HIGHER ORDER GENERALIZATION OF FUKAYA'S MORSE HOMOTOPY INVARIANT OF 3-MANIFOLDS II.

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Abstract. This note describes an attempt to defining a topological invariant for 3-manifold $M$ with $b_1(M) > 0$ by applying Fukaya's Morse homotopy theoretic approach for Chern–Simons perturbation theory to a local coefficient system on $M$ of rational functions associated to the maximal free abelian covering of $M$. The invariant takes values in a space of Jacobi diagrams whose edges are colored by rational functions. It is expected that the invariant gives nontrivial finite type invariants of 3-manifolds.

1. Introduction

By using an idea of his Morse homotopy theory, K. Fukaya constructed in [Fu] a topological invariant of 3-manifolds with flat bundles on them that is analogous to Chern–Simons perturbation theory ([AS, Ko1]). He considered a flat Lie algebra bundle over a 3-manifold $M$ and several Morse functions on $M$, and defined his invariant as the sum of the weights of some graphs (flow-graphs) in $M$ whose edges follow the gradients of the Morse functions. The weight is given by contracting the holonomies taken along the edges of a flow-graph by some tensor. Although Fukaya's construction was given for the 2-loop graphs, his construction also works for general 3-valent graphs at least when $M$ is a homology sphere with trivial connection ([Wa1]).

This paper is a continuation of [Wa1]. We construct a topological invariant $\hat{z}_{2k}$ for closed oriented 3-manifolds $M$ with $b_1(M) = 1$ and trivial Alexander polynomial by applying Fukaya’s construction to a local coefficient system of rational functions associated to the maximal free abelian covering of $M$. There are fundamental results of Lescop about an equivariant 2-loop invariant for closed oriented 3-manifolds with $b_1(M) = 1$ ([Les1, Les2]), by which we were significantly influenced in the construction of $\hat{z}_{2k}$. Indeed, the definition of $\hat{z}_{2k}$ can be rewritten by equivariant intersections in configuration spaces as given in [Les1, Les2]. Prior to Lescop’s works, Ohtsuki had given in [Oh1, Oh2] a considerable refinement of the LMO invariant for 3-manifolds $M$ with $b_1(M) = 1$, which is important in the study of equivariant perturbative invariant for non homology spheres. It is known that the LMO invariant ([LMO]) is very strong for homology spheres whereas it is rather weaker for non homology spheres. It is remarkable that Ohtsuki’s refined LMO
invariant is also very strong for 3-manifolds $M$ with $b_1(M) = 1$, and moreover his equivariant invariant is computable for some examples and yields some beautiful formulas. We expect that the invariant in this paper is closely related to Ohtsuki’s refined LMO invariant.

In [Wa2], we construct an invariant of some degree 1 maps from 3-manifolds to the 3-torus by a method similar to the construction of this paper and apply it to study finite type invariants. It will follow from a result of [Wa2] that the value of the 2-loop part $\hat{z}_2$ for some 3-manifolds with $b_1 = 1$ can be computed by clasper calculus of Goussarov and Habiro. In [Wa3], we define an invariant of fiberwise Morse functions on surface bundles over $S^1$, which can be considered as an analogue of the construction of the present paper for $S^1$-valued Morse theory.

In the case where a knot in $M$ is present, a construction similar to that of this paper gives a knot invariant, which would include many non-trivial finite type invariants of knots. We will explain this in a subsequent paper.

The aim of this paper is to explain how the results of [Wa1] can be used to obtain the result of this paper and will not repeat the details given in [Wa1], because all the signs needed are exactly the same as those in [Wa1].

This paper is organized as follows. In §2, we briefly review some necessary homological results on acyclic complex. In §3, we give a formula defining the invariant $\hat{z}_{2k}$ for $b_1(M) = 1$ and state the main theorem. In §4, we prove the main theorem. In §5, some remarks about invariants for $b_1(M) > 0$ and non-triviality of $\hat{z}_{2k}$.

2. Preliminaries

2.1. Acyclic Morse complex. For simplicity, we assume that $M$ is an oriented, connected, closed 3-manifold with $b_1(M) = 1$. We equip $M$ with a Riemannian metric $\mu$. Moreover, we assume that the Alexander polynomial $\Delta(M)$ of $M$ is 1, where $\Delta(M)$ is normalized so that $\Delta(M)(t^{-1}) = \Delta(M)(t)$ and $\Delta(M)(1) = 1$. Let $f : M \to \mathbb{R}$ be a Morse function and let $\Sigma \subset M$ be an oriented 2-submanifold that generates the oriented bordism group $\Omega_2(M) \cong H^1(M; \mathbb{Z}) = \text{Hom}(H_1(M; \mathbb{Z}), \mathbb{Z}) \cong \mathbb{Z}$. The pair $(f, \mu)$ gives the gradient $\xi$ on $M$. Let $P_i = P_i(f)$ be the set of critical points of $f$ of index $i$ and let $P_i(f) = \bigcup P_i(f)$. Let $(C_\cdot(f; \Lambda), \partial)$ be the twisted Morse complex for $\xi$ with $\Lambda$-coefficients. Namely, $C_i(f; \Lambda) = \Lambda P_i$ ($\Lambda = \mathbb{Q}[t, t^{-1}]$) and the boundary $\partial : C_i(f) \to C_{i-1}(f)$ is given as follows. We assume that $\xi$ is Morse–Smale, namely, ascending and descending manifolds of $\xi$ are mutually transversal. In this case, the set $\mathcal{M}^i(\xi; p, q)$ of unparametrized flow-lines of $-\xi$ between a critical point $p$ of index $i$ and a critical point $q$ of index $i - 1$ is a finite set.

$$\partial(p) = \sum_{q \in P_{i-1}} n(\xi; p, q) q, \quad n(\xi; p, q) = \sum_{\gamma \in \mathcal{M}^i(\xi; p, q)} \varepsilon(\gamma),$$

where $\varepsilon(\gamma) = \pm t^n$, $n = \gamma \cdot \Sigma$ (the intersection number), and the sign of $\varepsilon(\gamma)$ is determined by the orientation of the flow-line (see [Wa1, §2.4] for the definition of the sign, or [BZ, Appendix]). It can be checked that $(C_\cdot(f; \Lambda), \partial)$ is a chain
complex, namely, \( \partial^2 = 0 \). It is well-known that the homology of the complex is identified with \( H_*(\bar{M}; \Lambda) = H_*(\bar{M}; \mathbb{Q}) \) as a \( \Lambda \)-module, where \( \bar{M} \to M \) is the infinite cyclic covering (e.g., [BZ, Appendix], [HL, §1]).

Let \( \bar{\Lambda} = \Lambda[\frac{1}{\delta t}] = \mathbb{Q}[t^\pm 1, \frac{1}{\delta t}] \), which is a Dedekind domain (e.g., [Cl, Lemma 1-1]) and thus is a hereditary ring (e.g., [CE, §L5]). Put
\[
C_i(\bar{f}; \bar{\Lambda}) = C_i(f; \Lambda) \otimes_{\Lambda} \bar{\Lambda} = \bar{\Lambda}^P.
\]
This together with the boundary \( \partial \otimes 1 \) forms a chain complex. Since \( \Lambda \) is a PID and \( \bar{\Lambda} \) is a flat \( \Lambda \)-module, we have, as \( \Lambda \)-modules,
\[
H_i(C_*(\bar{f}; \bar{\Lambda})) \cong H_i(\bar{M}; \mathbb{Q}) \otimes_{\Lambda} \bar{\Lambda}
\]
by the universal coefficient theorem (e.g., [CE, Theorem VI.3.3]). From the fact that \( H_1(\tilde{M}; \mathbb{Q}) = H_3(\tilde{M}; \mathbb{Q}) = 0 \) and \( H_0(\tilde{M}; \mathbb{Q}) \cong H_2(\tilde{M}; \mathbb{Q}) \cong \Lambda/(t - 1) \) if \( \Delta(M) = 1 \) (e.g., [Les1, Lemma 2.2]), it follows that \( (C_*(\bar{f}; \bar{\Lambda}), \partial \otimes 1) \) is acyclic.

**Remark 2.1.** If \( \Delta(M) \) is non-trivial, one can still obtain an acyclic complex \( C_*(\bar{f}; \bar{\Lambda}) \) by letting \( \bar{\Lambda} = \mathbb{Q}[t^\pm 1, \frac{1}{\delta t}, \frac{1}{\Delta(M)}] \).

### 2.2. Combinatorial propagators

Let \( \text{End}_{\bar{\Lambda}}(C) = \text{Hom}_{\bar{\Lambda}}(C, C) \) and let \( \text{End}_{\bar{\Lambda}}(C)_k \) be its degree \( k \) part. We define \( \delta : \text{End}_{\bar{\Lambda}}(C)_k \to \text{End}_{\bar{\Lambda}}(C)_{k-1} \) by the following formula.
\[
\delta g = \partial g - (-1)^k g \circ \partial.
\]
This satisfies \( \delta^2 = 0 \). By the acyclicity and \( \bar{\Lambda} \)-freeness of \( C \), and by the Künneth theorem (e.g., [CE, Theorem VI.3.1a]), it follows that \( (\text{End}_{\bar{\Lambda}}(C), \delta) \) is acyclic, too.

Since the identity \( 1 \in \text{End}_{\bar{\Lambda}}(C)_0 \) is a \( \delta \)-cycle, there exists \( g \in \text{End}_{\bar{\Lambda}}(C)_1 \) such that \( \delta g = \partial g + g \partial = 1 \). Such a \( g \) is called a **combinatorial propagator** for \( C \) ([Fu]).

For two choices \( g, g' \) of combinatorial propagators for \( C \), \( g' - g \) is a \( \delta \)-cycle. Thus there exists \( h \in \text{End}_{\bar{\Lambda}}(C)_2 \) such that \( \partial h - h \partial = g' - g \).

### 3. Perturbation theory with holonomies in \( \bar{\Lambda} \)

#### 3.1. Moduli space \( \mathcal{M}_\Gamma(\zeta) \) of flow-graphs

Let \( f_1, f_2, \ldots, f_{3k} : M \to \mathbb{R} \) be a sequence of Morse functions and let \( \zeta_i \) be the gradient of \( f_i \). We consider a connected edge-oriented trivalent graph \( \Gamma \) such that

1. \( \Gamma \) has \( 2k \) vertices and \( 3k \) edges, and
2. the sets of vertices and edges of \( \Gamma \) are labelled by \( \{1, 2, \ldots, 2k\} \) and \( \{1, 2, \ldots, 3k\} \), respectively.

By the labelling \( \{1, 2, \ldots, 3k\} \to \text{Edges}(\Gamma) \) of \( \Gamma \), we identify edges with numbers. Choose some of the edges and split each chosen edge into two arcs. We attach elements of \( P_i(f_i) \) on the two 1-valent vertices (white-vertices) that appear after the splitting of the \( i \)-th edge. We call such obtained graph a **\( \bar{C} \)-graph** \( (\bar{C} = (C_1(\bar{f}; \bar{\Lambda}), \ldots, C_3(\bar{f}; \bar{\Lambda})) \), see Figure 1). A \( \bar{C} \)-graph may have two kinds of “edges” obtained from edges of a trivalent graph: a **compact edge**, which is a single arc between two black vertices, and a **separated edge**, which consists of two arcs. We call vertices that are not white vertices **black vertices**. If \( p_i \) (resp. \( q_i \)) is the critical point attached on the input (resp. output) white vertex of a separated edge \( i \), we
define the degree of \( i \) by \( \deg(i) = |p_i| - |q_i| \), where \( | \cdot | \) denotes the Morse index. We define the degree of a compact edge \( i \) by \( \deg(i) = 1 \). We define the degree of a \( \vec{C} \)-graph by \( \deg(\Gamma) = (\deg(1), \deg(2), \ldots, \deg(3k)) \).

We say that a continuous map \( I \) from a \( \vec{C} \)-graph \( \Gamma \) to \( M \) is a flow-graph for the sequence \( \vec{\xi} = (\xi_1, \xi_2, \ldots, \xi_{3k}) \) if it satisfies the following conditions (see Figure 2).

1. Every critical point \( p_i \) attached on the \( i \)-th edge is mapped by \( I \) to \( p_i \) in \( M \).
2. The restriction of \( I \) to each edge of \( \Gamma \) is a smooth embedding and at each point \( x \) of the \( i \)-th edge that is not on a white vertex, the tangent vector of \( I \) at \( x \) (chosen along the edge orientation) is \( a_x(-\xi_i)x \) for some positive real number \( a_x \in \mathbb{R} \) which depends smoothly on \( x \).

Two flow-graphs are considered equivalent if they are related by reparametrizations on edges. For a \( \vec{C} \)-graph \( \Gamma \), let \( \mathcal{M}_\Gamma(\vec{\xi}) \) be the set of equivalence classes of all flow-graphs for \( \vec{\xi} \) from \( \Gamma \) to \( M \). By extracting black vertices, a natural map from \( \mathcal{M}_\Gamma(\vec{\xi}) \) to the configuration space \( C_{2k}(M) \) of ordered tuples of \( 2k \) (distinct) points is defined. It follows from a property of the gradient that this map is injective. This induces a topology on the set \( \mathcal{M}_\Gamma(\vec{\xi}) \).

**Lemma 3.1** (Fukaya [Fu, Wa1]). If \( \vec{f} = (f_1, f_2, \ldots, f_{3k}) \) and \( \mu \) are generic, then for a \( \vec{C} \)-graph \( \Gamma \) with \( 2k \) black vertices and \( \deg(\Gamma) = (1, 1, \ldots, 1) \), the space \( \mathcal{M}_\Gamma(\vec{\xi}) \) is a compact 0-dimensional manifold. Moreover, we may choose a single \((\vec{f}, \mu)\) that satisfies this property simultaneously for all \( \vec{C} \)-graphs \( \Gamma \) with \( 2k \) black vertices and \( \deg(\Gamma) = (1, 1, \ldots, 1)^* \).

**3.2. The count of \( \mathcal{M}_\Gamma(\vec{\xi}) \).** When the assumption of Lemma 3.1 is satisfied, we may define an orientation of \( \mathcal{M}_\Gamma(\vec{\xi}) \) in a similar way as [Wa1]. Roughly, an orientation of \( \mathcal{M}_\Gamma(\vec{\xi}) \) is defined as follows. The space \( \mathcal{M}_\Gamma(\vec{\xi}) \) can be considered as the intersection of several smooth manifold strata in \( M^{2k} \) each corresponds to the moduli space of

\*In [Wa1], we considered flow-graphs on punctured homology sphere. Nevertheless, the proof of the corresponding lemma is essentially the same.
an edge of $\Gamma$. We define an orientation of $\mathcal{M}_T(\vec{\xi})$ by the coorientation $\bigwedge_{e \in \text{Edges}(\Gamma)} v_e$ of $\mathcal{M}_T(\vec{\xi})$ in $M^{2k}$ for some coorientations $v_e$ of the strata for $e$. If $e$ is a compact or separated, then $e$ has two black vertices and $v_e$ is a vector in $\Lambda^2(T_x M \oplus T_y M)$, where $x, y$ are the images from the black vertices of $e$.

In [Wa1], the number $\# \mathcal{M}_T(\vec{\xi})$ was defined as the sum of the signs determined by the orientations. The following definition of $\# \mathcal{T}_T(\vec{\xi})$ is different from that of [Wa1]. We assume without loss of generality that $\Sigma$ is disjoint from vertices of all the relevant flow-graphs. Then we count points of $\mathcal{M}_T(\vec{\xi})$ with weights in $\Lambda^\otimes 3k = Q[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_{3k}^{\pm 1}]$ as follows.

$$\# \mathcal{M}_T(\vec{\xi}) = \sum_{T \in \mathcal{T}_T(\vec{\xi})} \varepsilon(I), \quad \varepsilon(I) = \pm t_1^{n_1} t_2^{n_2} \cdots t_{3k}^{n_{3k}},$$

$$n_i = (\text{The intersection number of the } i\text{-th edge of } I \text{ with } \Sigma)$$

We take the sign $\pm$ as the one determined by the orientation of $I$. See [Wa1, §2.5] for the definition of the sign. The intersection number of the $i$-th edge with $\Sigma$ is determined by the orientations of the edge and of $\Sigma$ and that of $M$, by the local relations $\text{ori}(i\text{-th edge}) \wedge \text{ori}(\Sigma) = \pm \text{ori}(M)$.

### 3.3. The generating series and its trace

Let $R$ be either $\Lambda$ or $\hat{\Lambda}$. An $R$-colored $\hat{C}$-graph is a pair of a $\hat{C}$-graph $\Gamma$ and a map $\phi : \text{Edges}(\Gamma) \to R$. We will write an $R$-colored $\hat{C}$-graph as $\Gamma(\phi)$ or $\Gamma(\phi(1), \phi(2), \ldots, \phi(3k))$. We call $\phi$ an $R$-coloring of $\Gamma$. We define an “action” of a monomial $t_1^{n_1} t_2^{n_2} \cdots t_{3k}^{n_{3k}}$ on a $\hat{C}$-graph as follows.

$$t_1^{n_1} t_2^{n_2} \cdots t_{3k}^{n_{3k}} \cdot \Gamma = \Gamma(t_1^{n_1}, t_2^{n_2}, \ldots, t_{3k}^{n_{3k}}).$$

The right hand side is a $\Lambda$-colored $\hat{C}$-graph. By extending this by $Q$-linearity, the formal linear combination $\# \mathcal{T}_T(\vec{\xi}) \cdot \Gamma$ of $\Lambda$-colored $\hat{C}$-graphs is defined. Moreover, this operation can be extended to the case where $\Gamma$ is a $\Lambda$-colored $\hat{C}$-graph by $t_1^{n_1} t_2^{n_2} \cdots t_{3k}^{n_{3k}} \cdot \Gamma(p_1, p_2, \ldots, p_{3k}) = \Gamma(t_1^{n_1} p_1, t_2^{n_2} p_2, \ldots, t_{3k}^{n_{3k}} p_{3k}).$

Let $\mathcal{A}_{2k}(\Lambda)$ (resp. $\mathcal{A}_{2k}(\hat{\Lambda})$) be the vector space over $Q$ spanned by pairs $(\Gamma, \phi)$, where $\Gamma$ is an (unlabelled) edge-oriented trivalent graph with $2k$ vertices and with vertex-orientation and $\phi$ is a $\Lambda$-coloring (resp. $\hat{\Lambda}$-coloring) of $\Gamma$, quotiented by the relations AS, IHX, Orientation reversal, Linearity, Holonomy† (Figure 3) and automorphisms of oriented graphs.

**Lemma 3.2.** Let $(\Gamma, \phi)$ be a $\hat{\Lambda}$-colored trivalent graph with $2k$ vertices and with a self-loop edge. Then $[(\Gamma, \phi)] = 0$ in $\mathcal{A}_{2k}(\Lambda)$.

**Proof.** Let $e$ be a self-loop edge of $\Gamma$ and let $e'$ be the edge of $\Gamma$ adjacent to $e$. One may transform as $\phi(e') = \frac{1}{t_e} \phi(e) = \frac{1}{t_e} \phi(e') - \frac{1}{t_e} \phi(e')$ in $\hat{\Lambda}$. Then by Linearity and Holonomy relations, we have $(\Gamma, \phi) \sim 0$. \hfill \box

†This definition is by Garoufalidis and Rozansky [GR]. The AS and the IHX relations are due to Bar-Natan [BN]. In the Holonomy relation, it may happen that two of the three edges form a single self-loop, in which case we consider that the decoration on the self-loop gets multiplied by $t^{-1} t = 1$ on the right hand side.
Figure 3. The relations AS, IHX, Orientation reversal, Linearity and Holonomy. \( p, q, r \in \Lambda \) (or \( p, q, r \in \hat{\Lambda} \)), \( \alpha \in \mathbb{Q} \). The exponent \( \varepsilon_i \) is 1 if the \( i \)-th edge is oriented toward \( v \) and otherwise \( -1 \).

Now we shall define an element \( z_{2k}(\vec{\xi}) \in \mathfrak{A}_{2k}(\hat{\Lambda}) \). Let \( \vec{g} = (g^{(1)}, g^{(2)}, \ldots, g^{(3k)}) \) be a sequence of combinatorial propagators for \( \vec{C} = (C_*(f_1; \hat{\Lambda}), \ldots, C_*(f_{3k}; \hat{\Lambda})) \). Then we define

\[
(3.1) \quad z_{2k}(\vec{\xi}) = \text{Tr}_{\vec{g}} \left( \sum_{\Gamma} \# M(\vec{\xi}, \cdot) \cdot \Gamma \right).
\]

Here, the sum is taken over all \( \vec{C} \)-graphs \( \Gamma \) with \( 2k \) black vertices and \( \text{deg}(\Gamma) = (1,1,\ldots,1) \), and \( \text{Tr}_{\vec{g}} \) is defined as follows. For simplicity, we assume that the labels for the separated edges in a \( \Lambda \)-colored \( \vec{C} \)-graph \( \Gamma(u_1(t), u_2(t), \ldots, u_{3k}(t)) \) is \( 1,2,\ldots,r \). Let \( p_i, q_i \) be the critical points on the input and output of the \( i \)-th edge of \( \Gamma \), respectively and let \( g^{(i)}_{p_i, q_i} \in \hat{\Lambda} \) be the coefficient of \( p_i \) in \( g^{(i)}(q_i) \). Then \( \text{Tr}_{\vec{g}}(\Gamma(u_1(t), u_2(t), \ldots, u_{3k}(t))) \) is the equivalence class in \( \mathfrak{A}_{2k}(\hat{\Lambda}) \) of a \( \hat{\Lambda} \)-colored graph obtained by identifying each pair of the two white vertices of the separated edges in

\[
\Gamma(-g^{(1)}_{p_1, q_1} u_1(t), \ldots, -g^{(r)}_{p_r, q_r} u_r(t), u_{r+1}(t), \ldots, u_{3k}(t)).
\]

The definition of \( \text{Tr} \) can be generalized to graphs with other degrees in the same manner.

**Lemma 3.3.** \( z_{2k}(\vec{\xi}) \) does not depend on the choices of \( \vec{g} \) and of the hypersurface \( \Sigma \subset M \) within the oriented bordism class.

**Proof.** That \( z_{2k}(\vec{\xi}) \) does not depend on the choice of \( \vec{g} \) can be shown by the same argument as in [Wa1, §6].

For the rest, let \( \Sigma' \subset M \) be another oriented 2-submanifold that is oriented bordant to \( \Sigma \). Then by Morse theory, one may see that \( \Sigma' \) is obtained from \( \Sigma \) by a finite sequence of the following moves.

1. A homotopy in \( M \).
2. A 1- or 2-handle surgery in a small ball \( B \) in \( M \).

We may assume that for each move of type (2), the small ball \( B \) is disjoint from all the critical points of \( f_i \) and the images of the graphs in \( M(\vec{\xi}) \) for all \( \Gamma \). Thus a move of type (2) does not change \( z_{2k}(\vec{\xi}) \).

The sum in (3.1) may change under a move of type (1) when a homotopy intersects a black vertex of a flow-graph or intersects a critical point of some \( f_i \). Note that when a homotopy intersects interior of an edge in a way that an edge is
contact with the surface, the holonomy along the edge does not change. When a homotopy intersects a black vertex \(v\), a \(\Lambda\)-coloring for the three edges incident to \(v\) changes. The change of \(\Lambda\)-coloring is precisely the Holonomy relation. When a small homotopy intersects a critical point \(p\) of some Morse function, say of \(f_1\), the boundary operator of the twisted Morse complex \((C_\bullet^{(1)}, \partial^{(1)})\) for the first edge may change. Let \(\mathcal{T}^{(1)}\) be the be the resulting boundary operator. Let \(S_p : C_\bullet^{(1)} \rightarrow C_{\bullet-1}^{(1)}\) be the chain map of degree 0 defined for critical points \(x \in P_k^{\bullet}\) by

\[
S_p(x) = \begin{cases} 
 t^{\pm 1}p & \text{if } x = p \\
 x & \text{otherwise}
\end{cases}
\]

where the sign \(\pm 1\) depends on whether the homotopy crosses \(p\) from above or below.

Then we have \(\mathcal{T}^{(1)} = S_{\partial p} \circ \partial^{(1)} \circ S_p\) and that \(\mathcal{T}^{(1)} = S_{\partial p} \circ g^{(1)} \circ S_p : C_\bullet^{(1)} \rightarrow C_{\bullet+1}^{(1)}\) is a combinatorial propagator for \((C_\bullet^{(1)}, \mathcal{T}^{(1)})\). There is an analogous left/right action of \(S_{\partial p}^{\pm 1}\) on a \(\Lambda\)-colored \(\mathcal{C}\)-graph given as follows. For a \(\Lambda\)-colored \(\mathcal{C}\)-graph \(\Gamma(\phi)\), \(\Gamma(\phi) \circ S_p\) (resp. \(S_p \circ \Gamma(\phi)\)) is the \(\Lambda\)-colored \(\mathcal{C}\)-graph obtained from \(\Gamma(\phi)\) by replacing \(\phi(1)\) with \(t^{\pm 1}\phi(1)\) (resp. with \(t^{-1}\phi(1)\)) if \(p\) is the input (resp. the output) of the first edge of \(\Gamma\) and otherwise \(\Gamma(\phi) \circ S_p = \Gamma(\phi)\) (resp. \(S_p \circ \Gamma(\phi) = \Gamma(\phi)\)).

After the small homotopy that crosses \(p\), the flow graph that was counted as \(\Gamma(\phi)\) will be counted as \(S_{\partial p} \circ \Gamma(\phi) \circ S_p\). Now we have

\[
\text{Tr}_{\mathcal{T}^{(1)}, \ldots, (S_{\partial p}^{-1} \circ \Gamma(\phi) \circ S_p)} = \text{Tr}_{S_p \circ \mathcal{T}^{(1)} \circ S_p^{-1}, \ldots, (\Gamma(\phi))} = \text{Tr}_{\mathcal{T}^{(1)}, \ldots, (\Gamma(\phi))}.
\]

This completes the proof of the invariance under a move of type (1). \(\square\)

3.4. The invariant \(\hat{z}_{2k}\). For the independence of the choice of \(\vec{\xi}\), we shall define \(\hat{z}_{2k}\) by adding a correction term to \(z_{2k}(\vec{\xi})\). Though this could be done by the same method as [Wa1], we partially follow Shimizu’s definition of the correction term in ([Sh1]), which is nicer here. Take a compact oriented 4-manifold \(W\) with \(\partial W = M\) and with \(\chi(W) = 0\). By the condition \(\chi(W) = 0\), the outward normal vector field to \(M\) in \(TW\|\partial W\) can be extended to a nonsingular vector field \(\nu_W\) on \(W\). Let \(T^o W\) be the orthogonal complement of the span of \(\nu_W\) in \(T W\). Then \(T^o W\) is a rank 3 subbundle of \(TW\) that extends \(TM\). Take a sequence \(\vec{\xi} = (\gamma_1, \gamma_2, \ldots, \gamma_{3k})\) of generic sections of \(T^o W\) so that \(\gamma_i\) is an extension of \(-\xi_i\). We define

\[
z_{2k}^{\text{anomaly}}(\vec{\xi}) = \sum_i \# \mathcal{M}_T^{\text{local}}(\vec{\xi}) [\Gamma(1, 1, \ldots, 1)] \in \mathcal{A}_{2k}(\vec{\Lambda}).
\]

The sum is taken over all \(\mathcal{C}\)-graphs with \(2k\) vertices and with only compact edges\(^1\). Here, \(\mathcal{M}_T^{\text{local}}(\vec{\xi})\) is the moduli space of affine graphs in the fibers of \(T^o W\) whose \(i\)-th edge is an oriented straight line segment generated by \(\gamma_i\). The number \(\# \mathcal{M}_T^{\text{local}}(\vec{\xi}) \in \mathbb{Z}\) is the count of the signs of the affine graphs that are determined by transversal intersections of some codimension 2 chains in a configuration space bundle over \(W\).

See [Wa1, §2.7–2.8] for detail. By the same argument as in [Wa1, §7], it can be shown that \(z_{2k}^{\text{anomaly}}(\vec{\xi}) - \mu_k \text{sign} W\), where \(\mu_k \in \mathcal{A}_{2k}(\vec{\Lambda})\) is the constant given in [Wa1, Proposition 2.12], does not depend on the choices of \(W\), \(\nu_W\) and the extension \(\vec{\xi}\) of \(\vec{\xi}\). In this, the argument of existence of a framing on a (4 + 1)-dimensional

\(^1z_{2k}^{\text{anomaly}}(\vec{\xi})\) differs from \(X_k(\rho^o X; (W_i,))\) in [Sh1] by a constant multiple. In [Sh1], the same flow-line is counted twice according to the two choices of edge-orientation.
cobordism $V$ between two choices of $W$ ([Wa1, Lemma 7.3]) should be replaced by that of a rank 3 subbundle of $TV$ ([Sh1, Lemma 5.4]), which can be applied to any closed oriented 3-manifolds $M$. We define $\hat{z}_{2k}(\xi) \in \mathcal{A}_2(\Lambda)$ by the following formula.

$$\hat{z}_{2k}(\xi) = z_{2k}(\xi) - z_{2k}^{\text{anomaly}}(\gamma) + \mu_k \text{sign } W.$$  

**Theorem 3.4.** $\hat{z}_{2k}(\xi)$ is an invariant of the diffeomorphism type of $M$ and of the generating class $\text{PD}(\Sigma) \in H^1(M; \mathbb{Z}) \cong \mathbb{Z}$, where $\text{PD}$ denotes the Poincare dual.\(^3\)

**Remark 3.5.** 1. As mentioned in Remark 2.1, Theorem 3.4 also holds for $M$ with $b_1(M) = 1$ with nontrivial Alexander polynomial, by replacing $\Lambda$ with $\mathbb{Q}[t^{\pm 1}, \frac{1}{x_1}, \ldots, \frac{1}{x_\Delta(M)}]$.

2. The graph counting formula for $z_{2k}(\xi)$ can be rewritten by the “equivariant intersection” of some special codimension 2 chains in the “equivariant configuration space” of 2k points in $M$ with coefficients in $\Lambda^{\otimes 3k}$, as in [Les1, Les2]. We prefer the graph counting formula because it is convenient for later purposes and its proof of invariance can be obtained by a minimum change from that given in [Fu, Wa1].

4. **Proof of Theorem 3.4**

4.1. **Moduli space of graphs in 1-parameter family and bifurcations.** The outline of the proof is almost parallel to that of the main theorem of [Wa1]. We show that the value of $\hat{z}_{2k}$ does not change if one Morse function in the sequence $\vec{f}$ is replaced with another Morse function. Here we assume for simplicity that the first Morse function $f_1$ is replaced with another one $f'_1$. As usual in Cerf theory ([Ce]), we use the fact that there is a smooth 1-parameter family $\{h_s : M \to \mathbb{R}\}_{s \in [0, 1]}$ that restricts to $f_1$ and $f'_1$ on $s = 0, 1$ respectively, such that $h_s$ is Morse except for finitely many values of $s$ and at the excluded values the singularities of $h_s$ consist of birth-death singularities and Morse singularities. Moreover, there may be finitely many values of $s$ at which the Morse complex for $\xi_s = \text{grad } h_s$ changes, namely, at which there is a flow-line of $\xi_s$ between two Morse critical points of the same index $j$. Such a flow-line is called a $j/j$-intersection and corresponds to a handle-slide.

First, let $J = [s_0, s_1] \subset [0, 1]$ be an interval that does not have birth-death parameter and let $\xi_J = ((\xi_{s_1})_{s \in J}, \xi_2, \ldots, \xi_{3k})$. By replacing $f_1$ with the family $\{h_s\}$, the moduli spaces $\mathcal{M}_\Gamma(\xi_J)$ for flow-graphs mapped along the fiber $M$ in $J \times M$ is defined. Namely, since there are no birth-death point over $J$, the set $P_j(h_s)$ of critical points of $h_s$ for each $s \in J$ can be identified with that for $s = s_0$ in a canonical way through the critical loci. Let $\bar{C}_{s_0} = (C_s(h_{s_0}; \Lambda), C_s(f_2; \Lambda), \ldots, C_s(f_{3k}; \Lambda))$. For a $\bar{C}_{s_0}$-graph $\Gamma$, we define a flow-graph for $\xi_J$ in $J \times M$ as a map $I$ from $\Gamma$ into a fiber $\{s\} \times M$ that satisfies the following conditions.

(1) For $i \geq 2$, every critical point $p_i$ attached to the $i$-th edge is mapped by $I$ to $\{s\} \times p_i$ in $\{s\} \times M$.

\(^3\)There are only two possibilities for the class of $\Sigma$ that generates $\Omega_2(M) \cong \mathbb{Z}$, and the values of $z_{2k}$ for the two differ by turning every $\Lambda$-coloring $\phi(t)$ on edge into $\phi(t^{-1})$. Thus, they can be considered essentially the same.
Every critical point $p_1$ attached on the first edge is mapped by $I$ to the critical locus of $\{h_s\}_{s \in J}$ corresponding to $p_1$.

(3) The restriction of $I$ to each edge of $\Gamma$ is a smooth embedding and at each point $x$ of the $i$-th edge that is not on a white vertex, the tangent vector of $I$ at $x$ (chosen along the edge orientation) is a positive multiple of $(-\xi_i)_x$ if $i \geq 2$, and of $(-\xi_s)_x$ if $i = 1$.

Two flow-graphs are considered equivalent if they are related by reparametrizations on edges. Then we define $M_{\Gamma}(\vec{\xi}_J)$ as the set of equivalence classes of all flow-graphs $\Gamma \to J \times M$ for $\vec{\xi}_J$. It is equipped with the topology induced by the natural injection $M_{\Gamma}(\vec{\xi}_J) \to J \times C_{2k}(M)$, which extracts black vertices.

**Lemma 4.1** ([Wa1, Proposition 8.12]). After a small perturbation of the family $\xi_J = \{\xi_s\}_{s \in J}$ fixing the endpoints, we may arrange that for all $\tilde{C}_{\text{sn}}$-graph $\Gamma$ with $\deg(\Gamma) = (1, 1, \ldots, 1)$, the moduli space $M_{\Gamma}(\tilde{\xi}_J)$ is a smooth 1-submanifold of $J \times C_{2k}(M)$. Moreover, there is a natural compactification $\overline{M}_{\Gamma}(\tilde{\xi}_J)$ of $M_{\Gamma}(\tilde{\xi}_J)$ into a compact smooth 1-manifold with boundary. The boundary consists of degenerate flow-graphs as follows. (See Figure 4.)

(A) A subgraph (or the whole) of $\Gamma$ collapses into a point of $M$.

(B) An edge of $\Gamma$, either compact or separated, splits in the middle by a critical point.

We may assume that the bifurcations are independent, namely, no two bifurcations of different types or from different graphs overlap on a single time.

There is a natural immersion $\overline{M}_{\Gamma}(\tilde{\xi}_J) \to J \times \overline{C}_{2k}(M)$, where $\overline{C}_{2k}(M)$ is the differential geometric analogue of the Fulton–MacPherson compactification of the configuration space $C_{2k}(M)$ ([AS, Ko1]). The bifurcation (A) is a point where $\overline{M}_{\Gamma}(\tilde{\xi}_J)$ meets $J \times \partial \overline{C}_{2k}(M)$. The bifurcation (B) is induced from the boundaries of the trajectory spaces for edges.

**Assumption 4.2.** In the following, we assume the genericity of $\tilde{\xi}_J$ in Lemma 4.1 and independence of bifurcations.

### 4.2. Strategy for the proof.

The general strategy for the proof of Theorem 3.4 is quite simple and the same as [Wa1, §10]. Thanks to Lemma 4.1, the change of the value of $z_{2k}$ in an interval $K = [a, b]$ is given by the counts of degenerate graphs.
that occur on $\text{Int } K$, as

$$z_{2k}(\vec{\xi}_b) - z_{2k}(\vec{\xi}_a) = \sum \text{(terms of degenerate graphs)},$$

where $\vec{\xi}_s = (\xi_1, \xi_2, \ldots, \xi_{3k})$. Thus we only need to check that the sum of contributions of degenerate graphs is zero, or is cancelled after passing to $\hat{z}_{2k}$.

Here, if, for every $\Gamma$, $M_\Gamma(\vec{\xi}_K)$ does not have boundaries except on the endpoints of $K$, then they give cobordisms between the moduli spaces on the endpoints of $K$ and it follows that the value of $z_{2k}$ does not change between $s_0$ and $s_1$. Here, the value of $\# M_\Gamma(\vec{\xi}_s)$ may change when a vertex of $\Gamma$ intersects $\Sigma$, but the difference is killed by the Holonomy relation or by the $\text{Tr}$ as in Lemma 3.3 and the trace is invariant. This is the only bifurcation that is caused by the nontriviality of the holonomy of the local coefficient system.

In general, $M_\Gamma(\vec{\xi}_K)$ may have boundaries on $\text{Int } K$ as given in Lemma 4.1. In §4.3, 4.4, we shall check that $\hat{z}_{2k}$ is invariant under each bifurcation in $\text{Int } K$. Then we shall discuss in §4.5 about the invariance of $\hat{z}_{2k}$ at a birth-death parameter and complete the proof of Theorem 3.4. These kinds of bifurcations have been already discussed in [Fu] and [Wa1].

4.3. Invariance for bifurcation of type (A). Among the degenerations of type (A), if the whole of a graph collapses, then $z_{2k}$ may change. However, $\hat{z}_{2k}$ does not change because the change is cancelled by the change of the correction term, as shown in [Wa1, Lemma 10.1]. When a proper subgraph with at least 3 black vertices or with a double edge between 2 black vertices collapses, then the sum of the changes is shown to vanish by the same arguments of symmetries as given in [Ko1]. When a subgraph with exactly 2 vertices collapses, the sum of the changes vanishes by the IHX relation. Since the degeneration can be made apart from $\Sigma$, the presence of the nontrivial holonomy does not affect the argument. See [Wa1, Lemmas 6.1, 10.1] for detail about this paragraph.

4.4. Invariance for bifurcation of type (B). The degenerations of type (B) can be treated by the same argument as in [Wa1, Lemmas 10.1, 10.2, 10.5]. There are no changes at all in the proof except that $Z$ is replaced with $\Lambda$ and that the coefficients of graphs belong to $\Lambda^{\otimes 3k}$. In the rest of the proof, we copy the corresponding part from [Wa1] for readers’ convenience. Namely, let $u \in \text{Int } J$ be a parameter where the boundary of $\mathcal{M}_\Gamma(\vec{\xi}_J)$ of type (B) occurs and let $\epsilon > 0$ be a small number such that there is only one bifurcation over $J_u = [u - \epsilon, u + \epsilon]$ (at $u$).

There are two cases for the degenerate flow-graphs at the bifurcation as follows.

(B₁) A flow-graph with an edge of degree 0, possibly with a flow-line of degree 1 attached to a white vertex.

(B₂) A flow-graph with a separated edge with a flow-line of degree 0, i.e., a $j/j$-intersection, attached to a white vertex.

Note that a splitting of a compact edge by a critical point is of type (B₁).

4.4.1. A bifurcation of type (B₁). This corresponds to [Wa1, Lemma 10.1, 10.2]. When the $i$-th edge of a $\tilde{C}$-graph $\Gamma$ is a separated edge on which $x, y \in P(i)$ are attached on the input/output respectively, we will write $\Gamma = \Gamma(x, y)$. This notation enables us to express the graph $\Gamma(x, y)$, with $x, y$ replaced with $x', y'$ respectively,
as $\Gamma(x', y')$. The notation $\Gamma(\emptyset, \emptyset)$ will denote the graph obtained from $\Gamma(x, y)$ by replacing the $i$-th edge with a compact edge.

For a $C_{u-\varepsilon}$-graph $\Gamma$, we put $d\Gamma = \sum_{i=1}^{3k} d_i \Gamma$, where

$$d_i \Gamma(p_i, q_i) = \sum_{r_i \in P_i} (-1)^{|r_i|} \# d_{p_i} \Gamma(r_i, q_i) + \sum_{s_i \in P_i} (-1)^{|s_i|} \# d_{s_i} \Gamma(p_i, s_i),$$

where the labelling of $d\Gamma$ is the naturally induced one. We consider $\hat{p}_i$ etc. as an element of $\Lambda^{3k}$ by identifying $\Lambda$ with $\mathbb{1}^{(i-1)} \otimes \Lambda \otimes 1^{(3k-i)}$. We extend $\# \mathcal{M}$ to $\Lambda^{3k}$-linear combinations of graphs by $\Lambda^{3k}$-linearity. Then for $p_i, q_i \in P_i$ with $|p_i| = |q_i| + 1$, the count $\# \mathcal{M}_{d_i \Gamma(p_i, q_i)}(\hat{\xi}_u \circ \epsilon)$ is equal to

$$\sum_{r_i \in P_i} (-1)^{|r_i|} \# d_{p_i} \Gamma(r_i, q_i) \cdot \# \mathcal{M}_{\hat{\epsilon}_u \circ \epsilon} \cdot \# \mathcal{M}_{d_i \Gamma(p_i, q_i)}(\hat{\xi}_u \circ \epsilon).$$

This is the count of the type (B$_1$) degeneracy at the $i$-th edge of $\Gamma(p_i, q_i)$. Let $g$ be a combinatorial propagator for $C_u(h_{u-\varepsilon}; \Lambda)$ and let $\tilde{g} = (g, g^{(2)}, \ldots, g^{(3k)})$. Since there are no other bifurcation on $J_u$ except one of type (B$_1$), we have

$$z_{2k}(\hat{\xi}_{u+\varepsilon}) - z_{2k}(\hat{\xi}_{u-\varepsilon}) = -\text{Tr}_{\tilde{g}} \left( \sum_{\Gamma} \# \mathcal{M}_{d\Gamma}(\hat{\xi}_u \circ \epsilon) \cdot \Gamma \right)$$

$$= -\text{Tr}_{\tilde{g}} \left( \sum_{i=1}^{3k} \sum_{r_i \in P_i} (-1)^{|r_i|} \# d_{p_i} \Gamma(r_i, q_i) \cdot \# \mathcal{M}_{\hat{\epsilon}_u \circ \epsilon} \cdot \# \mathcal{M}_{d_i \Gamma(p_i, q_i)}(\hat{\xi}_u \circ \epsilon) \right),$$

where the second sum is taken over uncolored graphs of the form $\Gamma'(\tilde{p}_i, \tilde{q}_i)$ of degree $(\eta_1, \eta_2, \ldots, \eta_{3k})$ with $\eta_\ell = 1$ for $\ell \neq i$ and $\eta_i = 0$, and

$$d_i \Gamma'(\tilde{p}_i, \tilde{q}_i) = \sum_{s_i \in P_i} \partial_{\tilde{p}_i} s_i \cdot \Gamma'(x_i, \tilde{q}_i) + \sum_{w_i \in P_i} \partial_{\tilde{q}_i} w_i \cdot \Gamma'(\tilde{p}_i, y_i) + \delta_{\tilde{p}_i, \tilde{q}_i} \Gamma'(0, 0).$$

We put $W_{\Gamma'}(\tilde{p}_i, \tilde{q}_i) = (-1)^{|\tilde{p}_i|} \# d_{\tilde{p}_i} \mathcal{M}_{\hat{\epsilon}_u \circ \epsilon} \cdot \# \mathcal{M}_{d_i \Gamma(p_i, q_i)}(\hat{\xi}_u \circ \epsilon)$. For each fixed pair $\tilde{p}_i, \tilde{q}_i \in P_i$ with $|\tilde{p}_i| = |\tilde{q}_i|$, we have

$$\text{Tr}_{\tilde{g}} \left( W_{\Gamma'}(\tilde{p}_i, \tilde{q}_i) \cdot \left( \sum_{s_i \in P_i} \partial_{\tilde{p}_i} s_i \cdot \Gamma'(x_i, \tilde{q}_i) + \sum_{w_i \in P_i} \partial_{\tilde{q}_i} w_i \cdot \Gamma'(\tilde{p}_i, y_i) + \delta_{\tilde{p}_i, \tilde{q}_i} \Gamma'(0, 0) \right) \right)$$

$$= \text{Tr}_{\tilde{g}} \left( \prod_{i=1}^{3k} \prod_{\eta_i = 0} \left( W_{\Gamma'}(\tilde{p}_i, \tilde{q}_i) \cdot \Gamma'(\tilde{p}_i, \tilde{q}_i) \right) \right) = \prod_{i=1}^{3k} \prod_{\eta_i = 0} \left( W_{\Gamma'}(\tilde{p}_i, \tilde{q}_i) \cdot \Gamma'(\tilde{p}_i, \tilde{q}_i) \right) = 0.$$

Hence we have $z_{2k}(\hat{\xi}_{u+\varepsilon}) = z_{2k}(\hat{\xi}_{u-\varepsilon})$. Moreover, by the independence of bifurcation, the anomaly correction term does not change on $J_u$, and we have $\hat{z}_{2k}(\hat{\xi}_{u+\varepsilon}) = \hat{z}_{2k}(\hat{\xi}_{u-\varepsilon})$.

*The minus sign comes from [Wal1, Proposition 9.5].
4.4.2. A bifurcation of type \((B_2)\). This corresponds to [Wa1, Lemma 10.5]. We may identify the underlying \(\Lambda\)-modules of \(C_*(h_{u+\varepsilon};\bar{\Lambda})\) and \(C_*(h_{u-\varepsilon};\bar{\Lambda})\) in a natural way through critical loci. We put \(C_0(\Gamma_p) = C_*(h_{u-\varepsilon};\bar{\Lambda}) = C_*(h_{u+\varepsilon};\bar{\Lambda})\) and \(P_2(\Gamma_p) = P_*(h_{u-\varepsilon}) = P_*(h_{u+\varepsilon})\). We denote the boundary operators of \(C_*(h_{u-\varepsilon};\bar{\Lambda})\) and \(C_*(h_{u+\varepsilon};\bar{\Lambda})\) by \(\partial\) and \(\partial'\) respectively. Suppose that a \(j/j\)-intersection between critical points (loci) \(p\) and \(q\) occurs at \(s = u\). Let \(\varphi: C_0(\Gamma_p) \to C_0(\Gamma_q)\) be the \(\Lambda\)-linear map of homogeneous degree \(0\), defined for each critical point (locus) \(x \in P(\Gamma_p)\) by

\[
\varphi(x) = \sum_{y \in P(\Gamma_p)} \varphi_{xy} \cdot y,
\]

where \(\mathcal{M}'(\xi; x, y)\) is the set of flow-lines of \(-\xi_u\) between \(x\) and \(y\), and \(\varepsilon(\gamma) = \pm t^n\), \(n = 1 \cdot \Sigma\). The sign of \(\varepsilon(\gamma)\) is determined by the orientation of the flow-line (see [Wa1, §9.1] for the definition of the sign). By assumption, \(\varphi(x)\) is non-zero only if \(x = p\), in which case \(\varphi(x) = \varphi(p) = \pm t^n q\).

**Lemma 4.3.** We have \(\partial' = (1 + \varphi) \partial (1 - \varphi)\). The endomorphism \(g' = (1 + \varphi) g (1 - \varphi) \in \text{End}(C_*(h_{u+\varepsilon};\bar{\Lambda}))\) is a combinatorial propagator for \((C_*(h_{u+\varepsilon};\bar{\Lambda}), \partial')\), and we have \(g' - g = g f - g f = g' g - g'\).

The identity \(\partial' = (1 + \varphi) \partial (1 - \varphi)\) in Lemma 4.3 for general local coefficient system is well-known and its proof is the same as [BZ, Appendix, Proposition 8] or [Wa1, Lemma 9.3]. Let

\[
d'' \Gamma(p_1, q_1) = \sum_{r_1 \in P(\Gamma(p_1, q_1))} \varphi_{p_1 r_1} \cdot \Gamma(r_1, q_1) - \sum_{s_1 \in P(\Gamma(p_1, q_1))} \varphi_{s_1 q_1} \cdot \Gamma(p_1, s_1),
\]

and the labellings for graphs in the right side hand are the natural induced ones. Then \(\# \mathcal{M}'(\xi_u; \xi_u)\) is the count of the type \((B_2)\) degeneracy at the first edge of \(\Gamma(p_1, q_1)\). By Lemma 4.1 and by the assumption of independence of different bifurcations, we have

\[
(4.1) \quad \# \mathcal{M}'(\xi_u; \xi_u) - \# \mathcal{M}' \Gamma(\xi_u; \xi_u) + \# \mathcal{M}'' \Gamma(\xi_u; \xi_u) = 0.
\]

For \(\tilde{g} = (g, g^{(2)}, \ldots, g^{(3k)}), \tilde{g}' = (g', g^{(2)}, \ldots, g^{(3k)})\), we have

\[
\sum_{\Gamma} \text{Tr}_{g'} (\# \mathcal{M}' \Gamma(\xi_u; \xi_u) - \# \mathcal{M}' \Gamma(\xi_u; \xi_u) \cdot \Gamma) = \sum_{\Gamma} \text{Tr}_{g'} (\# \mathcal{M}' \Gamma(\xi_u; \xi_u) - \# \mathcal{M}' \Gamma(\xi_u; \xi_u) \cdot \Gamma) = \sum_{\Gamma} \text{Tr}_{g'} (\# \mathcal{M}' \Gamma(\xi_u; \xi_u) \cdot \Gamma).
\]

This implies that \(z_{2k}(\xi_u; \xi_u) = z_{2k}(\xi_u; \xi_u)\) and hence \(\tilde{z}_{2k}(\xi_u; \xi_u) = \tilde{z}_{2k}(\xi_u; \xi_u)\) by the independence of bifurcations.
4.5. Invariance at birth-death bifurcation. When \( s \) crosses a parameter of a birth-death singularity, a separated edge will be glued together into a compact edge, or its reverse. The explicit form of the gluing has been given in [Wa1, §8.4, 8.5]. Then the invariance of \( z_{2k} \) at a birth-death parameter can be proved similarly as [Wa1, Lemma 10.4] because the gluing is local.

4.6. Completing the proof of Theorem 3.4. Now we have seen that \( \hat{z}_{2k} \) is invariant at all possible bifurcations that may occur in a 1-parameter family, namely, at the ones listed in Lemma 4.1 and birth-death bifurcations. This completes the proof. \( \square \)

5. Concluding remarks

5.1. Invariants for \( M, b_1(M) > 0, \) with free abelian acyclic local coefficient system. One may generalize the main result to the local coefficient system associated to the free abelianization \( \tau_1(M) \rightarrow H_1(M)/\text{Torsion}. \) In this case, let \( M \) be an oriented, connected closed 3-manifold and let \( f : M \rightarrow \mathbb{R} \) be a Morse function. Choose a \( \mathbb{Z} \)-basis \( \{h_1, \ldots, h_r\} \) of \( H_1(M)/\text{Torsion} \) and let \( \Lambda = \mathbb{Q}[H_1(M)/\text{Torsion}] = \mathbb{Q}[h_i^{1 \pm 1}, \ldots, h_i^{1 \pm 1}] \) and \( \tilde{\Lambda} = \Lambda[\frac{1}{h_1}, \ldots, \frac{1}{h_r}]. \) There is a collection \( \Sigma_1, \ldots, \Sigma_r \) of oriented surfaces in \( M \) that gives a dual basis for \( h_1, \ldots, h_r. \) Then the intersection of a generic flow-line with \( \Sigma_1, \ldots, \Sigma_r \) gives a holonomy in \( \Lambda \) and one can define the twisted Morse complexes \( C_\ast(f; \Lambda) \) and \( C_\ast(f; \tilde{\Lambda}) \) by counting flow-lines with holonomies, as in §2.1. If \( C_\ast(f; \Lambda) \) is acyclic, then one can define the invariant \( \hat{z}_{2k} \) by the same way as §3. It is an invariant of diffeomorphism type of \( M \) and oriented bordism classes of \( \Sigma_1, \ldots, \Sigma_r. \) The proof of invariance of \( \hat{z}_{2k} \) is exactly the same as that for \( b_1(M) = 1. \)

5.2. About non-triviality of \( \hat{z}_{2k} \). We have no information about non-triviality of \( \hat{z}_{2k}. \) In fact, the natural map \( i : \mathcal{O}_{2k}(\Lambda) \rightarrow \mathcal{O}_{2k}(\tilde{\Lambda}) \) induced by the inclusion \( \Lambda \rightarrow \tilde{\Lambda} \) is not injective. Because of Lemma 3.2, the invertibility of \( t - 1 \) in \( \tilde{\Lambda} \) may not be convenient. For example, we have the following.

**Proposition 5.1.** If \( \tilde{\Lambda} = \mathbb{Q}[\ell^{1 \pm 1}, \frac{1}{\ell - 1}], \) then the image of \( i : \mathcal{O}_2(\Lambda) \rightarrow \mathcal{O}_2(\tilde{\Lambda}) \) is at most the span of \( \Theta(1, 1, 1), \) where \( \Theta(P, Q, R) \) is the \( \tilde{\Lambda} \)-colored theta-graph of Figure 1 with edges colored by \( P, Q, R \in \tilde{\Lambda}. \)

**Proof.** By Lemma 3.2 and IHX relation, we have \( \Theta(t^k, 1, P) \sim \Theta(t^{-k}, 1, P) \) for \( k \in \mathbb{Z} \) and \( P \in \tilde{\Lambda}. \) Moreover, by Holonomy relation, we have \( \Theta(t^k, 1, t^\ell) \sim \Theta(t^{k+\ell}, 1, t^\ell) \) and \( \Theta((1 - t^n) t^{-1}, 1, t^n) \sim 0 \) \( (k, \ell, n \in \mathbb{Z}). \) Using these relations, one can show that \( \Theta(t^k, 1, t^\ell) \sim 0 \) if \( k \neq 0 \) or \( \ell \neq 0, \) by induction on \( \max\{|k|, |\ell|\}. \) Since \( \mathcal{O}_2(\Lambda) \) is generated by the theta-graphs and the dumbbell-graphs \( \square \square \) colored by monomials in \( \Lambda \) and by Lemma 3.2, the image of \( i \) is generated by \( \Theta(t^k, 1, t^\ell) \) \( (k, \ell \in \mathbb{Z}). \) This completes the proof. \( \square \)

Although it is known that the space \( \prod_{k \geq 1} \mathcal{O}_{2k}(\Lambda) \) is far from trivial and some calculation by hand suggests that the image of \( i \) would be non-trivial for \( r > 1, \) we do not know whether it is certainly non-trivial or not.
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