{2\pi}{\sqrt{k^2+c}}$. Since f_2^1 is an isometric embedding, we find that $f_2^1\circ\gamma$ is simple and closed with length $\dfrac{2\pi}{\sqrt{k^2+\frac{c}{2}}}$.

As the embedding f_2^1 is holomorphic we have

$$\tau_{13} = \langle V_1, JV_3\rangle = \frac{2}{\sqrt{c}}\langle X, J\cdot\sigma(X,Y)\rangle = 0.$$

By routine calculations we get the desired expression for other τ_{ij}. □

By virtue of Propositions 10 and 11 we obtain

Theorem 4 ([3])**.** *For a circle γ of curvature $k(>0)$ on $S^2(c/2)$, the helix $f_2^1\circ\gamma$ in Example 3 is closed with length $2\pi/\sqrt{k^2+(c/2)}$, and it is a homogeneous curve of order 4 in $\mathbb{C}P^2(c)$. More precisely, when $k\neq\sqrt{c}/(2\sqrt{2})$, this homogeneous curve is of proper order 4, and $k=\sqrt{c}/(2\sqrt{2})$, it is of proper order 3.*

Remark 4. When $k>\sqrt{c}/(2\sqrt{2})$, the holomorphic torsions of the helix $f_2^1\circ\gamma$ in Theorem 4 satisfy (1.2), and when $k<\sqrt{c}/(2\sqrt{2})$, the holomorphic torsions of the helix $f_2^1\circ\gamma$ in Theorem 4 satisfy (1.3)

5. Totally real immersions and helices

We first recall the totally real minimal parallel isotropic embedding f of the flat surface N into $\mathbb{C}P^2(4)$ which is given in Section 2. The second fundamental form

σ_f of this embedding is expressed as follows for every unit vector $w \in TS^1(1)$ of the second component;

$$\sigma_f(u,u) = -\frac{1}{\sqrt{2}}Ju, \quad \sigma_f(w,w) = \frac{1}{\sqrt{2}}Ju, \quad \sigma_f(u,w) = \frac{1}{\sqrt{2}}Jw, \tag{5.1}$$

where the tangent vector $u \in TS^1(1)$ of the first component is the normalized vector of $\partial/\partial\theta$.

We shall study images of circles of positive curvature on N through this isometric embedding f. Note that the embedding f maps each geodesic on N to a circle of curvature $1/\sqrt{2}$ in $\mathbb{C}P^2(4)$ (see Theorem A(1)). This circle does not have self-intersections, but it is not necessarily closed in $\mathbb{C}P^2(4)$. For images of circles on N through f we have the following.

Proposition 12 ([3])**.** *For a circle γ of curvature $k(>0)$ on N, the curve $f \circ \gamma$ is a helix of order 4 in $\mathbb{C}P^2(4)$. More precisely,*

(1) *when $k = \dfrac{1}{2}$, it is a helix of proper order 3 with curvatures $k_1 = \dfrac{\sqrt{3}}{2}$, $k_2 = \sqrt{\dfrac{3}{2}}$,*

(2) *when $k \neq \dfrac{1}{2}$, it is a helix of proper order 4 with curvatures*

$$k_1 = \sqrt{k^2 + \frac{1}{2}}, \; k_2 = \frac{3k}{\sqrt{2k^2+1}}, \; k_3 = \frac{|4k^2-1|}{\sqrt{2(2k^2+1)}}.$$

Proof. We denote by $\widetilde{\nabla}$ and ∇ the Riemannian connections of $\mathbb{C}P^2(4)$ and N, respectively. We take a circle γ of curvature $k(>0)$ on N satisfying the following equations:

$$\nabla_X X = kY \quad \text{and} \quad \nabla_X Y = -kX \quad \text{with} \quad X = V_1 = \dot{\gamma}. \tag{5.2}$$

We can represent the orthonormal pair $\{X, Y\}$ as:

$$\begin{cases} X = \cos\phi \cdot u + \sin\phi \cdot w, \\ Y = -\sin\phi \cdot u + \cos\phi \cdot w \quad (0 \leqq \phi < 2\pi), \end{cases} \tag{5.3}$$

at each point $\gamma(s)$. So we have $\sigma_f(X,X) = -\sigma_f(Y,Y)$. By the Gauss formula we see that the curve $f \circ \gamma$ satisfies

$$\widetilde{\nabla}_{V_1} V_1 = k_1 V_2, \text{ where } \quad k_1 = \sqrt{k^2 + \frac{1}{2}} \quad \text{and} \quad V_2 = \frac{1}{k_1}(kY + \sigma(X,X)). \tag{5.4}$$

Since f has parallel second fundamental form, we find by the Gauss formula and (5.2) that

$$\widetilde{\nabla}_{V_1} V_2 = -k_1 V_1 + k_2 V_3,$$

where

$$k_2 = \frac{3k}{\sqrt{2k^2+1}} \quad \text{and} \quad V_3 = \sqrt{2}\,\sigma(X,Y).$$

When $k \neq 1/2$, by routine calculations and (5.3) we have

$$\widetilde{\nabla}_{V_1} V_3 = -k_2 V_2 + k_3 V_4 \quad \text{and} \quad \widetilde{\nabla}_{V_1} V_4 = -k_3 V_3,$$

where

$$\begin{cases} k_3 = \dfrac{|4k^2-1|}{\sqrt{2(2k^2+1)}}, \\ V_4 = \dfrac{\sqrt{2(2k^2+1)}}{|4k^2-1|}\left\{\left(\dfrac{3\sqrt{2}\,k^2}{2k^2+1} - \dfrac{1}{\sqrt{2}}\right)Y + \sqrt{2}\,k\left(\dfrac{3}{2k^2+1} - 2\right)\sigma(X,X)\right\}. \end{cases}$$

When $k = 1/2$, since our calculations go through without the field V_4, we obtain the conclusion. □

We next study other geometric properties of these helices.

Theorem 5 ([3])**.** *Let $f : N \to \mathbb{C}P^2(4)$ denote the embedding given in Section 2 and γ be a circle of curvature $k(>0)$ on N. Then we have the following:*
(1) *The helix $f \circ \gamma$ is closed of length $2\pi/k$.*
(2) *The helix $f \circ \gamma$ has self-intersections if and only if $k \leqq 3/(\sqrt{2}\,\pi)$. The number of intersection points is greater than 2.*
(3) *The helix $f \circ \gamma$ is not a homogeneous curve, i.e., it is not generated by any Killing vector field on $\mathbb{C}P^2(4)$.*

Proof. (1), (2): We first consider the universal Riemannian covering $p : \mathbb{R}^2 \to N$. Regarding the Riemannian metric of N, we can choose a fundamental region for N in $\mathbb{R}^2$ as $\mathfrak{F} = \left[0, \dfrac{2\sqrt{2}}{3}\pi\right) \times \left[0, \dfrac{\sqrt{6}}{3}\pi\right)$. Two points (x_1, y_1) and (x_2, y_2) on $\mathbb{R}^2$ satisfies $p((x_1, y_1)) = p((x_2, y_2))$ if and only if either

i) $x_1 - x_2 = \dfrac{\sqrt{2}}{3}(2m_1)\pi,\ y_1 - y_2 = \dfrac{\sqrt{6}}{3}(2m_2)\pi$ for some $m_1, m_2 \in \mathbb{Z}$, or

ii) $x_1 - x_2 = \dfrac{\sqrt{2}}{3}(2m_1+1)\pi,\ y_1 - y_2 = \dfrac{\sqrt{6}}{3}(2m_2+1)\pi$ for some $m_1, m_2 \in \mathbb{Z}$.

We set the equivalence relation "$\sim$" on $\mathbb{R}^2$ defined by $p((x_1, y_1)) = p((x_2, y_2))$. In the following, we shall study on the above fundamental region $\mathfrak{F}$ instead of the flat surface $N(=\mathbb{R}^2/\sim)$ in Section 2. However note that in the region $\mathfrak{F}$ we identify the points $\left(\dfrac{2\sqrt{2}}{3}\pi, \dfrac{\sqrt{6}}{3}\pi\right)$ and $\left(\dfrac{\sqrt{2}}{3}\pi, 0\right)$. So we should regard this region $\mathfrak{F}$ as a parallelogram but not a rectangle. We denote by $\tilde{\gamma}$ a covering circle in $\mathbb{R}^2$, namely it is a circle with radius $1/k$ in the sense of Euclidean Geometry. This means that γ is a closed curve of length $2\pi/k$.

We shall show that γ has self-intersections in the case of $k \leqq 3/(\sqrt{2}\,\pi)$. The covering circle $\tilde{\gamma}(s) = (\tilde{\gamma}_1(s), \tilde{\gamma}_2(s))$ is represented as:

$$\begin{cases} \tilde{\gamma}_1(s) = \dfrac{1}{k}(v_2 \cos ks + v_1 \sin ks) - \dfrac{v_2}{k} + \tilde{\gamma}_1(0), \\ \tilde{\gamma}_2(s) = \dfrac{1}{k}(-v_1 \cos ks + v_2 \sin ks) + \dfrac{v_1}{k} + \tilde{\gamma}_2(0), \end{cases}$$

where $(v_1, v_2) \in \mathbb{R}^2$ denotes the unit tangent vector $\dot{\tilde{\gamma}}(0)$. If $\gamma(s_0) = \gamma(0)$ $(s_0 \neq 0)$, then $\tilde{\gamma}(s_0)$ and $\tilde{\gamma}(0)$ satisfy either the condition i) or ii). When they satisfy the condition i), we find

$$\begin{cases} \sin ks_0 = \dfrac{2\sqrt{2}}{3} m_1 \pi v_1 k + \dfrac{2\sqrt{6}}{3} m_2 \pi v_2 k, \\ \cos ks_0 = \dfrac{2\sqrt{2}}{3} m_1 \pi v_2 k - \dfrac{2\sqrt{6}}{3} m_2 \pi v_1 k + 1, \end{cases}$$

so that

$$\pi k = \frac{3(-\sqrt{2}\, v_2 m_1 + \sqrt{6}\, v_1 m_2)}{2(m_1^2 + 3m_2^2)} \tag{5.5}$$

for some integers m_1, m_2 with $(m_1, m_2) \neq (0,0)$. Similarly, when they satisfy the condition ii), we see

$$\pi k = \frac{3\{-\sqrt{2}\, v_2(2m_1+1) + \sqrt{6}\, v_1(2m_2+1)\}}{(2m_1+1)^2 + 3(2m_2+1)^2} \tag{5.6}$$

for some integers m_1, m_2 with $(m_1, m_2) \neq (0,0)$. Conversely, if there exists a pair of integers $(m_1, m_2)(\neq (0,0))$ satisfying either (5.5) or (5.6) for some $(v_1, v_2) \in \mathbb{R}^2$ with $v_1^2 + v_2^2 = 1$, we see that γ has a self-intersection. The number of intersection points corresponds to the cardinality of pairs (v_1, v_2) with such properties. That is, the circle γ has self-intersections if and only if some of the images of its covering circles in $\mathbb{R}^2$ cut each other in $\mathfrak{F}$. When $k > 3/(\sqrt{2}\,\pi)$, every covering circle $\tilde{\gamma}$ is contained in a fundamental region $\mathfrak{F}$, so that the circle γ does not have self-intersections in this case. Hence the circle γ does not have self-intersections. In the case of $k = 3/(\sqrt{2}\,\pi)$, the circle $\tilde{\gamma}$ has three points of contact (see Figure in p.140 in [3]). This corresponds to the fact that for $(v_1, v_2) = (\sqrt{3}/2, -1/2)$, $(\sqrt{3}/2,\, 1/2)$, $(0,1)$ the equation (5.5) holds with $(m_1, m_2) = (0,1)$, $(0,-1)$, $(-1,0)$, respectively. Thus we can find easily that γ has self-intersections if and only if $k \leqq 3/(\sqrt{2}\,\pi)$ and that the number of self-intersection points is greater than 2 in this case. Since $f : N \to \mathbb{C}P^2(4)$ is an isometric embedding, we can see that the curve $f \circ \gamma$ inherits these properties.

(3): To see that $f \circ \gamma$ is not homogeneous in $\mathbb{C}P^2(4)$, we compute the holomorphic torsion $\tau_{12} = \langle V_1, JV_2 \rangle$. As f is totally real, we find that

$$\tau_{12} = \frac{1}{k_1} \langle X, J \cdot \sigma_f(X,X) \rangle.$$

We here make use of the representation (5.3). We denote by ϕ_0 the angle between $\dot{\tilde{\gamma}}$ and the positive direction of the first component in $\tilde{N} = \mathbb{R}^2$. Then the angle between $\dot{\tilde{\gamma}}(s)$ and the positive direction of the first component is $\phi = ks + \phi_0$. Making use of this angle in (5.3), from (5.1) we have

$$\begin{cases} \sigma_f(X,X) = -\sigma_f(Y,Y) = \dfrac{1}{\sqrt{2}}(-\cos 2\phi \cdot Ju + \sin 2\phi \cdot Jw), \\ \sigma_f(X,Y) = \dfrac{1}{\sqrt{2}}(\sin 2\phi \cdot Ju + \cos 2\phi \cdot Jw). \end{cases} \tag{5.7}$$

Hence we can see that $\tau_{12} = \dfrac{1}{\sqrt{2k^2+1}}\cos 3\phi$. As τ_{12} is not constant, the curve $f \circ \gamma$ is not homogeneous in $\mathbb{C}P^2(4)$ (see Lemma 3). $\square$

We shall compute all the holomorphic torsions of the curve $f\circ\gamma$. When $k > 1/2$, we find that

$$\tau_{24} = \frac{1}{k_1 k_3}\left\{\sqrt{2}\left(\frac{3}{2k^2+1} - 2\right)k^2 - \left(\frac{3\sqrt{2}\,k^2}{2k^2+1} - \frac{1}{\sqrt{2}}\right)\right\}\langle Y, J\cdot\sigma_f(X,X)\rangle,$$

which, together with (5.3) and (5.7), shows that $\tau_{24} = \sin 3(ks+\phi_0)$. When $k < 1/2$, we see $\tau_{24} = -\sin 3(ks+\phi_0)$. By the same calculation we have the following.

Proposition 13 ([3]). *For a circle of curvature $k(>0)$ in the flat surface N the holomorphic torsions $\tau_{ij}(s) = \langle V_i(s), JV_j(s)\rangle$ $(1 \leqq i < j \leqq 4)$ of the curve $f\circ\gamma$ are described as follows*:

(1) *When $k > 1/2$, we have*

$$\tau_{12} = \tau_{34} = \frac{1}{\sqrt{2k^2+1}}\cos 3(ks+\phi_0),\ \ \tau_{13} = -\tau_{24} = -\sin 3(ks+\phi_0),$$
$$\tau_{14} = \tau_{23} = -\frac{\sqrt{2}\,k}{\sqrt{2k^2+1}}\cos 3(ks+\phi_0).$$

(2) *When $k = 1/2$, we have*

$$\tau_{12} = \sqrt{\frac{2}{3}}\cos 3\Big(\frac{1}{2}s+\phi_0\Big),\ \ \tau_{13} = -\sin 3\Big(\frac{1}{2}s+\phi_0\Big),$$
$$\tau_{23} = -\frac{1}{\sqrt{3}}\cos 3\Big(\frac{1}{2}s+\phi_0\Big).$$

(3) *When $k < 1/2$, we have*

$$\tau_{12} = -\tau_{34} = \frac{1}{\sqrt{2k^2+1}}\cos 3(ks+\phi_0),\ \ \tau_{13} = \tau_{24} = -\sin 3(ks+\phi_0),$$
$$\tau_{14} = -\tau_{23} = -\frac{\sqrt{2}\,k}{\sqrt{2k^2+1}}\cos 3(ks+\phi_0).$$

Here, ϕ_0 is the angle between $\dot{\tilde\gamma}(0)$ and the unit vector u tangent to the first component of N.

Inspired by Proposition 12, we shall investigate the image of a circle γ of curvature 1/2 on the flat surface N through the totally real minimal parallel embedding $f : N \to \mathbb{C}P^m(4)$.

Let $\tilde\gamma$ denote a covering circle in $\mathbb{R}^2$ of a circle of curvature 1/2 on N. Then the curve $\tilde\gamma$ is a circle of radius 2 in the sense of Euclidean Geometry. This, together with the fact that f is an isometric embedding, implies that the curve $f\circ\gamma$ is a closed curve of length 4π in $\mathbb{C}P^2(4)$. Moreover, since the curvature $1/2$ of the circle γ is less than $3/(\sqrt{2}\,\pi)(=0.67\ldots)$, it has self-intersection points so does the curve $f\circ\gamma$ (see Theorem 5(1), (2)). Suppose that $\gamma(s_0)$ is a self-intersection point. Denoting the tangential vector $\dot\gamma(s_0)$ by $(v_1,v_2)\in\mathbb{R}^2\cong T_{\gamma(s_0)}N$, we have

$$\tilde\gamma(s)=2\begin{pmatrix} v_1\sin\frac{s-s_0}{2}+v_2\Big(\cos\frac{s-s_0}{2}-1\Big)\\ v_2\sin\frac{s-s_0}{2}-v_1\Big(\cos\frac{s-s_0}{2}-1\Big)\end{pmatrix}+\tilde\gamma(s_0).$$

If $\gamma(s_0+s_1)=\gamma(s_0)$, then we see in the case i) in the proof of Theorem 5 that

$$\begin{cases} v_1\sin\dfrac{s_1}{2}+v_2\Big(\cos\dfrac{s_1}{2}-1\Big)=\dfrac{\sqrt{2}}{3}m_1\pi,\\ v_2\sin\dfrac{s_1}{2}-v_1\Big(\cos\dfrac{s_1}{2}-1\Big)=\dfrac{\sqrt{6}}{3}m_2\pi.\end{cases}$$

Since $v_1^2+v_2^2=1$, we find in this case that

$$\begin{cases} \sin\dfrac{s_1}{2}=\dfrac{\sqrt{2}}{3}\pi(m_1v_1+\sqrt{3}\,m_2v_2),\\ \cos\dfrac{s_1}{2}=\dfrac{\sqrt{2}}{3}\pi(m_1v_2-\sqrt{3}\,m_2v_1)+1.\end{cases}$$

Next, if $\gamma(s_0+s_1)=\gamma(s_0)$, then we see in the case ii) in the proof of Theorem 5 that

$$\begin{cases} v_1\sin\dfrac{s_1}{2}+v_2\Big(\cos\dfrac{s_1}{2}-1\Big)=\dfrac{\sqrt{2}}{6}(2m_1+1)\pi,\\ v_2\sin\dfrac{s_1}{2}-v_1\Big(\cos\dfrac{s_1}{2}-1\Big)=\dfrac{\sqrt{6}}{6}(2m_2+1)\pi.\end{cases}$$

Since $v_1^2+v_2^2=1$, we find in this case that

$$\begin{cases} \sin\dfrac{s_1}{2}=\dfrac{\sqrt{2}}{6}\pi\{(2m_1+1)v_1+\sqrt{3}\,(2m_2+1)v_2\},\\ \cos\dfrac{s_1}{2}=\dfrac{\sqrt{2}}{6}\pi\{(2m_1+1)v_2-\sqrt{3}\,(2m_2+1)v_1\}+1.\end{cases}$$

Thus we find that (v_1,v_2) satisfies either
I) $\pi(m_1^2+3m_2^2)=3(-\sqrt{2}\,v_2m_1+\sqrt{6}\,v_1m_2)$, or
II) $\pi\{(2m_1+1)^2+3(2m_2+1)^2\}=6\{-\sqrt{2}\,v_2(2m_1+1)+\sqrt{6}\,v_1(2m_2+1)\}$
for some integers $(m_1,m_2)\neq(0,0)$ corresponding to the conditions i) and ii) in the proof of Theorem 5. In [6], there is an error in the case II). In our case, as the curvature $1/2$ of γ is greater than $3/(\sqrt{6}\,\pi)$, such conditions might occur only for $(m_1,m_2)=(\pm1,0)$ for the case I) and for $(m_1,m_2)=(0,-1)$, $(0,0)$, $(-1,0)$, $(-1,-1)$, $(1,-1)$, $(1,0)$ for the case II) (see Figure 1 in p.241 in [6]). Checking these conditions (or looking Figure 1 carefully), we can see that the circle γ does not have self intersection points corresponding to $(m_1,m_2)=(1,-1)$, $(1,0)$ for II), and has self-intersection points corresponding to other conditions. Thus we find that the circle γ has 6 self-intersection points, and so does $f\circ\gamma$.

Summing up we obtain the following

Theorem 6 ([6]). *Let $f : N \to \mathbb{C}P^2(4)$ denote the embedding given in Section 2 and γ be a circle of curvature $1/2$ on N. Then the curve $f \circ \gamma$ satisfies the following:*
(1) It is a closed helix of proper order 3 of length 4π with curvatures $k_1 = \sqrt{3}/2,\ k_2 = \sqrt{3/2}$ on $\mathbb{C}P^2(4)$;
(2) It is not a homogeneous curve, i.e., it is not generated by any Killing vector field on $\mathbb{C}P^2(4)$, and has 6 self-intersetion points.

6. Geodesics on geodesic spheres and homogeneous curves

We first recall the congruence theorem for geodesics γ on a geodesic sphere $G(r)$ of radius r $(0 < r < \pi/\sqrt{c}\,)$ in $\mathbb{C}P^n(c)$. To do this, we denote by ρ_γ the *structure torsion* of the geodesic γ, which is defined by $\rho_\gamma(s) := \langle \dot\gamma(s), \xi_{\gamma(s)} \rangle$, where ξ is the characteristic vector field on $G(r)$. Note that the function $\rho_\gamma = \rho_\gamma(s)$ is constant along the curve γ. Indeed, from the well-known equalities $\nabla_X \xi = \phi A X$ for each vector X on $G(r)$ and $\phi A = A\phi$ we find

$$\begin{aligned}\dot\gamma\rho_\gamma &= \dot\gamma\langle\dot\gamma,\xi\rangle = \langle\dot\gamma,\nabla_{\dot\gamma}\xi\rangle = \langle\dot\gamma,\phi A\dot\gamma\rangle = \langle\dot\gamma, A\phi\dot\gamma\rangle \\ &= \langle A\dot\gamma,\phi\dot\gamma\rangle = -\langle\phi A\dot\gamma,\dot\gamma\rangle = 0.\end{aligned}$$

Then we have the following:

Lemma 14 ([7]). *For geodesics γ_1 an γ_2 on $G(r)\,(0 < r < \pi/\sqrt{c}\,)$ in $\mathbb{C}P^n(c)$, they are congruent with respect to an isometry on $G(r)$ if and only if their structure torsions ρ_{γ_1} and ρ_{γ_2} satisfy $|\rho_{\gamma_1}| = |\rho_{\rho_{\gamma_2}}|$.*

We set $\iota_{G(r)} : G(r) \to \mathbb{C}P^n(c)$, which is a natural isometric (equivariant) embedding. We shall show the following fact.

Fact 1. *For every geodesic γ on $G(r)$ the curve $\iota_{G(r)} \circ \gamma$ is a homogeneous curve of order 4 in a totally geodesic Kähler submanifold $\mathbb{C}P^2(c)$ of the ambient space $\mathbb{C}P^n(c)$, $n \geqq 2$.*

In order to prove this fact we need the following.

Proposition 15 ([7]). *The extrinsic shape $\iota_{G(r)} \circ \gamma$ of a geodesic γ on a geodesic sphere $G(r)$ of radius r $(0 < r < \pi/\sqrt{c}\,)$ in $\mathbb{C}P^n(c)$, $n \geqq 2$ is as follows:*
(1) Suppose the radius r satisfies $\pi/(2\sqrt{c}\,) \leqq r < \pi/\sqrt{c}$. If the structure torsion of γ is $\pm\cot(\sqrt{c}\,r/2)$, then the curve $\iota_{G(r)} \circ \gamma$ is a geodesic.
(2) When $r \neq \pi/(2\sqrt{c}\,)$, if the structure torsion of γ is ± 1 (i.e., $\dot\gamma = \pm\xi$), then the curve $\iota_{G(r)} \circ \gamma$ is a circle of curvature $\sqrt{c}\,|\cot(\sqrt{c}\,r)|$ and of holomorphic torsion $\mp\mathrm{sgn}(\cot(\sqrt{c}\,r))\cdot 1$ in $\mathbb{C}P^n(c)$, where $\mathrm{sgn}(a)$ denotes the signature of a real number a. This circle lies on a totally geodesic holomorphic line $\mathbb{C}P^1(c)$.
(3) If γ has null structure torsion (i.e., $\dot\gamma$ is orthogonal to ξ), then the curve $\iota_{G(r)} \circ \gamma$ is a circle of curvature $(\sqrt{c}/2)\cot(\sqrt{c}\,r/2)$ and of null holomorphic torsion in $\mathbb{C}P^n(c)$. This circle lies on a totally real totally geodesic $\mathbb{R}P^2(c/4)$.
(4) Generally, if the structiure torsion of γ is of the form $\sin\theta\,(0 < |\theta| <$

$\pi/(2\sqrt{c}\,))$, $\sin\theta \neq \pm\cot(\sqrt{c}\,r/2)$, *then the curve* $\iota_{G(r)}\circ\gamma$ *is a homogeneous curve of proper order* 4 *whose curvatures are described as*:

$$k_1 = \frac{\sqrt{c}}{2}\Big|\cot\frac{\sqrt{c}\,r}{2} - \tan\frac{\sqrt{c}\,r}{2}\sin^2\theta\Big|, k_2 = \frac{\sqrt{c}}{2}\tan\frac{\sqrt{c}\,r}{2}|\sin\theta|\cos\theta,$$
$$k_3 = \frac{\sqrt{c}}{2}\cot\frac{\sqrt{c}\,r}{2}.$$

Its holomorphic torsions are described as:

$$\tau_{12} = \begin{cases} -\sin\theta & \textit{if} \cot\frac{\sqrt{c}r}{2} - \tan\frac{\sqrt{c}\,r}{2}\sin^2\theta > 0, \\ \sin\theta & \textit{if} \cot\frac{\sqrt{c}r}{2} - \tan\frac{\sqrt{c}\,r}{2}\sin^2\theta < 0, \end{cases}$$

$$\tau_{14} = \begin{cases} -\mathrm{sgn}(\sin\theta)\cos\theta & \textit{if} \cot\frac{\sqrt{c}r}{2} - \tan\frac{\sqrt{c}\,r}{2}\sin^2\theta > 0, \\ \mathrm{sgn}(\sin\theta)\cos\theta & \textit{if} \cot\frac{\sqrt{c}r}{2} - \tan\frac{\sqrt{c}\,r}{2}\sin^2\theta < 0, \end{cases}$$

$$\tau_{23} = \mathrm{sgn}(\sin\theta)\cos\theta,\ \tau_{14} = \sin\theta,\ \tau_{13} = \tau_{24} = 0.$$

This helix $\iota_{G(r)}\circ\gamma$ *lies on a totally geodesic* $\mathbb{C}P^2(c)$.

Proof. It suffices to prove our Proposition in the case of $c = 4$.
(1) We denote by $\widetilde{\nabla}$ and ∇ the Riemannian connections of $\mathbb{C}P^n(4)$ and $G(r)$, respectively. For simplicity we also denote the curve $\iota_{G(r)}\circ\gamma$ by γ. By Gauss formula $\widetilde{\nabla}_X Y = \nabla_X Y + \langle AX, Y\rangle\mathcal{N}$ we have

$$\widetilde{\nabla}_{\dot\gamma}\dot\gamma = \nabla_{\dot\gamma}\dot\gamma + \langle A\dot\gamma, \dot\gamma\rangle\mathcal{N} = \langle A\dot\gamma, \dot\gamma\rangle\mathcal{N}.$$

Next, for a geodesic $\gamma = \gamma(s)$ on $G(r)$ we can set the initial vector $\dot\gamma(0)$ as:

$$\dot\gamma(0) = \rho_\gamma\xi_{\gamma(0)} + \sqrt{1-\rho_\gamma^2}\,u,$$

where $A\xi_{\gamma(0)} = 2\cot(2r)\xi_{\gamma(0)}$ and u is a unit vector orthogonal to $\xi_{\gamma(0)}$, so that it satisfies $Au = (\cot r)u$. Since $2\cot(2r) = \cot r - \tan r$, thanks to the above equalities we see that $\langle A\dot\gamma(0), \dot\gamma(0)\rangle = \cot r - \tan r\cdot\rho_\gamma^2$, which, combined with the constancy of ρ_γ along γ, implies that the curve $\iota_{G(r)}\circ\gamma$ is a geodesic in the ambient space $\mathbb{C}P^n(4)$ if and only if $\cot r - \tan r\cdot\rho_\gamma^2 = 0$ holds. Hence we have $\rho_\gamma = \pm\cot r$, which, together with $0 \leqq \rho_\gamma \leqq 1$, shows $\pi/4 \leqq r < \pi/2$.
(2), (3): When γ is not of the case (1), we set

$$k_1 = |\langle A\dot\gamma, \dot\gamma\rangle| = |\cot r - \tan r\cdot\sin^2\theta|,$$
$$V_2 = \begin{cases} \mathcal{N} & \text{if } \langle A\dot\gamma, \dot\gamma\rangle > 0, \\ -\mathcal{N} & \text{if } \langle A\dot\gamma, \dot\gamma\rangle < 0. \end{cases}$$

Since we have

$$\widetilde{\nabla}_{\dot\gamma}\mathcal{N} = -A\dot\gamma = -\langle A\dot\gamma, \dot\gamma\rangle\dot\gamma + (\langle A\dot\gamma, \dot\gamma\rangle\dot\gamma - A\dot\gamma),$$

and $\|A\dot\gamma\|^2 = \cot^2 r + (\tan^2 r - 2)\sin^2\theta$ is constant along γ, we can see that

$$\widetilde{\nabla}_{\dot\gamma}V_2 = -k_1\dot\gamma + k_2V_3,$$

where

$$k_2 = \sqrt{\|A\dot\gamma\|^2 - \langle A\dot\gamma, \dot\gamma\rangle^2} = \tan r|\sin\theta|\cos\theta,$$

$$V_3 = \begin{cases} (1/k_2)(\langle A\dot\gamma, \dot\gamma\rangle\dot\gamma - A\dot\gamma) & \text{if } \langle A\dot\gamma, \dot\gamma\rangle > 0, \\ -(1/k_2)(\langle A\dot\gamma, \dot\gamma\rangle\dot\gamma - A\dot\gamma) & \text{if } \langle A\dot\gamma, \dot\gamma\rangle < 0. \end{cases}$$

thus, when $\theta = 0$ or $\theta = \pm\pi/2$, we see $k_2 = 0$ and find that the curve $\iota_{G(r)} \circ \gamma$ is a circle of curvature $\cot r$ or $2|\cot 2r|$, respectively.
(4) It follows from the following formula on the covariant derivative of the shape operator A of $G(\pi/4)$ in $\mathbb{C}P^n(4)$

$$(\nabla_X A)Y = \mp\{\langle\phi X, Y\rangle\xi + \eta(Y)\phi X\}$$

that

$$\widetilde\nabla_{\dot\gamma}(\langle A\dot\gamma, \dot\gamma\rangle\dot\gamma - A\dot\gamma) = -k_2^2\mathcal{N} + \langle\dot\gamma, \xi\rangle\phi(\dot\gamma).$$

This, together with a fact that $\|\phi\dot\gamma\|^2 = 1 - \langle\dot\gamma, \xi\rangle^2 = 1 - \sin^2\theta$ is constant along γ, shows

$$\widetilde\nabla_{\dot\gamma}V_3 = -k_2V_2 + k_3V_4,$$

where

$$k_3 = \frac{1}{k_2}|\langle\dot\gamma, \xi\rangle|\sqrt{1 - \langle\dot\gamma, \xi\rangle^2} = \cot r,$$

$$V_4 = \begin{cases} \left(1/\sqrt{1 - \langle\dot\gamma, \xi\rangle^2}\right)\phi\dot\gamma = (1/\cos\theta)\phi\dot\gamma & \text{if } \langle A\dot\gamma, \dot\gamma\rangle \cdot \langle\dot\gamma, \xi\rangle > 0, \\ \left(1/\sqrt{1 - \langle\dot\gamma, \xi\rangle^2}\right)\phi\dot\gamma = (-1/\cos\theta)\phi\dot\gamma & \text{if } \langle A\dot\gamma, \dot\gamma\rangle \cdot \langle\dot\gamma, \xi\rangle < 0. \end{cases}$$

Finally we have

$$\widetilde\nabla_{\dot\gamma}(\phi\dot\gamma) = \langle\dot\gamma, \xi\rangle A\dot\gamma - \langle A\dot\gamma, \dot\gamma\rangle\xi.$$

As we find

$$|\langle\langle\dot\gamma, \xi\rangle A\dot\gamma - \langle A\dot\gamma, \dot\gamma\rangle\xi, V_3\rangle| = k_3\cos\theta,$$

$$\|\langle\dot\gamma, \xi\rangle A\dot\gamma - \langle A\dot\gamma, \dot\gamma\rangle\xi\|^2 = \cot^2 r \cdot \cos^2\theta = k_3^2\cos^2\theta,$$

so that

$$\widetilde\nabla_{\dot\gamma}V_4 = -k_3V_3,$$

and know that $\iota_{G(\pi/4)} \circ \gamma$ is a helix of proper order 4 when $\sin\theta \neq 0, \pm 1, \pm\cot r$. By direct computation we can obtain the assertion on holomorphic torsions. Thus we have established our assertion.

Fact 1 is an immediate consequence of Proposition 15 and Lemma 3. □

The following gives information on lengths of closed geodesics on geodesic spheres in a complex projective space.

Theorem 7 ([7]). *Let γ be a geodesic on a geodesic sphere $G(r)$ of radius r ($0 < r < \pi/\sqrt{c}$) in $\mathbb{C}P^n(c)$.*
(1) *If the structure torsion of γ is ± 1, then γ is closed and its length is $(2\pi/\sqrt{c})\sin(\sqrt{c}\, r)$.*

(2) *If γ has null structure torsion, then γ is also closed and its length is* $(4\pi/\sqrt{c}\,)\sin(\sqrt{c}\,r/2)$.
(3) *When the structure torsion of γ is of the form* $\sin\theta$ $(0<|\theta|<\pi/2)$, *it is closed if and only if*

$$\sin\theta = \frac{\pm q}{\sin(\sqrt{c}\,r/2)\sqrt{p^2\tan^2(\sqrt{c}\,r/2)+q^2}}$$

with some relatively prime positive integers p and q with $q < p\tan^2(\sqrt{c}\,r/2)$. In this case, its length is

$$\text{length}(\gamma) = \begin{cases} (4\pi/\sqrt{c}\,)\sqrt{p^2\sin^2(\sqrt{c}\,r/2)+q^2\cos^2(\sqrt{c}\,r/2)} \\ \qquad\qquad \textit{if } pq \textit{ is even}, \\ (2\pi/\sqrt{c}\,)\sqrt{p^2\sin^2(\sqrt{c}\,r/2)+q^2\cos^2(\sqrt{c}\,r/2)} \\ \qquad\qquad \textit{if } pq \textit{ is odd}. \end{cases}$$

$$\text{LSpec}(G(r)) = \big\{(2\pi/\sqrt{c}\,)\sin(\sqrt{c}\,r)\big\} \cup \big\{(4\pi/\sqrt{c}\,)\sin(\sqrt{c}\,r/2)\big\}$$
$$\cup \left\{ \frac{4\pi}{\sqrt{c}}\sqrt{p^2\sin^2(\sqrt{c}\,r/2)+q^2\cos^2(\sqrt{c}\,r/2)} \;\middle|\; \begin{array}{c} p \text{ and } q \text{ are relatively prime} \\ \text{positive integers which satisfy} \\ pq \text{ is even and } q < p\tan^2(\sqrt{c}\,r/2) \end{array} \right\}$$
$$\cup \left\{ \frac{2\pi}{\sqrt{c}}\sqrt{p^2\sin^2(\sqrt{c}\,r/2)+q^2\cos^2(\sqrt{c}\,r/2)} \;\middle|\; \begin{array}{c} p \text{ and } q \text{ are relatively prime} \\ \text{positive integers which satisfy} \\ pq \text{ is odd and } q < p\tan^2(\sqrt{c}\,r/2) \end{array} \right\}.$$

Therefore we obtain the following:

Theorem 8 ([7])**.** *On a geodesic sphere $G(r)$ $(0<r<\pi/\sqrt{c}\,)$ in $\mathbb{C}P^n(c)$, there exist countably infinite congruency classes of closed geodesics. Moreover the length spectrum* $\text{LSpec}(G(r))$ *of $G(r)$ is a discrete unbounded subset in the real line $\mathbb{R}$.*

For detailed properties of the length spectrum $\text{LSpec}(G(r))$, see [7].

7. Appendix: Berger spheres and their definitions

We state the back ground in this last section. To do this, we first review the following due to Klingenberg ([13]): Let M be an even dimensional compact simply connected Riemannian manifold having sectional curvature K with $0 < K \leqq L$ on M, where L is a constant. Then the length ℓ of every closed geodesic on M satisfies $\ell \geqq 2\pi/\sqrt{L}$.

Berger ([8]) gave examples of metrics on S^3 for which this inequality does not hold. This 3-sphere is called a *Berger sphere* with a Riemannian metric from a one-parameter family, which can be obtained from the standard metric by shrinking along fibers of a Hopf fibration. Chavel ([11]) constructed similar metrics on higher odd-dimensional spheres.

Weinstein ([26]) gave a description of these Berger and Chavel examples as geodesic spheres of radius r $(0<r<\pi/\sqrt{c}\,)$ with $\tan^2(\sqrt{c}\,r/2) > 2$ in $\mathbb{C}P^n(c)$, $n \geqq 2$.

In this context it is natural to study geometric properties of $G(r)$ with $\tan^2(\sqrt{c}\,r/2) > 2$. The shape operator A of every $G(r)$ $(0<r<\pi/\sqrt{c}\,)$ is written as: $A\xi = \sqrt{c}\cot(\sqrt{c}\,r)\xi$ and $AX = (\sqrt{c}/2)\cot(\sqrt{c}\,r/2)X$ for each X orthogonal to ξ. To estimate the sectional curvature K of every $G(r)$ it suffices to compute $K(\sin\theta\cdot X + \cos\theta\cdot\xi, Y)$ for a pair of orthonormal vectors X and Y that are orthogonal to ξ. Hence we have

$$(c/4)\cot^2(\sqrt{c}\,r/2) \leqq K \leqq c + (c/4)\cot^2(\sqrt{c}\,r/2).$$

Here, $K_{\min} = (c/4)\cot^2(\sqrt{c}\,r/2) = K(X,\xi)$ and $K_{\max} = c + (c/4)\cot^2(\sqrt{c}\,r/2) = K(X,\phi X)$ for each unit vector X perpendicular to ξ. Then, solving $K_{\min}/K_{\max} < 1/9$, we get $\tan^2(\sqrt{c}\,r/2) > 2$ and vice versa.

Next, we take an integral curve $\gamma_\xi = \gamma_\xi(s)$ of the characteristic vector field ξ on $G(r)$ $(0 < r < \pi/\sqrt{c}\,)$. As the curve γ_ξ lies on a holomorphic line $\mathbb{C}P^1(c)(= S^2(c))$ as a small circle of positive curvature $k = \sqrt{c}\,|\cot(\sqrt{c}\,r)|$ on $S^2(c)$, it is closed with length

$$\ell = \frac{2\pi}{\sqrt{k^2+c}} = \frac{2\pi}{\sqrt{c\cot^2(\sqrt{c}\,r)+c}} = \frac{2\pi}{\sqrt{c}}\sin(\sqrt{c}\,r).$$

This, together with an equality $\nabla_X\xi = \phi AX$, implies $\nabla_\xi\xi = \phi A\xi = 0$, so that the curve $\gamma_\xi = \gamma_\xi(s)$ is a geodesic on $G(r)$. We set the following inequality:

$$\frac{2\pi}{\sqrt{K_{\max}}} = \frac{2\pi}{\sqrt{c + \frac{c}{4}\cot^2\left(\frac{\sqrt{c}\,r}{2}\right)}} > \frac{2\pi}{\sqrt{c}}\sin(\sqrt{c}\,r).$$

Then, solving this inequality, we get $\tan^2(\sqrt{c}\,r/2) > 2$ and vice versa. Hence we obtain the following which clarifies geometric properties of geodesic spheres with sufficiently big radii in a complex projective space.

Proposition 16 ([18])**.** *Let $G(r)$ be a geodesic sphere of radius r $(0 < r < \pi/\sqrt{c}\,)$ in $\mathbb{C}P^n(c)$, $n \geqq 2$. Then the following three conditions are mutually equivalent*:

(1) *The radius r satisfies an inequality* $\tan^2(\sqrt{c}\,r/2) > 2$;

(2) *The sectional curvature K of $G(r)$ satisfies sharp inequalities $\delta L \leqq K \leqq L$ for some $\delta \in (0, 1/9)$ at its each point*;

(3) *The length of every integral curve of the characteristic vector field ξ on $G(r)$ is shorter than $2\pi/\sqrt{L}$, where L is the maximal sectional curvature of $G(r)$.*

Needless to say, each of geodesic sphere $G(r)$ $(0 < r < \pi/\sqrt{c}\,)$ in $\mathbb{C}P^n(c)$ is a Riemannian homogeneous manifold.

Inspired by Proposition 16, we redefine Berger spheres.

Definition. An odd dimensional Riemannian homogeneous manifold M is called a *Berger sphere* if M satisfies the following three conditions:

(1) M is diffeomorphic to a Euclidean sphere;
(2) The sectional curvature K of M satisfies sharp inequalities $0 < \delta L \leqq K \leqq L$ on M for some $\delta \in (0, 1/9)$;
(3) M has a closed geodesic whose length is shorter than $2\pi/\sqrt{L}$, where L is given by (2).

Roughly speaking, Sasakian space forms can be considered as space forms in the sense of contact geometry. Classically, every complete simply connected Sasakian space form $N(k) := N^{2n-1}(k)$ of constant ϕ-sectional curvature k is obtained by constructing contact metric structures on a standard sphere S^{2n-1}, on a Euclidean space $\mathbb{R}^{2n-1}$ and on a product $D_{n-1}(\mathbb{C})\times\mathbb{R}$ of a Kähler ball of constant holomorphic sectional curvature and a real line (see pp. 114–115 in [9]).

On the other hand, in the sense of submanifold geometry Sasakian space forms are realized as totally η-umbilic real hypersurfaces M^{2n-1} in a complete simply connected complex space forms $\widetilde{M}_n(c)$, i.e., they are congruent to either a complex projective space $\mathbb{C}P^n(c)$, complex hyperbolic space $\mathbb{C}H^n(c)$ or complex Euclidean space $\mathbb{C}^n$. Precisely, the space $N(k)$ is regarded as a real hypersurface M^{2n-1} having the shape operator A as either $A = -\mathrm{I} + (c/4)\eta\otimes\xi$ or $A = \mathrm{I} - (c/4)\eta\otimes\xi$ in the ambient space $\widetilde{M}_n(c)$, where $k = c + 1$ (cf. [4]).

We shall focus our attention on Berger spheres from the viewpoints of contact geometry and submanifold geometry.

Theorem 9 ([18]). *Every complete simply connected Sasakian space form $N^{2n-1}(k)$, $n \geqq 2$ whose ϕ-sectional curvature k is greater than 9 is a Berger sphere. In particular, when $k = 8n+5$, the space $N^{2n-1}(k)$ can be realized as a homogeneous submanifold with nonzero parallel mean curvature vector with respect to the normal connection in some Euclidean sphere $S^N(\tilde{c})$ of sectional curvature $\tilde{c}$, where $N = n(n+2)-1$ and $\tilde{c} = 2(n+1)(2n+1)/n$.*

Sketch of proof. We first show that the space $N^{2n-1}(k)$ with $k > 9$ is congruent to some geodesic sphere $G(r)$ in $\mathbb{C}P^n(c)$, where all of $\tan^2(\sqrt{c}\,r/2) > 2$, $(\sqrt{c}/2)\cot(\sqrt{c}\,r/2) = 1$ and $k = c + 1$ hold.

Next, in order to establish the second half of our Theorem we take the minimal equivariant parallel embeding $f_1 : \mathbb{C}P^n(c) \to S^{n(n+2)-1}((n+1)c/(2n))$, which is defined by eigenfunctions associated to the first eigenvalue of the Laplacian Δ of $\mathbb{C}P^n(c)$ (see [25]). We consider the class of submanifolds $\{(G(r), f_1\circ\iota_{G(r)})|\ 0 < r < \pi/\sqrt{c}\}$ of the ambient sphere $S^N(\tilde{c})$, where $\iota_{G(r)} : G(r) \to \mathbb{C}P^n(c)$ is the natural inclusion mapping. Then by direct computation we can see that the isometric embedding $f_1\circ\iota_{G(r)}$ has parallel mean curvature vector with respect to the normal connection in the sphere $S^{n(n+2)-1}((n+1)c/(2n))$ if and only if $\tan^2(\sqrt{c}\,r/2) = 2n + 1$.

□

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