Theta-graph and diffeomorphisms of some 4-manifolds

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Abstract

In this article, we construct countably many mutually non-isotopic diffeomorphisms of some closed non simply-connected 4-manifolds that are homotopic to but not isotopic to the identity, by surgery along Θ-graphs. As corollaries of this, we obtain some new results on codimension 1 embeddings and pseudo-isotopies of 4-manifolds. In the proof of the non-triviality of the diffeomorphisms, we utilize a twisted analogue of Kontsevich’s characteristic class for smooth bundles, which is obtained by extending a higher dimensional analogue of Marché–Lescop’s “equivariant triple intersection” in configuration spaces of 3-manifolds to allow Lie algebraic local coefficient system.

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1 Introduction

In [Ga], David Gabai proposed a remarkable viewpoint that several important problems in 4-dimensional topology can be interpreted by 4-dimensional light bulb problem. There is a version of 4D light bulb problem, which asks whether spanning \( k \)-disk of the unknotted \( S^{k-1} \) in \( S^4 \) is unique up to isotopy fixing the boundary. Gabai gave a positive resolution of this problem for \( k = 2 \) ("the 4D light bulb theorem") in [Ga], and pointed out that a positive resolution for \( k = 3 \) implies the smooth Schoenflies conjecture and that the converse is true up to taking a lift in some finite branched covering of \( S^4 \) over the unknotted \( S^2 \).

He came up with a beautiful idea to study the 4D light bulb problem for 3-disk. Let \( \text{Emb}_0(D^3, D^3 \times S^1) \) denote the space of embeddings \( D^3 \to D^3 \times S^1 \) that agree with the standard inclusion \( \iota: D^3 \subset D^3 \times S^1 \) near the boundary and that are relatively homotopic to \( \iota \). There is a fibration sequence

\[
\text{Diff}(D^4, \partial) \to \text{Diff}(D^3 \times S^1, \partial) \to \text{Emb}_0(D^3, D^3 \times S^1).
\]

He showed that if spanning 3-disk of the unknot is unique up to isotopy rel boundary, then

\[
\pi_0 \text{Diff}(D^3 \times S^1, \partial)/\iota_\ast \pi_0 \text{Diff}(D^4, \partial) = 1.
\]

Hence nontriviality of the left hand side of this identity implies non-uniqueness of spanning 3-disk of the unknot in \( S^4 \). In [BG], Ryan Budney and Gabai studied the group \( \pi_0 \text{Diff}(D^3 \times S^1, \partial) \) in detail and gave a framework for approaching the smooth Schoenflies conjecture, and in particular, proved that the left hand side of the above identity is nontrivial, by explicit construction of elements and by computing homotopy groups of the space of knots in \( S^3 \times S^1 \), based on a work of Gregory Arone and Markus Szymik given in [ArSz]. More precisely, they considered the fibration sequence

\[
\text{Diff}(D^3 \times S^1, \partial) \to \text{Diff}_0(S^3 \times S^1) \to \text{Emb}_0^f(S^1, S^3 \times S^1), \tag{1.1}
\]

where \( \text{Emb}_0^f(S^1, S^3 \times S^1) \) is the space of framed embeddings of \( S^1 \) into \( S^3 \times S^1 \) homotopic to the standard inclusion, and studied the image of the composition

\[
\pi_0 \text{Emb}_0^f(S^1, S^3 \times S^1) \to \pi_0 \text{Diff}(D^3 \times S^1, \partial)
\to \pi_0 \text{Diff}(D^3 \times S^1, \partial)/\iota_\ast \pi_0 \text{Diff}(D^4, \partial) \tag{1.2}
\]

of the isotopy extension and the projection, by an argument of embedding calculus and by their explicit construction of a 1-parameter family.

In this paper, we attempt to extend some of the results of [BG] to 4-manifolds that may not be diffeomorphic to \( D^3 \times S^1 \). More precisely, we use the technique of graph surgery given in [Wa1] to construct 4-manifold bundles, and show their
nontriviality by using invariants. Here, the invariant we use is a twisted analogue of Kontsevich’s characteristic class of smooth bundles ([Kon]), and differs from that in [BG]. The invariant in this paper is defined by extending the 3-manifold invariant of Julien Marché and Christine Lescop that counts “equivariant triple intersection” in configuration space ([Mar, Les1, Les2]) to 4-manifolds with more general local coefficient system including those of Lie algebra. Main ideas in the definition and computation of the invariant of this paper are included in [Les1, Les2], which is analogous to those by Greg Kuperberg and Dylan Thurston [KT] for $\mathbb{Z}$ homology 3-spheres. Similar invariants of 3-manifolds with Lie algebra local coefficient system were defined in [AxSi, Kon, Fuk, BC, CS].

Our approach can be considered different from that of [BG] in the sense that ours can give nontrivial elements that cannot be obtained from homotopy groups of the space of embeddings from $S^1$ into a closed 4-manifold by isotopy extension. It should be mentioned that our construction is obtained by the “barbell implantation” given in [BG, §6], as written there, which looks quite simpler than our description.

1.1 Main results

For a parallelizable closed manifold $X$ with a base point $x_0$, let $X^\bullet$ be the complement of a small open ball around $x_0$. For a compact manifold $Z$ and its submanifold $Y$, let $\text{Diff}(Z,Y)$ denote the group of self-diffeomorphisms of $Z$ which fix a neighborhood of $Y$ pointwise, let $\text{Diff}(Z,\partial) = \text{Diff}(Z,\partial Z)$, and let $\text{Diff}(Z) = \text{Diff}(Z,\emptyset)$. Let $\text{Diff}_0(Z,Y)$ denote the subgroup of $\text{Diff}(Z,Y)$ consisting of diffeomorphisms relatively homotopic to the identity. Let $\mathcal{F}_\ast(X)$ denote the space of all framings on $X$ that are standard near $x_0$. Let $\text{Map}^\ast_{\text{deg}}(X,X)$ denote the space of all degree 1 maps $(X,x_0) \to (X,x_0)$ that maps a neighborhood of $x_0$ onto $x_0$ and that are relatively homotopic to the identity. The groups $\text{Diff}(X^\bullet,\partial)$, $\text{Diff}_0(X^\bullet,\partial)$ act on these spaces respectively. Let

$$\widehat{\text{BDiff}}(X^\bullet,\partial) = E\text{Diff}(X^\bullet,\partial) \times_{\text{Diff}(X^\bullet,\partial)} \mathcal{F}_\ast(X),$$
$$\widehat{\text{BDiff}}_{\text{deg}}(X^\bullet,\partial) = E\text{Diff}_0(X^\bullet,\partial) \times_{\text{Diff}_0(X^\bullet,\partial)} (\mathcal{F}_\ast(X) \times \text{Map}^\ast_{\text{deg}}(X,X)).$$

The space $\widehat{\text{BDiff}}(X^\bullet,\partial)$ is the classifying space for framed $X$-bundles $\pi: E \to B$ that are standard near $x_0$, and $\widehat{\text{BDiff}}_{\text{deg}}(X^\bullet,\partial)$ is the classifying space for such $X$-bundles equipped with a fiberwise pointed degree 1 map $E \to X$. For $p \geq 1$, the groups $\pi_p \mathcal{F}_\ast(X)$ and $\pi_p \text{Map}^\ast_{\text{deg}}(X,X)$ are both abelian groups of finite ranks.

**Theorem 1.1** (Theorem 5.1, 6.2). Let $\Sigma^3$ be the Poincaré homology 3-sphere $\Sigma(2,3,5)$ and $X = \Sigma^3 \times S^1$. Suppose that the local coefficient system of the adjoint action of a representation $\rho: \pi^1(\Sigma^3) \to SU(2)$ on $\mathfrak{g} = \mathfrak{sl}_2$ is acyclic. A homomorphism

$$Z^0_\Theta: \pi_1 \widehat{\text{BDiff}}_{\text{deg}}(X^\bullet,\partial) \to \mathcal{O}_\Theta^0(\mathfrak{g} \otimes \mathbb{Z}^2[t^{\pm 1}]; \rho(\pi^1) \times \mathbb{Z})$$

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The space $\widehat{\text{BDiff}}(X^\bullet,\partial)$ is the classifying space for framed $X$-bundles $\pi: E \to B$ that are standard near $x_0$, and $\widehat{\text{BDiff}}_{\text{deg}}(X^\bullet,\partial)$ is the classifying space for such $X$-bundles equipped with a fiberwise pointed degree 1 map $E \to X$. For $p \geq 1$, the groups $\pi_p \mathcal{F}_\ast(X)$ and $\pi_p \text{Map}^\ast_{\text{deg}}(X,X)$ are both abelian groups of finite ranks.

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to some space of anti-symmetric tensors is defined, and the image of $Z^0_{\Theta}$ has at least countable infinite rank.

**Theorem 1.2** (Theorem 5.3, 6.2). Let $\Sigma^4 = \Sigma^3 \times S^1$ and let $X = \Sigma^4 \times S^1$. Let $\rho_1 : \pi' = \pi_1(\Sigma^3 \times S^1) \to SU(2)$ be the extension of $\rho$ in Theorem 1.1 by mapping the generator of $\pi_1(S^1)$ into the center $\mathbb{Z}_2$ of $SU(2)$. Then a homomorphism

$$Z^0_{\Theta} : \pi_2 \text{BDiff}_{\text{deg}}(X^\bullet, \partial) \to \mathcal{A}^1_{\Theta}(\mathbb{R}^{2[t^\pm1]}; \rho_1(\pi') \times \mathbb{Z})$$

to some space of symmetric tensors is defined, and the image of $Z^0_{\Theta}$ has at least countable infinite rank.

The nontrivial elements detected in Theorem 1.1, 1.2 are constructed by surgery along $\Theta$-graphs, which is analogous to Goussarov and Habiro’s graph surgery in 3-manifolds.

Let $\text{Emb}_0(\Sigma^n, \Sigma^n \times S^1)$ denote the space of embeddings $\Sigma^n \to \Sigma^n \times S^1$ homotopic to the standard inclusion $\Sigma^n = \Sigma^n \times \{1\} \subset \Sigma^n \times S^1$. There is a fibration sequence

$$\text{Diff}_0(\Sigma^n \times I, \partial) \xrightarrow{i} \text{Diff}_0(\Sigma^n \times S^1) \xrightarrow{j} \text{Emb}_0(\Sigma^n, \Sigma^n \times S^1),$$

where $j$ is defined by the action on the standard inclusion $\Sigma^n = \Sigma^n \times \{1\} \subset \Sigma^n \times S^1$.

**Theorem 1.3** (Theorem 7.4). Let $\Sigma^3 = \Sigma(2,3,5)$ and $\Sigma^4 = \Sigma(2,3,5) \times S^1$.

1. The quotient set $\pi_0 \text{Diff}_0(\Sigma^3 \times S^1)/i_0 \pi_0 \text{Diff}_0(\Sigma^3 \times I, \partial)$ is at least countable.

Thus the set $\pi_0 \text{Emb}_0(\Sigma^3, \Sigma^3 \times S^1)$ is at least countable.

2. The abelianization of the group $\pi_1 \text{Diff}_0(\Sigma^4 \times S^1)/i_0 \pi_1 \text{Diff}_0(\Sigma^4 \times I, \partial)$ has at least countable infinite rank. Thus the abelianization of the group $\pi_1 \text{Emb}_0(\Sigma^4, \Sigma^4 \times S^1)$ has at least countable infinite rank.

Understanding $\pi_0 \text{Emb}_0(\Sigma^n, \Sigma^n \times S^1)$ or $\pi_1 \text{Emb}_0(\Sigma^n, \Sigma^n \times S^1)$ is important in relation with the following problems.

**Problem 1.4.**

1. Is every $h$-cobordism of $\Sigma^n$ diffeomorphic to $\Sigma^n \times I$?

2. Let $\mathcal{C}(\Sigma^n) = \text{Diff}(\Sigma^n \times I, \Sigma^n \times \{0\})$. Is $\pi_0 \mathcal{C}(\Sigma^n) = 0$? Namely, does pseudo-isotopy of $\Sigma^n$ imply isotopy?

The answers of these problems are well understood for $n \geq 5$ in the sense of the $h$-cobordism theorem of Smale ([Sm]), $s$-cobordism theorem of Barden, Mazur, Stallings ([Ker]), Cerf’s pseudo-isotopy theorem ([Ce]), and a theorem of Hatcher–Wagoner and K. Igusa ([HW, Hat]). On the other hand, not much is known about this problem for $n = 3, 4$ in the smooth category, except some deep results on gauge theory ([Don] etc.).

Let $\text{Emb}_0^{\text{fib}}(\Sigma^n, \Sigma^n \times S^3)$ be the image of the fibration $j : \text{Diff}_0(\Sigma^n \times S^1) \to \text{Emb}_0(\Sigma^n, \Sigma^n \times S^1)$ mentioned above.
Proposition 1.5. 1. If every $h$-cobordism of $\Sigma^n$ is diffeomorphic to $\Sigma^n \times I$, then every element of $\pi_0 \text{Emb}_{0}^h(\Sigma^n, \Sigma^n \times S^1)/\text{Diff}_0(\Sigma^n)$ can be trivialized after taking the lift in a finite cyclic covering of $\Sigma^n \times S^1$ in the $S^1$ direction.

2. If $\pi_0 \mathcal{C}(\Sigma^n) = 0$, then every element of $\pi_1 \text{Emb}_0(\Sigma^n, \Sigma^n \times S^1)/\text{Diff}_0(\Sigma^n)$ can be trivialized after taking the lift in a finite cyclic covering of $\Sigma^n \times S^1$ in the $S^1$ direction, fixing the endpoints.

Proof. 1. Each element of $\text{Emb}_{0}^h(\Sigma^n, \Sigma^n \times S^1)/\text{Diff}_0(\Sigma^n)$ has a lift $S$ in the infinite cyclic cover $\Sigma^n \times \mathbb{R}$ of $\Sigma^n \times S^1$. Then $S$ is a fiber of a possibly nonstandard $\Sigma^n$-bundle structure on $\Sigma^n \times \mathbb{R}$, which splits $\Sigma^n \times \mathbb{R}$ into two parts. Let $W_-$ be the part that includes $\Sigma^n \times (-\infty, a]$ for some $a \in \mathbb{R}$. By combining the two $\Sigma^n$-bundle structures, $W_-$ can be deformation retracted onto $\Sigma^n \times (-\infty, a]$ for some $a \in \mathbb{R}$ that is far below $S$. This shows that the inclusion of $\Sigma^n \times (-\infty, a]$ into $W_-$ is a homotopy equivalence. If every $h$-cobordism of $\Sigma^n$ is trivial, then so is the $h$-cobordism between $S$ and $\Sigma^n \times \{a\}$.

2. A loop in $\text{Emb}_0(\Sigma^n, \Sigma^n \times S^1)/\text{Diff}_0(\Sigma^n)$ gives rise to a path of maps $\phi_t : \Sigma^n \to \Sigma^n \times S^1$, which lifts to a path of maps $\tilde{\phi}_t : \Sigma^n \to \Sigma^n \times \mathbb{R}$ by the homotopy lifting property. The endpoints of the path are maps onto $\Sigma^n \times \{a\}$, $\Sigma^n \times \{a'\}$ for some $a, a' \in \mathbb{R}$, respectively. For some $b \in \mathbb{R}$ far below $a$ and $a'$, the part of $\Sigma^n \times \mathbb{R}$ between $\Sigma^n \times \{b\}$ and the image of $\tilde{\phi}_t$ is diffeomorphic to $\Sigma^n \times I$, and hence the path $\{\tilde{\phi}_t\}_t$ gives an element of $\pi_1 \mathcal{B}(\Sigma^n) = \pi_0 \mathcal{C}(\Sigma^n)$. Then the rest is similar to 1.

If there is a (family of) embedding(s) $\Sigma^n \to \Sigma^n \times S^1$ that does not satisfy the property of Proposition 1.5, then the answer to Problem 1.4 is NO! This kind of relation of codimension 1 embedding to Problem 1.4 were proposed by Gabai in the case of $\pi_0 \text{Emb}_0(D^3, D^3 \times S^1)$, in which case the lift property is equivalent to the 4-dimensional smooth Schoenflies conjecture ([Ga]. See also [BG]).

Remark 1.6. Although the groups $\pi_1 \text{Emb}_0(\Sigma^n, \Sigma^n \times S^1)$ and $\pi_0 \mathcal{C}(\Sigma^n)$ could be related indirectly by the connecting homomorphism $\pi_1 \text{Emb}_0(\Sigma^n, \Sigma^n \times S^1) \to \pi_0 \text{Diff}_0(\Sigma^n \times I, \partial)$ and the natural map $\pi_0 \text{Diff}_0(\Sigma^n \times I, \partial) \to \pi_0 \mathcal{C}(\Sigma^n)$, it is different from the relation mentioned above, at least on the definition. Hence, for example, we cannot immediately say that every element in the kernel of $\pi_1 \text{Emb}_0(\Sigma^n, \Sigma^n \times S^1) \to \pi_0 \text{Diff}_0(\Sigma^n \times I, \partial)$ (or mapped from $\pi_1 \text{Diff}_0(\Sigma^n \times S^1)$) is not a counterexample of Problem 1.4 (2).

Proposition 1.7 (Proposition 8.1). The nontrivial elements of $\pi_0 \text{Emb}_0(\Sigma^3, \Sigma^3 \times S^1)$ and $\pi_1 \text{Emb}_0(\Sigma^4, \Sigma^4 \times S^1)$ found in Theorem 1.3 can be all trivialized after taking the lift in a finite cyclic covering in the $S^1$ direction. Namely, none of the nontrivial elements of Theorem 1.3 contradict the conclusions of Proposition 1.5.
This suggests that it may be difficult to find counterexample to the conclusions of Proposition 1.5. On the other hand, Peter Teichner pointed out the following ([Tei]).

**Theorem 1.8** (Teichner, Theorem 9.3). The nontrivial elements of \( \pi_0 \Diff_0(\Sigma^3 \times S^1) \) found in Theorem 1.3 are included in the image of the natural map

\[
\pi_0 C(\Sigma^3 \times S^1) \to \pi_0 \Diff_0(\Sigma^3 \times S^1).
\]

**Corollary 1.9.** If \( \Sigma^3 = \Sigma(2, 3, 5) \), we have \( \pi_0 C(\Sigma^3 \times S^1) \neq 0 \).

**Proposition 1.10** (Proposition 9.7). Let \( \Sigma^4 = \Sigma(2, 3, 5) \times S^1 \). There exists an element of \( \pi_1 \Emb_0(\Sigma^4, \Sigma^4 \times S^1) / \Diff_0(\Sigma^4) \) that cannot be trivialized after taking the lift in any finite cyclic covering of \( \Sigma^4 \times S^1 \) in the \( S^1 \) direction.

Our \( \Theta \)-graph surgery construction gives diffeomorphisms of \( \Sigma^3 \times S^1 \) that are homotopic to the identity but smoothly nontrivial (Proposition 9.5). We do not know whether these elements are also topologically nontrivial. We add that there are several works ([Ru, BK, KM]) giving diffeomorphisms of many 4-manifolds \( X \) whose isotopy classes are in the kernel of \( \pi_0 \Diff(X) \to \pi_0 \Top(X) \).

One can consider similar problems for \( X = D^3 \times S^1 \).

**Theorem 1.11** (Theorem 10.1). For \( n \geq 3 \), \( \varepsilon = n + 1 \mod 2 \), a homomorphism

\[
Z_{\Theta} : \pi_{n-2} \widetilde{B} \Diff(D^n \times S^1, \partial) \to \mathcal{A}_{\Theta}(C[t^{\pm 1}]; \mathbb{Z})
\]

is defined, and the image of \( Z_{\Theta} \) has at least countable infinite rank.

The following result of [BG] can be obtained as a corollary of Theorem 1.11.

**Corollary 1.12** (Budney–Gabai). The abelian group

\[
\pi_0 \Diff(D^3 \times S^1, \partial)/i_* \pi_0 \Diff(D^4, \partial)
\]

has at least countable infinite rank.

According to [BG], it follows from Corollary 1.12 that the abelian group \( \pi_0 \Emb_0(D^3, D^3 \times S^1) \) has at least countable infinite rank. Hence there are many distinct spanning 3-disks of the unknot in \( S^4 \) that are not relatively isotopic to the standard one. A statement analogous to Theorem 1.8 follows in this case as well, as also shown in [BG].

Budney and Gabai also showed that the image of the isotopy extension induced map (1.2) is nontrivial, which is mapped to 0 of \( \pi_0 \Diff_0(S^3 \times S^1) \) by the natural map. Nevertheless, they conjectured that there is an element of \( \pi_0 \Diff(D^3 \times S^1, \partial)/i_* \pi_0 \Diff(D^4, \partial) \) that is not in the image from \( \pi_1 \Emb_0(S^4, S^3 \times S^1) \) ([BG, Conjecture 8.1]). Their conjecture implies existence of a nontrivial element of \( \pi_0 \Emb_0(S^3, S^3 \times S^1) \).
The example of Theorem 1.3 shows that their conjecture is true if $S^3$ is replaced with $\Sigma^3$. For $\Sigma^3 = \Sigma(2, 3, 5)$, we consider the following fibration sequence.

\[ \text{Diff}_0(\Sigma^3 \times S^1, \partial) \to \text{Diff}_0(\Sigma^3 \times S^1) \to \text{Emb}^f_0(S^1, \Sigma^3 \times S^1) \]

**Corollary 1.13.** The nontrivial elements of Theorem 1.3 for $\Sigma^3 \times S^1$ all lift to nontrivial elements of

\[ \pi_0 \text{Diff}_0(\Sigma^3 \times S^1, \partial)/i_* \pi_0 \text{Diff}_0(\Sigma^3 \times I, \partial). \]

Hence this set has elements that are not in the image from $\pi_1 \text{Emb}^f_0(S^1, \Sigma^3 \times S^1)$.

In the proof of Theorem 1.3, it is essential that $\Sigma^3$ is not simply connected. For $D^3 \times S^1$, there is still a possibility that the $\Theta$-graph construction in this paper could be obtained from $\pi_1 \text{Emb}^f_0(S^1, S^3 \times S^1)$.

1.2 Plan of the paper

In §2, we describe the spaces of (anti-)symmetric tensors, which are the target spaces of the invariants considered in this paper.

In §3, we make assumptions on manifolds with local coefficient systems and define their configuration spaces, and propagator.

In §4, we make assumptions on manifold bundles with local coefficient systems and define their fiberwise configuration spaces, and propagator for families. Also, we consider an intersection form between chains of configuration space with local coefficients.

In §5, we define the invariant and prove its bordism invariance.

In §6, we construct fiber bundles by graph surgery, following [Wa1], and give a formula of the invariant for the bundles obtained by graph surgery. The proof of the surgery formula boils down to Proposition 6.3. The outline of its proof is almost the same as given in [Les1].

In §7, we describe a restriction on the value of the invariant for $\Sigma^n \times S^1$-bundles that has support in $\Sigma^n \times I$. Then we prove Theorem 1.3.

In §8, we see that the embedding $\Sigma^3 \to \Sigma^3 \times S^1$ constructed by $\Theta$-graph surgery can be trivialized after taking the lift in a finite cyclic covering.

In §9, we describe the trivalent graph surgery by means of family of framed links. This description allows us to show that the $X$-bundle obtained by trivalent graph surgery extends to a bundle of pseudo-isotopy.

In §10, we prove Theorem 1.11 about diffeomorphisms of $D^n \times S^1$.

1.3 Notations and conventions

For a sequence of submanifolds $A_1, A_2, \ldots, A_r \subset W$ of a smooth Riemannian manifold $W$, we say that the intersection $A_1 \cap A_2 \cap \cdots \cap A_r$ is *transversal* if for
each point \( x \) in the intersection, the subspace \( N_xA_1 + N_xA_2 + \cdots + N_xA_r \subset T_xW \) spans the direct sum \( N_xA_1 \oplus N_xA_2 \oplus \cdots \oplus N_xA_r \), where \( N_xA_i \) is the orthogonal complement of \( T_xA_i \) in \( T_xW \) with respect to the Riemannian metric.

For manifolds with corners and their (strata) transversality, we follow [BT, Appendix].

As chains in a manifold \( X \), we consider \( \mathbb{C} \)-linear combinations of finitely many smooth maps from compact oriented manifolds with corners to \( X \). We say that two chains \( \sum n_i \sigma_i \) and \( \sum m_j \tau_j \) (\( n_i, m_j \in \mathbb{C} \), \( \sigma_i, \tau_j \): smooth maps from compact manifolds with corners) are strata transversal if for every pair \( i, j \), the terms \( \sigma_i \) and \( \tau_j \) are strata transversal. Strata transversality among two or more chains can be defined similarly. The intersection number \( \langle \sigma, \tau \rangle_X \) of strata transversal two chains \( \sigma = \sum n_i \sigma_i \) and \( \tau = \sum m_j \tau_j \) with \( \dim \sigma_i + \dim \tau_j = \dim X \) is defined by \( \sum_{i,j} n_i m_j \langle \sigma_i, \tau_j \rangle \), where we impose the orientation on the intersection by the wedge product of coorientations of the two pieces. We also consider intersection \( \langle \sigma_1, \ldots, \sigma_n \rangle_X \) of strata transversal chains \( \sigma_1, \ldots, \sigma_n \) for \( n \geq 2 \), which is defined similarly.

The diagonal \( \{(x, x) \in X \times X \mid x \in X\} \) is denoted by \( \Delta_X \).

For a fiber bundle \( \pi : E \to B \), we denote by \( T^vE \) the (vertical) tangent bundle along the fiber \( \text{Ker } d\pi \subset TE \). Let \( ST^vE \) denote the subbundle of \( T^vE \) of unit spheres. Let \( \partial^\text{fib} E \) denote the fiberwise boundaries: \( \bigcup_{b \in B} \partial (\pi^{-1}\{b\}) \).

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2 Spaces of (anti-)symmetric tensors

In this section, we define the target spaces of the invariants and give some ways to extract information from these spaces.

2.1 Anti-symmetric tensors for even dimensional manifolds

Let $G = SU(m)$, $\mathfrak{g} = \text{Lie}(G) \otimes \mathbb{C} = \mathfrak{sl}_m$. Lie algebra $\mathfrak{g}$ has an $\text{Ad}(G)$-invariant symmetric nondegenerate bilinear form $B : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$. If $\mathfrak{g} = \mathfrak{sl}_m$, then $B$ can be given by $B(X,Y) = \text{Tr}(XY)$. We equip $\mathfrak{g} \otimes \mathfrak{g}$ with an algebra structure over $\mathbb{C}$ by the following product.

$$(X_1 \otimes Y_1) \cdot (X_2 \otimes Y_2) = B(Y_1, X_2) \cdot X_1 \otimes Y_2$$

The multiplicative unit for this product is the Casimir element $c_\mathfrak{g} = \sum_i e_i \otimes e_i^*$, where $e = \{e_i\}$ a basis of $\mathfrak{g}$, $e^* = \{e_i^*\}$ is the dual basis for $e$ with respect to $B(\cdot, \cdot)$: $B(e_i, e_j^*) = \delta_{ij}$. The two algebras $\mathfrak{g}^{\otimes 2}$ and $\text{End}(\mathfrak{g})$ are identified by $u \otimes v^* \mapsto (x \mapsto v^*(x)u)$. Then the product in $\mathfrak{g}^{\otimes 2}$ corresponds to the composition of endomorphisms, and $B$ corresponds to the trace. The element $\text{Ad}(g) \in \text{End}(\mathfrak{g})$ corresponds to $(1 \otimes \text{Ad}(g^*))((c_\mathfrak{g}) = (\text{Ad}(g) \otimes 1)(c_\mathfrak{g}) \in \mathfrak{g}^{\otimes 2}$, which we denote by the same symbol $\text{Ad}(g)$. An $\text{Ad}(G)$-invariant skew-symmetric 3-form $\text{Tr} : \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$ is defined by the following formula.

$$\text{Tr}(X \otimes Y \otimes Z) = B([X, Y], Z)$$

**Definition 2.1.**

1. Let $\mathfrak{g}^\mathfrak{o}(\mathbb{C}[t^{\pm 1}]/\mathbb{Z}) = \wedge^3 \mathbb{C}[t]^{\pm 1}/\mathbb{Z}$, where the alternating tensor product is over $\mathbb{C}$, and the quotient by $\mathbb{Z}$ is given by the relation:

   $$a \wedge b \wedge c = t^na \wedge t^nb \wedge t^nc \quad (a,b,c \in \mathbb{C}[t^{\pm 1}], n \in \mathbb{Z}).$$

2. Let $\mathfrak{g}^\mathfrak{g}(\mathbb{C}[t^{\pm 1}]/\text{Ad}(G) \times \mathbb{Z}) = \wedge^3 (\mathfrak{g}^{\otimes 2}[t^{\pm 1}]/(\text{Ad}(G)^{\otimes 2} \times \mathbb{Z})$, where the alternating tensor product is over $\mathbb{C}$, and the quotient by $\text{Ad}(G)^{\otimes 2} \times \mathbb{Z}$ is given by the relations:

   $$a \wedge b \wedge c = t^na \wedge t^nb \wedge t^nc,$$

   $$a \wedge b \wedge c = \text{Ad}(g) a \wedge \text{Ad}(g) b \wedge \text{Ad}(g) c,$$

   $$a \wedge b \wedge c = a \text{Ad}(g') \wedge b \text{Ad}(g') \wedge c \text{Ad}(g') \quad (a,b,c \in \mathfrak{g}^{\otimes 2}[t^{\pm 1}], g,g' \in G).$$

3. Let $\mathfrak{g}^\mathfrak{g} = \wedge^3 \mathbb{C}[t^{\pm 1}]/\sim, [t^p \wedge t^q \wedge t^r] \sim [t^{-p} \wedge t^{-q} \wedge t^{-r}]$. This space is embedded into a $\mathbb{C}$ subspace of $\mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}]$ by the map

   $$[t^p \wedge t^q \wedge t^r] \mapsto \sum_{\sigma \in \mathfrak{S}_3} \text{sgn}(\sigma)(t_{\sigma(1)}^pt_{\sigma(2)}^qt_{\sigma(3)}^r + t_{\sigma(1)}^{-p}t_{\sigma(2)}^{-q}t_{\sigma(3)}^{-r}).$$
Let $\Theta(a, b, c) = a \land b \land c$. Let $W_{\Theta}^0(\Theta(t^a, t^b, t^c)) \in \mathcal{S}_3^0$ be defined by the following formula.

$$W_{\Theta}^0(\Theta(t^a, t^b, t^c)) = [t^{a-b} \land t^{b-c} \land t^{c-a}]$$

For $P^a, Q^b, R^c \in \mathfrak{sl}_2[t^\pm 1]$, let $\Theta(P^a, Q^b, R^c)$ denote the element $P^a \land Q^b \land R^c$ in $\mathcal{S}_3^0(\mathfrak{sl}_2[t^\pm 1]; \text{Ad}(G) \times \mathbb{Z})$, and let $W_{\Theta}^0(\Theta(P^a, Q^b, R^c)) \in \mathcal{S}_3^0$ be defined by the following formula:

$$W_{\Theta}^0(\Theta(P^a, Q^b, R^c)) = (\text{Tr} \otimes \text{Tr}) \sigma_\Theta (P \otimes Q \otimes R)[t^{b-a} \land t^{a-c} \land t^{c-b}],$$

where $\sigma_\Theta : (\mathfrak{sl}_2)^\otimes 3 \rightarrow (\mathfrak{sl}_2)^\otimes 3$ is the permutation of factors that is determined by the combinatorial structure of the $\Theta$-graph:

$$(x_P \otimes y_P) \otimes (x_Q \otimes y_Q) \otimes (x_R \otimes y_R) \overset{\sigma_\Theta}{\rightarrow} (y_R \otimes y_Q \otimes y_P).$$

The coefficient $(\text{Tr} \otimes \text{Tr}) \sigma_\Theta (P \otimes Q \otimes R)$ is precisely the sl$_2$-weight system of the $\Theta$-graph with decorations $P, Q, R$ on edges.

**Proposition 2.2.** The $W_{\Theta}^0(\Theta(t^a, t^b, t^c))$ as above induce well-defined $\mathbb{C}$-linear maps

$$W_{\Theta}^0(\Theta(t^a, t^b, t^c)) : \mathcal{S}_3^0[\mathbb{C}[t^\pm 1]; \mathbb{Z}] \rightarrow \mathcal{S}_3^0,$$

$$W_{\Theta}^0(\Theta(t^a, t^b, t^c)) : \mathcal{S}_3^0(\mathfrak{sl}_2[t^\pm 1]; \text{Ad}(G) \times \mathbb{Z}) \rightarrow \mathcal{S}_3^0.$$

**Proof.** First, we see that $W_{\Theta}^0(\Theta(t^a, t^b, t^c))$ is well-defined. The part $[t^{b-a} \land t^{a-c} \land t^{c-b}]$ is invariant under the action of $\mathbb{Z}_3$, and the transpositions $a \leftrightarrow b$, $b \leftrightarrow c$, $c \leftrightarrow a$ turn this into

$$[t^{a-b} \land t^{b-c} \land t^{c-a}] = [t^{b-a} \land t^{c-b} \land t^{a-c}] = -[t^{b-a} \land t^{a-c} \land t^{c-b}],$$

$$[t^{c-a} \land t^{a-b} \land t^{b-c}] = [t^{a-c} \land t^{b-a} \land t^{c-b}] = -[t^{b-a} \land t^{a-c} \land t^{c-b}],$$

$$[t^{b-c} \land t^{c-a} \land t^{a-b}] = [t^{c-b} \land t^{a-c} \land t^{b-a}] = -[t^{b-a} \land t^{a-c} \land t^{c-b}].$$

Also, it follows from the $S_3$-antisymmetry and the Ad($G$)-invariance of $\text{Tr}$ that the coefficient part $(\text{Tr} \otimes \text{Tr}) \sigma_\Theta (P \otimes Q \otimes R)$ is $S_3$-symmetric and Ad($G$)$^\otimes S_3$-invariant. This together with the $\mathbb{Z}$-invariance and $S_3$-antisymmetry of the part $[t^{b-a} \land t^{a-c} \land t^{c-b}]$ proves that $W_{\Theta}^0(\Theta(t^a, t^b, t^c))$ is well-defined. \hfill $\square$

**Example 2.3.** We have $W_{\Theta}^0(\Theta(1, t, t^p)) = [t \land t^{-p} \land t^{-p-1}]$. This element corresponds to the following element in $\mathbb{C}[t^\pm 1, t^\pm 1, t^\pm 1]$.

$$t_1t_2^{-p}t_3^{-p-1} - t_1t_3^{-p}t_2^{-p-1} - t_2t_3^{-p}t_1^{-p-1} + t_2t_3^{-p}t_1^{-p-1} + t_3^{-p}t_1^{-p-1} - t_3^{-p}t_2^{-p-1} - t_3^{-p}t_2^{-p-1} + t_3^{-p}t_1^{-p-1} + t_3^{-p}t_2^{-p-1} - t_3^{-p}t_1^{-p-1} + t_3^{-p}t_2^{-p-1} + t_3^{-p}t_1^{-p-1} - t_3^{-p}t_2^{-p-1}$$

We denote this polynomial by $f_p(t_1, t_2, t_3)$. 

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Proposition 2.4. $[\Theta(1, t, t^n)]$ $(p \geq 3)$ spans a countable infinite dimensional $\mathbb{C}$-subspace of $\mathfrak{s}_0(\mathbb{C}[t^{\pm 1}]; \mathbb{Z})$.

Proof. We have $f_p(1, x, x^3) = x^{3p-1} - x^{3p-2} - x^{2p+1} + x^{2p-3} + x^{p+2} - x^{p-3} - x^{-p+3} + x^{-p-2} + x^{-2p+3} - x^{-2p-1} + x^{-3p+1}$, and for $p \geq 3$, its maximal degree term is $x^{3p-1}$, minimal degree term is $x^{-3p+1}$. Thus $\{f_p(1, x, x^3) \mid p \geq 3\}$ is linearly independent over $\mathbb{C}$ in $\mathbb{C}[x^{\pm 1}]$.

\[ \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} \]

\[ \begin{array}{c}
W_0^\Theta(\mathfrak{su}(2), \mathbb{Z}) = (\mathfrak{su}(2) \otimes \mathbb{C}) = \mathfrak{sl}_2(\mathbb{C}) \]

\[ \begin{array}{c}
\text{Proof. For } g = \mathfrak{su}(2) \otimes \mathbb{C} = \mathfrak{sl}_2, \text{ the following identity holds in } \text{End}_{\mathbb{C}}(\mathfrak{g}\otimes^2) \text{ (}[CVa]\).}
\end{array} \]

Here, the left hand side is the composition of the Lie bracket $b : \mathfrak{g}\otimes^2 \rightarrow \mathfrak{g}$ and its dual $b^* : \mathfrak{g} \rightarrow \mathfrak{g}\otimes^2$. The two terms in the right hand side represent the identity morphism, and the transposition $x \otimes y \mapsto y \otimes x$, respectively.

If we apply this relation to the $\Theta$-graph with one edge decorated by 1, then we obtain a disjoint union of circles with decorations. The first term in the right hand side of (2.1) gives disjoint union of two circles decorated by $\text{Ad}(x)$ and $\text{Ad}(y)$, respectively. The second term in the right hand side of (2.1) gives one circle decorated by $\text{Ad}(x)\text{Ad}(y) = \text{Ad}(xy)$.

2.2 Symmetric tensors for odd dimensional manifolds

Definition 2.6. 1. Let $\mathfrak{sl}_0^3(\mathbb{C}[t^{\pm 1}]; \mathbb{Z}) = \text{Sym}^3(\mathbb{C}[t^{\pm 1}]/\mathbb{Z})$, where the alternating tensor product is over $\mathbb{C}$, and the quotient by $\mathbb{Z}$ is given by the relation:

\[ a \cdot b \cdot c = t^a a \cdot t^b b \cdot t^c c \quad (a, b, c \in \mathbb{C}[t^{\pm 1}], n \in \mathbb{Z}). \]

2. Let $\mathfrak{s}_0^3(\mathfrak{g}\otimes^2[t^{\pm 1}]; \text{Ad}(G) \times \mathbb{Z}) = \text{Sym}^3(\mathfrak{g}\otimes^2[t^{\pm 1}])/(\text{Ad}(G)^{\otimes 2} \times \mathbb{Z})$, where the tensor product is over $\mathbb{C}$, and the quotient by $\text{Ad}(G)^{\otimes 2} \times \mathbb{Z}$ is given by the relations:

\[ a \cdot b \cdot c = t^a a \cdot t^b b \cdot t^c c, \]
\[ a \cdot b \cdot c = \text{Ad}(g) a \cdot \text{Ad}(g) b \cdot \text{Ad}(g) c, \]
\[ a \cdot b \cdot c = a \text{Ad}(g') \cdot b \text{Ad}(g') \cdot c \text{Ad}(g') \quad (a, b, c \in \mathfrak{g}\otimes^2[t^{\pm 1}], g, g' \in G). \]
3. Let \( \mathcal{S}_\Theta^1 = \text{Sym}^3 \mathbb{C}[t^{\pm 1}]/\sim \), \([tp \cdot tq \cdot tr] \sim [t^{-p} \cdot t^{-q} \cdot t^{-r}]\). This space is embedded into a \( \mathbb{C} \) subspace of \( \mathbb{C}[[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}]] \) by the map

\[
[t^p \cdot t^q \cdot t^r] \mapsto \sum_{\sigma \in \mathfrak{S}_3} (t^p_{\sigma(1)} t^q_{\sigma(2)} t^r_{\sigma(3)} + t^{-p}_{\sigma(1)} t^{-q}_{\sigma(2)} t^{-r}_{\sigma(3)}).
\]

Let \( \Theta(a, b, c) \) denote the symmetric product \( a \cdot b \cdot c \), abusing the notation. Let \( W^1_{\mathbb{C}[t^{\pm 1}]}(\Theta(t^a, t^b, t^c)) \in \mathcal{S}_\Theta^1 \) be defined by the following formula:

\[
W^1_{\mathbb{C}[t^{\pm 1}]}(\Theta(t^a, t^b, t^c)) = [t^{b-a} \cdot t^{a-c} \cdot t^{c-b}]
\]

For \( Pt^a, Qt^b, Rt^c \in \mathfrak{g}^{\otimes 2}[t^{\pm 1}] \), let \( \Theta(Pt^a, Qt^b, Rt^c) \) denote the element \( Pt^a \cdot Qt^b \cdot Rt^c \) in \( \mathfrak{g}^1(\mathfrak{g}^{\otimes 2}[t^{\pm 1}] ; \text{Ad}(G) \times \mathbb{Z}) \), and let \( W^1_{\mathfrak{g}[t^{\pm 1}]}(\Theta(Pt^a, Qt^b, Rt^c)) \in \mathcal{S}_\Theta^1 \) be defined by the following formula:

\[
W^1_{\mathfrak{g}[t^{\pm 1}]}(\Theta(Pt^a, Qt^b, Rt^c)) = (\text{Tr} \otimes \text{Tr})(\sigma_{\Theta}(P \otimes Q \otimes R)[t^{b-a} \cdot t^{a-c} \cdot t^{c-b}]).
\]

**Proposition 2.7.** The \( W^1_{\mathbb{C}[t^{\pm 1}]} \) and \( W^1_{\mathfrak{g}[t^{\pm 1}]} \) for \( \Theta(Pt^a, Qt^b, Rt^c) \) as above induce well-defined \( \mathbb{C} \)-linear maps

\[
W^1_{\mathbb{C}[t^{\pm 1}]} : \mathcal{S}_\Theta^1(\mathbb{C}[t^{\pm 1}] ; \mathbb{Z}) \to \mathcal{S}_\Theta^1,
\]

\[
W^1_{\mathfrak{g}[t^{\pm 1}]} : \mathcal{S}_\Theta^1(\mathfrak{g}^{\otimes 2}[t^{\pm 1}] ; \text{Ad}(G) \times \mathbb{Z}) \to \mathcal{S}_\Theta^1.
\]

**Proof.** First, we see that the part \([t^{b-a} \cdot t^{a-c} \cdot t^{c-b}]\) is symmetric since it is invariant under the action of \( \mathbb{Z} \) at vertices, and the transpositions \( a \leftrightarrow b, b \leftrightarrow c, c \leftrightarrow a \) do not affect the class. This shows that \( W^1_{\mathbb{C}[t^{\pm 1}]} \) is well-defined. Also, we have seen in the proof of Proposition 2.2 that the coefficient part \((\text{Tr} \otimes \text{Tr})(\sigma_{\Theta}(P \otimes Q \otimes R))\) is \( \mathfrak{S}_3 \)-symmetric and \( \text{Ad}(G) \times \mathbb{Z} \)-invariant. This together with the \( \mathbb{Z} \)-invariance and \( \mathfrak{S}_3 \)-symmetry of the part \([t^{b-a} \cdot t^{a-c} \cdot t^{c-b}]\) proves that \( W^1_{\mathfrak{g}[t^{\pm 1}]} \) is well-defined.

The following proposition can be proved by the same manner as Proposition 2.4.

**Proposition 2.8.** \( [\Theta(1, t, t^p)] \ (p \geq 1) \) spans a countable infinite dimensional \( \mathbb{C} \)-subspace of \( \mathcal{S}_\Theta^1(\mathbb{C}[t^{\pm 1}] ; \mathbb{Z}) \).

### 3 Manifolds and configuration spaces

We make an assumption on the manifold \( X \) with a local coefficient system \( A \), and we compute the homology of the configuration space \( \text{Conf}_2(X) \) of two points with local coefficients. Based on the computation of the homology, we define propagator, which is an analogue of “equivariant propagator” in [Les1]. We will define in the next section propagator in families of configuration spaces, which plays an important role in the definition of the main perturbative invariant.
3.1 Acyclic complex

Let $X$ be a manifold with universal cover $\tilde{X}$. The space $\tilde{X}$ can be considered as the set of pairs $(x, [γ])$, where $x$ is a point of $X$ and $[γ]$ is the homotopy class relative to the endpoints of a path $γ : [0, 1] → X$ from $x$ to the base point $x_0 ∈ X$. We take a cellular chain complex $C_*(X)$ and take $C_*(\tilde{X})$ for the CW decomposition of $\tilde{X}$ compatible with that of $X$. Let $S_*(X)$, $S_*(\tilde{X})$ be the complexes of piecewise smooth singular chains in $X$ and $\tilde{X}$, respectively. We assume that the coefficients are in $\mathbb{C}$, unless otherwise noted.

Let $π = π_1(X)$, $A$ be a left $\mathbb{C}[π]$-module, and let $ρ_A : π → \text{End}_\mathbb{C}(A)$ be the corresponding $\mathbb{C}$-linear representation. We assume that $A$ has a nondegenerate $\mathbb{C}[π]$-invariant symmetric $\mathbb{C}$-bilinear form $B(·, ·) : A ⊗_\mathbb{C} A → \mathbb{C}$. Let $c_A ∈ A ⊗_\mathbb{C} A$ be the $\mathbb{C}[π]$-invariant symmetric 2-tensor defined by the following formula.

$$c_A = \sum_i e_i ⊗ e_i^*$$

Here, $\{e_i\}$ is a $\mathbb{C}$-basis of $A$, $\{e_i^*\}$ is the dual basis for $\{e_i\}$ with respect to $B$.

Let $C_*(X; A) = C_*(\tilde{X}) ⊗_{\mathbb{C}[π]} A$, $S_*(X; A) = S_*(\tilde{X}) ⊗_{\mathbb{C}[π]} A$. The boundary operators of these complexes are given by $∂_A = ∂ ⊗ 1$. When $X = S^n × S^1$, $C_*(X; A) = C_*(\tilde{X}) ⊗_{\mathbb{C}[π]} A$ has a structure of a free $(\mathbb{C}[t^±1], \mathbb{C}[t^±1])$-bimodule if we let $π' = π_1(Σ^n)$ and take $A = A_1 ⊗_\mathbb{C} \mathbb{C}[t^±1]$ for a $\mathbb{C}[π']$-module $A_1$. In this case, we have a symmetric $\mathbb{C}[t^±1]$-bilinear form $B(·, ·) : A ⊗_R A → R$, where $R = \mathbb{C}[t^±1]$, and the invariant 2-tensor $c_{A_1} ∈ A_1^{⊗2}[t^±1] = A ⊗_R A$.

**Assumption 3.1.** We assume that $H_*(X; A) = 0$.

**Lemma 3.2.** Let $Λ = \mathbb{C}[t^±1]$. We assume Assumption 3.1. The following hold.

1. $C_*(X × X; A ⊗_\mathbb{C} A) = C_*(X; A) ⊗_\mathbb{C} C_*(X; A)$ is acyclic.

2. If moreover $X = Σ^n × S^1$, $A = A_1 ⊗_\mathbb{C} Λ$, then $C_*(X × X; A ⊗_\Lambda A) = C_*(X; A) ⊗_Λ C_*(X; A)$ is acyclic.

**Proof.** The assertion 1 follows from the Künneth formula. For 2, since $Λ$ is a PID, the exact sequence for $Λ$-modules (Künneth formula (e.g., [CE, Theorem VI.3.2]))

$$0 → \bigoplus_{p+q=n} H_p(C)⊗_Λ H_q(C) → H_n(C⊗_Λ C) → \bigoplus_{p+q=n-1} \text{Tor}_1^\Lambda(H_p(C), H_q(C)) → 0$$

($C = C_*(X; A)$) holds. Then 2 is an immediate corollary of this. □

**Remark 3.3.** 1. To apply the Künneth formula, some restriction on the coefficient ring or on the chain complex $C$ is necessary, and it is not always possible to replace the tensor product in Lemma 3.2 (2) with that of $\mathbb{C}[π]$-modules. For example, for $π = \mathbb{Z} × \mathbb{Z}$, the Künneth formula for $\mathbb{C}[π]$-modules fails.
2. When $X$ is a closed manifold, $\chi(X) = 0$ is necessary for the complex $C_*(X; g)$ of the adjoint representation of a flat $G$-connection to be acyclic. Indeed, the Euler characteristic of this complex depends only on the dimensions of the modules $C_i(X; g)$, and not on the twisted differential. Thus the identity

$$\sum_i (-1)^i \dim C_i(X; g) = \chi(X) \dim g$$

holds. If $C_*(X; g)$ is acyclic, $\chi(X)$ must be 0. For $\dim X = 4$, $\chi(X) = 0$ is satisfied if $X$ is a homology $S^3 \times S^1$, but not if $X$ is a homology $S^4$.

**Example 3.4.** Let $\Sigma^3 = \Sigma(2, 3, 5)$ and $g = su(2) \otimes \mathbb{C}$. The fundamental group $\pi'$ of $\Sigma^3$ has the following presentation.

$$\pi' = \langle x_1, x_2, x_3, h \mid h \text{ central}, x_1^2 = h, x_2^3 = h^{-1}, x_3^3 = h^{-1}, x_1x_2x_3 = 1 \rangle$$

There are exactly two conjugacy classes of irreducible representations $\rho : \pi' \rightarrow SU(2)$. One is such that

$$\rho(h) = -I, \quad \rho(x_1) = \text{diag}(i, -i),$$

$$\rho(x_2) = \begin{pmatrix} \frac{1}{2} - \alpha i & \beta \\ -\beta & \frac{1}{2} + \alpha i \end{pmatrix} \sim \text{diag}(e^{\pi i/3}, e^{-\pi i/3}),$$

$$\rho(x_3) = \begin{pmatrix} \alpha - \frac{1}{2} i & -\beta i \\ -\beta i & \alpha + \frac{1}{2} i \end{pmatrix} \sim \text{diag}(e^{\pi i/5}, e^{-\pi i/5}),$$

(3.1)

where $\alpha = \cos \frac{\pi}{5} = \frac{1 + \sqrt{5}}{4}$, $\beta = -\frac{1 + \sqrt{5}}{4}$, and $\sim$ is the conjugacy relation ([Sa, Bod] etc.). For the local coefficient system $g_\rho = g$ on $\Sigma^3$ for the adjoint action of $\rho$, we have $H^0(\Sigma^3; g_\rho) = 0$ by the irreducibility of the representation. Moreover, it is known that $H^1(\Sigma^3; g_\rho) = 0$ holds for any irreducible $SU(2)$-representation of $\Sigma^3 = \Sigma(p, q, r)$ ([FS, Bod] etc.). This together with the Poincaré duality implies the acyclicity.

$$H_*(\Sigma^3; g_\rho) = 0$$

By taking the tensor product over $\mathbb{C}$ with the homology of the universal cover of $S^1$ with local coefficients

$$H_*(S^1; \mathbb{C}[t^{\pm 1}]) = H_*(\mathbb{R}^1; \mathbb{Z} \otimes \mathbb{Z}[t^{\pm 1}]) \cong \mathbb{C},$$

we have

$$H_*(\Sigma^3 \times S^1; g_\rho[t^{\pm 1}]) = 0.$$ 

Also, by mapping the generator of $\pi_1(S^3) = \mathbb{Z}$ into the center $\mathbb{Z}_2$ of $SU(2)$, an irreducible extension $\rho_1 : \pi_1(\Sigma^3 \times S^1) \rightarrow SU(2)$ is obtained. Since the component of $\rho_1$ in the moduli space of representations is a point, we have

$$H_*(\Sigma^3 \times S^1; g_{\rho_1}) = 0.$$
The same results hold for another irreducible representation $\rho : \pi' \to SU(2)$ defined by the same formula as (3.1) with $\alpha = \cos \frac{3\pi}{5} = \frac{1 - \sqrt{5}}{4}$, $\beta = \frac{1 + \sqrt{5}}{4}$.

In relation to the local coefficient system $g_{\rho[t^{\pm 1}]}$ above, let us compute the value of

$$W_0^0 g_{[t^{\pm 1}]}(\Theta(1, \Ad(\alpha_3) t, \Ad(\alpha_3^2) t^p)) \quad (\alpha_3 = \rho(x_3))$$

for example. By applying Proposition 2.5, the value of this weight is

$$2(w(\alpha_3) w(\alpha_3^2) - w(\alpha_3^3)) [t \wedge t^{-p} \wedge t^{p-1}],$$

where $w(\alpha)$ is the weight of the oriented circle decorated by $\Ad(\alpha)$. If $V$ is the standard representation of $SU(2)$, the adjoint $SU(2)$-representation $g$ can be considered as the codimension 1 subrepresentation of $V \otimes V^* = \mathfrak{gl}_2$, where the $SU(2)$-action is given by $v \otimes w^* \mapsto g v \otimes gw^*$, $gw^*(\cdot) = w^*(g^{-1}(\cdot)) \quad (g \in SU(2))$. This shows that

$$w(\alpha) = \Tr(\alpha) \Tr(\alpha^*) - 1 = |\Tr(\alpha)|^2 - 1.$$

By $w(\alpha_3) = \frac{1 + \sqrt{5}}{2}$, $w(\alpha_3^2) = w(\alpha_3^3) = \frac{1 - \sqrt{5}}{2}$, the value of the weight (3.2) is

$$(-3 + \sqrt{5}) [t \wedge t^{-p} \wedge t^{p-1}].$$

Here, by Proposition 2.4, we know that $[t \wedge t^{-p} \wedge t^{p-1}] \neq 0$ for $p \geq 3$. Also, replacing $\alpha_3$ with $I$ in the above formula gives the weight of $\Theta(1, t, t^p)$, and its value is

$$12 [t \wedge t^{-p} \wedge t^{p-1}].$$

Hence $[\Theta(1, t, t^p)]$ and $[\Theta(1, \Ad(\alpha_3) t, \Ad(\alpha_3^2) t^p)]$ are linearly independent over $\mathbb{Q}$. \hfill \Box

### 3.2 Propagator in a fiber for $n$ odd

Let $X$ be a parallelizable $(n + 1)$-manifold with a local coefficient system $A$ satisfying the acyclicity Assumption 3.1. Let $\Delta_X$ be the diagonal of $X \times X$. The configuration space of two points of $X$ is

$$\text{Conf}_2(X) = X \times X \setminus \Delta_X.$$ 

The Fulton–MacPherson compactification of $\text{Conf}_2(X)$ is

$$\overline{\text{Conf}}_2(X) = B\ell(X \times X, \Delta_X).$$

The boundary $\partial \overline{\text{Conf}}_2(X)$ is identified with the normal sphere bundle of $\Delta_X$, which is canonically identified with the unit tangent bundle $ST(X)$. Since $X$ is parallelizable, there is a diffeomorphism $\partial \overline{\text{Conf}}_2(X) \cong S^n \times X$.

We also denote by $A^\otimes 2$ the pullback of the local coefficient system $A^\otimes 2 = A \otimes_R A$, $R = \mathbb{C}$ or $\mathbb{C}[t^{\pm 1}]$, on $X \times X$ to $\overline{\text{Conf}}_2(X)$, and its restriction to...
Lemma 3.6. Abusing the notation. Also, we write $\partial A$ for $\partial_{A^\otimes 2}$ for simplicity. There is an $R$-submodule of $A^\otimes 2$ spanned by $c_A$, and the diagonal action of the holonomy on $\Delta_X$ is trivial on it. Since the natural map $H_*(X;R) = H_*(X;\mathbb{C}) \otimes R \to H_*(\Delta_X;A^\otimes 2)$ given by $\sigma \mapsto \sigma \otimes c_A$ is a section of the projection, $H_*(\Delta_X;A^\otimes 2)$ has a direct summand isomorphic to $H_*(X;R)$.

**Assumption 3.5.** We assume that $H_*(\Delta_X;A^\otimes 2) \cong H_*(X;R)$ and generated over $R$ by $\sigma \otimes c_A$ for cycles $\sigma$ of $X$.

We will see later in Proposition 3.13 that Assumption 3.5 is satisfied by the local coefficient system $A = g_\rho[t^{\pm}]$ on $X = \Sigma(2,3,5) \times S^1$ of Example 3.4. Also, for $X = D^3 \times S^1$, $A = \Lambda = \mathbb{C}[t^{\pm}]$, we have $A^\otimes 2 = A \otimes A = \Lambda$ and Assumption 3.5 is satisfied.

**Lemma 3.6.** For an odd integer $n \geq 3$, let $X$ be a parallelizable $\mathbb{Z}$ homology $S^n \times S^1$. Let $K$ be an oriented knot in $X$ that generates $H_1(X;\mathbb{Z})$, and $\Sigma$ be an oriented $n$-submanifold of $X$ that generates $H_n(X;\mathbb{Z})$ and such that $(\Sigma,K) = 1$.

Under Assumptions 3.1 and 3.5, we have

$$H_i(\text{Conf}_2(X);A^\otimes 2) = \begin{cases} R[ST(*) \otimes c_A] & (i = n) \\ R[ST(K) \otimes c_A] & (i = n+1) \\ R[ST(\Sigma) \otimes c_A] & (i = 2n) \\ R[ST(X) \otimes c_A] & (i = 2n+1) \\ 0 & \text{(otherwise)} \end{cases}$$

where for a submanifold $\sigma$ of $X$, we denote by $ST(\sigma)$ the restriction of $ST(X)$ to $\sigma$.

**Proof.** We consider the exact sequence

$$0 = H_{i+1}(X^x;A^\otimes 2) \to H_{i+1}(X^x,\text{Conf}_2(X);A^\otimes 2) \to H_i(\text{Conf}_2(X);A^\otimes 2) \to H_i(X^x;A^\otimes 2) = 0.$$ 

Here, letting $N(\Delta_X)$ be a closed tubular neighborhood of $\Delta_X$, we have

$$H_{i+1}(X^x,\text{Conf}_2(X);A^\otimes 2) \cong H_{i+1}(N(\Delta_X),\partial N(\Delta_X);A^\otimes 2)$$

by excision. Since $X$ is parallelizable, the normal bundle of $\Delta_X$ is trivial. By Assumption 3.5, we have

$$H_{i+1}(N(\Delta_X),\partial N(\Delta_X);A^\otimes 2) = H_{n+1}(D^{n+1},\partial D^{n+1};R) \otimes_R H_{i-n}(\Delta_X;A^\otimes 2) 
\cong H_{n+1}(D^{n+1},\partial D^{n+1};R) \otimes_R H_{i-n}(X;R) \cong H_{i-n}(X;R).$$

Here, $H_{i-n}(X;R)$ is rank 1 for $i-n = 0,1,n,n+1$, and its generator is $*,K,\Sigma,X$, respectively. \hfill \square

**Lemma 3.7.** Let $X,K,A$ be as in Lemma 3.6. Let $s_{\tau_0} : X \to ST(X)$ be the section given by the first vector of the standard framing $\tau_0$. Then we have

$$H_{i+1}(\partial \text{Conf}_2(X);A^\otimes 2) = R[ST(K) \otimes c_A] \oplus R[s_{\tau_0}(X) \otimes c_A].$$
Proposition 3.12. Under the above assumption, we have

\[ \text{For each point } (x, y) \in X \times X, \]  

Proof. This follows from the trivialization \( \partial \overline{\text{Conf}}_2(X) \cong S^n \times X \) induced by \( \tau_0 \), the Künneth formula, and Assumption 3.5.

Since \( H_{n+2}(\overline{\text{Conf}}_2(X); A^\otimes 2) = 0 \) by Lemma 3.6, we have the following exact sequence of \( R \)-modules.

\[
0 \to H_{n+2}(\overline{\text{Conf}}_2(X), \partial \overline{\text{Conf}}_2(X); A^\otimes 2) \to H_{n+1}(\partial \overline{\text{Conf}}_2(X); A^\otimes 2) \to H_{n+1}(\overline{\text{Conf}}_2(X); A^\otimes 2) \to 0.
\]

Corollary 3.8. Let \( X, K, A \) be as in Lemma 3.6. There exists an element \( \mathcal{O}(X, A) \in R \) such that

\[
i([s_{\tau_0}(X) \otimes c_A]) = \mathcal{O}(X, A)[ST(K) \otimes c_A].
\]

Hence we have

\[
[s_{\tau_0}(X) \otimes c_A] - \mathcal{O}(X, A)[ST(K) \otimes c_A] \in \text{Ker } i = \text{Im } r.
\]

Proof. This follows from the identity \( H_{n+1}(\overline{\text{Conf}}_2(X); A^\otimes 2) = R[ST(K) \otimes c_A] \) of Lemma 3.6.

Definition 3.9 (Propagator). Let \( X, K \) be as in Lemma 3.6. A propagator is an \((n + 2)\)-chain \( P \) of \( \overline{\text{Conf}}_2(X) \) with coefficients in \( A^\otimes 2 \) that is transversal to the boundary and that satisfies

\[
\partial A P = s_{\tau_0}(X) \otimes c_A - \mathcal{O}(X, A) ST(K) \otimes c_A.
\]

Remark 3.10. For 3-manifolds with \( b_1 = 1 \), Lescop described \( \mathcal{O}(X, C(t)) \) in terms of the logarithmic derivative of the Alexander polynomial \( (\lesc) \).

Example 3.11. Let \( X = \Sigma^n \times S^1 \) (\( \Sigma^n \): parallelizable homology \( S^n \), \( \pi' = \pi_1(\Sigma^n) \)). We consider \( A = A_1 \otimes C \Lambda \) as a \( C[\pi] \)-module by the \( \pi = \pi' \times \mathbb{Z} \)-action

\[
(g \times n)(X \otimes f(t)) = \rho_{A_1}(g)(X) \otimes t^n f(t).
\]

If \( C_*(\Sigma^n; A_1) \) is acyclic, so is \( C_*(X; A) \).

Proposition 3.12. Under the above assumption, we have \( \mathcal{O}(X, A) = 0 \).

Proof. For each point \((x, y)\) of \( \overline{\text{Conf}}_2(\Sigma^n) \), the chain \((x, y) \times S^1 = \{(x, \theta) \times (y, \theta) \mid \theta \in S^1\} \) of \( \overline{\text{Conf}}_2(X) \) is defined. Similarly, for each \( k \)-chain \( \sigma \) of \( \overline{\text{Conf}}_2(\Sigma^n) \), the \((k + 1)\)-chain \( \sigma \times S^1 \) of \( \overline{\text{Conf}}_2(X) \) is defined. It follows from the invariance of the coefficient \( A \otimes A = A_1^\otimes [t] \) under the diagonal action of \( \Lambda \) that this correspondence \( \sigma \mapsto \sigma \times S^1 \) induces a well-defined map

\[
\cdot \times S^1 : H_i(\overline{\text{Conf}}_2(\Sigma^n); A_1^\otimes) \to H_{i+1}(\overline{\text{Conf}}_2(X); A_1^\otimes[t]).
\]

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We consider the following commutative diagram.

\[
\begin{array}{ccc}
H_{n+2}(\text{Conf}_2(X); \partial \text{Conf}_2(X); A_1^{\otimes 2}[t^{\pm 1}]) & \overset{r}{\longrightarrow} & H_{n+1}(\partial \text{Conf}_2(X); A_1[t^{\pm 1}]) \\
\uparrow \times S^1 & & \uparrow \times S^1 \\
H_{n+1}(\text{Conf}_2(\Sigma^n); \partial \text{Conf}_2(\Sigma^n); A_1^{\otimes 2}) & \overset{r'}{\longrightarrow} & H_n(\partial \text{Conf}_2(\Sigma^n); A_1^{\otimes 2})
\end{array}
\]

We may see by an argument similar to Lemma 3.6 that \( H_n(\partial \text{Conf}_2(\Sigma^n); A_1^{\otimes 2}) \) is spanned over \( R \) by \([s_{\tau_0}(\Sigma^n) \otimes c_{A_1}]\). The following holds.

\[
[(s_{\tau_0}'(\Sigma^n) \times S^1) \otimes c_{A_1}] = [s_{\tau_0}(X) \otimes c_{A_1}] \in H_{n+1}(\partial \text{Conf}_2(X); A_1^{\otimes 2}[t^{\pm 1}])
\]

Moreover, by an argument similar to Corollary 3.8, it follows that there exists an \((n + 1)\)-chain \( P_{\Sigma^n} \) of \( \text{Conf}_2(\Sigma^n) \) with coefficients in \( A_1^{\otimes 2} \) that satisfies

\[
r'(\{P_{\Sigma^n}\}) = [s_{\tau_0}'(\Sigma^n) \otimes c_{A_1}].
\]

Then by the commutativity of the above diagram, we have

\[
r(\{P_{\Sigma^n} \times S^1\}) = [s_{\tau_0}(X) \otimes c_{A_1}].
\]

Since this belongs to the kernel of the map \( i : H_{n+1}(\partial \text{Conf}_2(X); A_1^{\otimes 2}[t^{\pm 1}]) \to H_{n+1}(\text{Conf}_2(X); A_1^{\otimes 2}[t^{\pm 1}]) \), it follows that \( \partial'(X, A) = 0 \).

\[\square\]

**Proposition 3.13.** The local coefficient system \( A = \mathfrak{g}_p[t^{\pm}] \) on \( X = \Sigma(2, 3, 5) \times S^1 \) of Example 3.4 satisfies Assumption 3.5. Namely, we have \( H_*(\Delta X; \mathfrak{g}_p^{\otimes 2}[t^{\pm}]) \cong H_*(X; \mathbb{C}[t^{\pm 1}]) \) and generated by \( \sigma \otimes c_A \) for cycles \( \sigma \) of \( X \).

**Proof.** It is well-known that the adjoint representation of \( SU(2) \) on \( \mathfrak{g} \) is irreducible and \( \mathfrak{g}^{\otimes 2} \) equipped with the diagonal adjoint \( SU(2) \)-action \( \text{Ad} \otimes \text{Ad} \) has the following decomposition into irreducible modules:

\[
\mathfrak{g} \otimes \mathfrak{g} \cong V_0 \oplus V_2 \oplus V_4,
\]

where \( V_m \) is the irreducible representation of \( \mathfrak{sl}_2 \) of highest weight \( m \) and of dimension \( m + 1 \). More explicitly, \( V_0 \cong \mathbb{C} \) is the trivial representation spanned by \( c_0 \), and \( V_2 \cong \mathfrak{g} \). This together with a result of Example 3.4 shows that

\[
H_*(\Delta X; \mathfrak{g}_p^{\otimes 2}[t^{\pm 1}]) \cong H_*(X; \mathbb{C}[t^{\pm 1}]) \oplus H_*(X; \mathfrak{g}_p[t^{\pm 1}]) \oplus H_*(X; V_0[t^{\pm 1}])
\]

\[
\cong H_*(X; \mathbb{C}[t^{\pm 1}]) \oplus H_*(X; V_4[t^{\pm 1}]).
\]

Hence it suffices to prove that \( H_*(\Sigma(2, 3, 5); (V_4)_\rho) = 0 \).

The module \( V_m = \text{Sym}^m V, \ V = \mathbb{C}^2 \), can be considered as the space of homogeneous (commutative) polynomials of degree \( m \) in two variables, on which \( g \in SU(2) \) acts by

\[
(g \cdot f)(x, y) = f(g^{-1}(x, y)^T).
\]
One can see that $H^0(\Sigma(2,3,5);(V_4)_\rho) \cong V_4^{\pi'} = \{v \in V_4 \mid g \cdot v = v \ (g \in \pi')\}$ is zero. Indeed, by a direct computation with (3.1), the intersection of the eigenspaces of eigenvalue 1 of the actions of $x_1$ and $x_2$ of $\pi'$ on $V_4$ is zero. Thus we also have $H_0(\Sigma(2,3,5);(V_4)_\rho) = 0$.

Since $\pi' = \pi_1(\Sigma(2,3,5)$ is finite, $\mathbb{C}[\pi']$ is semisimple in the sense of [CE, §I.4] by Maschke’s theorem. Hence by the universal coefficient theorem, which is valid if the ring is hereditary, the sequence

$$0 \to H_j(C) \otimes_R W \to H_j(C \otimes_R W) \to \text{Tor}_1^R(H_{j-1}(C), W) \to 0$$

is exact for $R = \mathbb{C}[\pi']$, $C = S_*(S^3;\mathbb{C})$ and any $\mathbb{C}[\pi']$-module $W$ (e.g., [CE, Theorem VI.3.3]). Hence we have $H_2(\Sigma(2,3,5);(V_4)_\rho) = 0$. Then by Poincaré duality and duality between homology and cohomology, we have $H_*(\Sigma(2,3,5);(V_4)_\rho) = 0$. \hfill $\square$

### 3.3 Propagator in a fiber for $n$ even

**Lemma 3.14.** For an even integer $n \geq 4$, let $\Sigma^{n-1}$ be a parallelizable $\mathbb{Z}$ homology $S^{n-1}$, let $X = \Sigma^n \times S^1$, where $\Sigma^n = \Sigma^{n-1} \times S^1$. Let $K = (\{x\} \times S^1) \times \{1\}$, $x \in \Sigma^{n-1}$, and let $L = \{y\} \times S^1$, $y = (x,1) \in \Sigma^n$, which are oriented knots in $X$. Under the Assumptions 3.1 and 3.5 for $X, A$, we have

$$H_i(\overline{\text{Conf}_2}(X); A^{\otimes 2}) = \begin{cases} R[ST(*) \otimes c_A] & (i = n) \\ R[ST(K) \otimes c_A] \oplus R[ST(L) \otimes c_A] & (i = n + 1) \\ R[ST(K \times L) \otimes c_A] & (i = n + 2) \\ R[ST(\Sigma^{n-1}) \otimes c_A] & (i = 2n - 1) \\ R[ST(\Sigma^n) \otimes c_A] \oplus R[ST(\Sigma^{n-1} \times L) \otimes c_A] & (i = 2n) \\ R[ST(X) \otimes c_A] & (i = 2n + 1) \\ 0 & (\text{otherwise}) \end{cases}$$

where $K \times L$ is the torus $\{x\} \times S^1 \times S^1$ in $X$, and $\Sigma^{n-1} \times L$ is $(\Sigma^{n-1} \times \{1\}) \times S^1$.

**Proof.** The proof is almost the same as Lemma 3.6. Namely, we have

$$H_i(\overline{\text{Conf}_2}(X); A^{\otimes 2}) \cong H_{i-n}(X; R).$$

Now $H_{i-n}(X; R)$ is nonzero for $i - n = 0, 1, 2, n - 1, n, n + 1$, and the corresponding generators are $\ast, \{K, L\}, K \times L, \Sigma^{n-1}, \{\Sigma^n, \Sigma^{n-1} \times L\}, X$, respectively. \hfill $\square$

**Lemma 3.15.** Let $X, K, L, \Sigma^{n-1}, \Sigma^n, A$ be as in Lemma 3.14 that satisfies Assumption 3.5. Let $s_{\tau_0} : X \to ST(X)$ be the section given by the first vector of the standard framing $\tau_0$. Then we have

$$H_{n+1}(\partial\overline{\text{Conf}_2}(X); A^{\otimes 2}) = R[ST(K) \otimes c_A] \oplus R[ST(L) \otimes c_A] \oplus R[s_{\tau_0}(X) \otimes c_A].$$

**Proof.** Again, this follows from the trivialization $\partial\overline{\text{Conf}_2}(X) \cong S^n \times X$ induced by $\tau_0$ and from the Künneth formula. \hfill $\square$
It follows from Lemma 3.14 that the natural map \( H_{n+2}(\text{Conf}_2(X); A^{\otimes 2}) \to H_{n+2}(\text{Conf}_2(X), \partial \text{Conf}_2(X); A^{\otimes 2}) \) is zero. As for the odd \( n \) case, we have the following exact sequence.

\[
0 \to H_{n+2}(\text{Conf}_2(X), \partial \text{Conf}_2(X); A^{\otimes 2}) \overset{r}{\to} H_{n+1}(\partial \text{Conf}_2(X); A^{\otimes 2}) \overset{i}{\to} H_{n+1}(\text{Conf}_2(X); A^{\otimes 2})
\]

In \( H_{n+1}(\partial \text{Conf}_2(X); A^{\otimes 2}) \), the class \([s_{\tau_0}(X) \otimes c_A]\) can be written as \([(s'_{\tau_0}(\Sigma^n) \times L) \otimes c_A]\). By Proposition 3.12, we have \( \partial(\Sigma^n, A_1) = 0 \) and that

\[
i([s_{\tau_0}(X) \otimes c_A]) = 0.
\]

**Definition 3.16 (Propagator).** Let \( X, K, L, \Sigma^{n-1}, \Sigma^n, A \) be as in Lemma 3.14 that satisfies Assumption 3.5. A propagator is an \((n + 2)\)-chain \( P \) of \( \text{Conf}_2(X) \) with coefficients in \( A^{\otimes 2} \) that is transversal to the boundary and that satisfies

\[
\partial_A P = s_{\tau_0}(X) \otimes c_A.
\]

## 4 Framed fiber bundles and their fiberwise configuration spaces

We make an assumption on \( X \)-bundles \( \pi : E \to B \) with local coefficient system, and we compute the homology of the bundle \( E_{\text{Conf}_2(\pi)} \) associated to \( \pi \) with fiber the configuration space of two points. We also recall a geometric interpretation of chains with local coefficients and intersections among them.

### 4.1 Moduli spaces of manifolds with structures

Let \( \Sigma^n \) be a stably parallelizable \( n \)-manifold and let \( X = \Sigma^n \times S^1 \). We fix a base point \( x_0 \in X \). Let

\[
K = \{*\} \times S^1 \subset X,
\]

and we assume that \( K \) is disjoint from \( x_0 \). Let \( \tau_0 : TX \to \mathbb{R}^{n+1} \times X \) be the standard framing, i.e., the one obtained from a stable framing on \( \Sigma^n \times \{*\} \) by \( S^1 \)-symmetry. In the following, we assume that \( \pi : E \to B \) is an \( X \)-bundle with structure group \( \text{Diff}_0(X^*, \partial) \), equipped with the following data.

1. A trivialization of the restriction of \( \pi \) near the base point section \( \tilde{x}_0 \).
2. A smooth trivialization of the vertical tangent bundle \( T^v E \) over \( E \)

\[
\tau : T^v E \to \mathbb{R}^{n+1} \times E
\]

that agrees with \( \tau_0 \) on the base fiber, and that agrees with the trivialization of the normal bundle of \( \tilde{x}_0 \) induced from 1.
3. A fiberwise pointed degree 1 map $f : E \to X$.

The classifying space for $X$-bundles $\pi$ with such structures is given by $\widetilde{BDiff}_\text{deg}(X^\bullet, \partial)$, defined in §1.1. We have the following fibration sequences.

$$\mathcal{F}_*(X) \to \widetilde{BDiff}(X^\bullet, \partial) \to BDiff(X^\bullet, \partial)$$
$$\mathcal{F}_*(X) \times \text{Map}^\text{deg}_*(X, X) \to \widetilde{BDiff}_\text{deg}(X^\bullet, \partial) \to BDiff_0(X^\bullet, \partial)$$

**Proposition 4.1.** Suppose that $\Sigma^n$ is a homology $n$-sphere or a (homology $S^{n-1} \times S^1$). Then for $p \geq 1$, $\pi_p \mathcal{F}_*(X)$ and $\pi_p \text{Map}^\text{deg}_*(X, X)$ are finitely generated abelian groups.

**Proof.** This follows immediately from the following two lemmas.

**Lemma 4.2.** For $p \geq 1$, we have the following isomorphisms.

1. If $\Sigma^n$ is a homology $n$-sphere,
   $$\pi_p \mathcal{F}_*(X) \cong \pi_{p+1} SO_{n+1} + \pi_{p+n} SO_{n+1} + \pi_{p+n+1} SO_{n+1}.$$

2. If $\Sigma^n$ is a (homology $S^{n-1} \times S^1$),
   $$\pi_p \mathcal{F}_*(X) \cong (\pi_{p+1} SO_{n+1})^2 + \pi_{p+2} SO_{n+1} + \pi_{p+n-1} SO_{n+1} + \pi_{p+n} SO_{n+1} + \pi_{p+n+1} SO_{n+1}.$$ 

**Proof.** Fixing the standard framing $\tau_0$ gives an identification

$$\mathcal{F}_*(X) = \text{Map}_*(X, SO_{n+1}).$$

When $X = \Sigma^n \times S^1$, $\Sigma^n$ is a homology $n$-sphere, we take a pointed degree 1 map $c : X \to S^n \times S^1$. One can see that $\Omega^p \text{Map}_*(X, SO_{n+1}) \cong \text{Map}_*(S^p \wedge X, SO_{n+1})$, and the iterated suspension $S^p c : S^p \wedge X \to S^p \wedge (S^n \times S^1)$ is a $1$-connected homology equivalence. By Whitehead’s theorem, $S^p c$ is a homotopy equivalence. Hence $S^p c$ induces an isomorphism

$$\pi_0 \text{Map}_*(S^p \wedge X, SO_{n+1}) \cong \pi_0 \text{Map}_*(S^p \wedge (S^n \times S^1), SO_{n+1}).$$

The result follows by $S^p \wedge (S^n \times S^1) \cong S^p(S^n) \vee S^p(S^1) \vee S^p(S^n \wedge S^1) \cong S^{p+n} \vee S^{p+1} \vee S^{p+n+1}$. Proof of 2 is similar.

Proof of the next lemma is the same as that of Lemma 4.2.

**Lemma 4.3.** For $p \geq 1$, we have the following isomorphisms.

1. If $\Sigma^n$ is a homology $n$-sphere,
   $$\pi_p \text{Map}^\text{deg}_*(X, X) \cong \pi_{p+1} X + \pi_{p+n} X + \pi_{p+n+1} X.$$
2. If $\Sigma^n$ is a $(\text{homology } S^{n-1}) \times S^1$,

$$\pi_p \text{Map}^\text{deg}_* (X, X) \cong (\pi_{p+1}X)^2 + \pi_{p+2}X + \pi_{p+n-1}X + \pi_{p+n}X + \pi_{p+n+1}X.$$ 

**Corollary 4.4.**

1. If the abelianization of the group $\pi_1 \widetilde{\text{BDiff}}_{\text{deg}}(X^*, \partial)$ has at least countable infinite rank, then so is the abelianization of $\pi_1 \text{BDiff}_0(X^*, \partial)$.

2. For $p \geq 2$, if the abelian group $\pi_p \widetilde{\text{BDiff}}_{\text{deg}}(X^*, \partial)$ has at least countable infinite rank, then so is the abelian group $\pi_p \text{BDiff}_0(X^*, \partial)$.

**Proof.** For 1, we consider the exact sequence

$$\pi_1 (\mathcal{F}_s(X) \times \text{Map}^\text{deg}_* (X, X)) \to \pi_1 \widetilde{\text{BDiff}}_{\text{deg}}(X^*, \partial) \to \pi_1 \text{BDiff}_0(X^*, \partial) \to 0.$$ 

If we put $G = \pi_1 \widetilde{\text{BDiff}}_{\text{deg}}(X^*, \partial)$, and let $H \subset G$ be the image from $\pi_1 (\mathcal{F}_s(X) \times \text{Map}^\text{deg}_* (X, X))$, then by exactness, $H$ is a normal subgroup of $G$, $G/H$ is isomorphic to $\pi_1 \text{BDiff}_0(X^*, \partial)$, and we have the homology exact sequence

$$H_1(H; \mathbb{Z}) \to H_1(G; \mathbb{Z}) \to H_1(G/H; \mathbb{Z}) \to 0,$$

which is a part of the five term exact sequence for group homology. Since $H$ is the image from a finitely generated abelian group, and $H_1(G; \mathbb{Z})$ has at least countable infinite rank, the result follows.

The assertion 2 follows immediately from the long exact sequence for homotopy groups of fibration, and that the homotopy groups of the fiber is finitely generated abelian groups.

**Proposition 4.5.**

1. If the abelianization of the group $\pi_1 \text{BDiff}_0(X^*, \partial)$ has at least countable infinite rank, then so is the abelianization of $\pi_1 \text{BDiff}_0(X)$.

2. For $p \geq 2$, if the abelian group $\pi_p \text{BDiff}_0(X^*, \partial)$ has at least countable infinite rank, then so is the abelian group $\pi_p \text{BDiff}_0(X)$.

**Proof.** In the fibration sequence,

$$\text{Diff}_0(X^*, \partial) \to \text{Diff}_0(X) \to \text{Emb}^\text{fr}((x_0), X),$$

we have $\text{Emb}^\text{fr}((x_0), X) \simeq SO_{n+1} \times X$. Hence for $p \geq 2$, the cokernel of the homomorphism $\pi_p(SO_{n+1} \times X) \to \pi_p \text{BDiff}_0(X^*, \partial)$ has at least countable infinite rank. For $p = 1$, the result follows from the exact sequence for group homology as in Corollary 4.4.

**Remark 4.6.** The results in this subsection hold even if $X$ is replaced with a homology $\Sigma^n \times S^1$. In that case, one should choose $K$ and $\tau_0$, which may not be canonical.
4.2 Chains of $E$ with twisted coefficients

We shall recall a geometric interpretation of chains of $E$ with twisted coefficients. Let $f : X \to \mathbb{R}$ be a Morse function with exactly one critical point of index 0, and let $\xi$ be its gradient-like vector field that is Morse–Smale. In the handle decomposition of $X$ with respect to $\xi$, the 2-skeleton gives a presentation of $\pi_1(X)$. If we let $p$ denote the critical point of $f$ of index 0, the compactification $A_p(\xi)$ of its ascending manifold has a structure of manifold with corners, and its codimension 1 strata consists of the ascending manifolds of critical points of index 1. A generic smooth path in $X$ may intersect the codimension 1 strata of $A_p(\xi)$ transversally and finitely many times, and the sequence of intersections aligned on the path gives a word in the generator of the presentation of $\pi_1(X)$.

When an endpoint of a path is on a codimension 1 stratum, we push the endpoint slightly in the positive direction of the coorientation of the stratum to determine a word. Then a homomorphism

$$\text{Hol}_X : \Pi(X) \to \pi_1(X),$$

where $\Pi(X)$ is the fundamental groupoid of $X$, is obtained. Furthermore, by composing with the representation $\rho_A : \pi_1(X) \to \text{End}_C(A)$, we obtain a local coefficient system over $X$.

We assume that the $X$-bundle $\pi : E \to B$ is equipped with a fiberwise pointed degree 1 map $q : E \to X$. In this case, by pulling back the local coefficient system over $X$ by $q$, we obtain a local coefficient system on $E$. More precisely, taking the holonomy of a smooth path $\gamma$ in $E$ by that of a path $q \circ \gamma$ in $X$ gives a homomorphism

$$\text{Hol}_E : \Pi(E) \to \pi_1(X),$$

and by composing with the representation $\rho_A : \pi_1(X) \to \text{End}_C(A)$, we obtain a local coefficient system over $E$.

Let $\tilde{E}$ be the $\pi$-covering over $X$ defined by pullback of the universal cover $\tilde{X} \to X$ by $q$. A chain of $S_p(E; A) = S_p(\tilde{E}) \otimes_{\pi} A$ can be seen as that of $S_p(E) \otimes_C A$ as follows.

We take a base point $\delta_0$ at the barycenter of the standard $p$-simplex $\Delta^p$. A smooth simplex $\tilde{\sigma} : \Delta^p \to \tilde{E}$ of $\tilde{E}$ corresponds bijectively to the pair of a smooth simplex $\sigma : \Delta^p \to E$ of $E$ and the equivalence class of a path of $E$ from $\sigma(\delta_0)$ to $x_0$, where we consider two such paths are equivalent if their image under the homomorphism $q : \Pi(E) \to \Pi(X)$ agree. Moreover, through the holonomy homomorphism $\Pi(E) \to \pi_1(X)$, the equivalence class of a base path in $E$ corresponds bijectively to an element of $\pi$. This gives a bijective correspondence between $\tilde{\sigma}$ in $\tilde{E}$ and a pair $(\sigma, g)$ of a smooth simplex $\sigma$ and an element $g$ of $\pi_1(X)$. If we denote this pair by $\sigma \cdot g$, a $\mathbb{C}$-chain of $\tilde{E}$ is formally written as

$$\sum_{g \in \pi} \sigma \cdot g.$$
Figure 1: A singular simplex $\sigma$ with a path $\gamma$ to $x_0$

where $\sigma_g$ is a $\mathbb{C}$-chain of $E$. We obtain a bijective correspondence between chains of $S_p(\tilde{E}) \otimes_{\mathbb{C}\pi} A$ and those of $S_p(E) \otimes_{\mathbb{C}} A$ by the modification

$$\sum_g \sigma_g \cdot g \otimes a_g = \sum_g \sigma_g \otimes g \cdot a_g.$$  

This correspondence depends on the handle stratification of $X$ and may not be canonical.

The boundary operator $\partial_A = \partial \otimes 1$ of $S_p(\tilde{E}) \otimes_{\mathbb{C}\pi} A$ induces a twisted boundary operator on $S_p(E) \otimes_{\mathbb{C}} A$. This can be described as follows. For a smooth simplex with a base path $(\sigma, \gamma)$, an induced base path for a face $\sigma_j$ of $\sigma$ can be defined by connecting $\gamma$ and the segment $\eta_j$ between the barycenters of $\sigma$ and $\sigma_j$ (see Figure 1). The boundary of $(\sigma, \gamma)$ is the sum of such faces $(\sigma_j, \gamma \circ \eta_j)$:

$$\partial(\sigma, \gamma) = \sum_j (\sigma_j, \gamma \circ \eta_j).$$

If a segment $\eta_j$ between the barycenters hits a codimension 1 wall, the coefficient of $\sigma_j$ in $\partial(\sigma, \gamma)$ differs from that of $\sigma$ by the action of $\eta_j$. Thus the twisted boundary of $S_p(E) \otimes_{\mathbb{C}} A$ is given by the formula

$$\partial_A(\sigma \otimes a) = \sum_j \sigma_j \otimes \text{Hol}_E(\eta_j) \cdot a.$$  

4.3 Chains of $\mathcal{E Conf}_2(\pi)$ with twisted coefficients

We shall explain a geometric interpretation of chains of $\mathcal{E Conf}_2(\pi)$ with twisted coefficients.

The direct product of projections $\beta : \tilde{X} \times \tilde{X} \to X \times X$ is a $\pi \times \pi$-covering of $X \times X$. The space $\tilde{X} \times \tilde{X}$ can be interpreted as the set of pairs $([\gamma_1], [\gamma_2])$ of equivalence classes of paths $\gamma_1, \gamma_2$ to $x_0$. 

$$\gamma_1 \quad \gamma_2 \quad x_0$$

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The group $\text{Diff}(X^\bullet, \partial)$ acts diagonally on $\bar{X} \times \bar{X}$ by pulling back the diagonal action on $X \times X$, and on $\bar{\text{Conf}}(X) = B\ell(\bar{X} \times \bar{X}, \beta^{-1}\Delta_X)$.

We denote by $\bar{\text{Conf}}_2(\pi) : E\bar{\text{Conf}}_2(\pi) \to B$ and $\bar{\text{Conf}}(\pi) : E\bar{\text{Conf}}(\pi) \to B$ the associated $\bar{\text{Conf}}_2(X)$-bundle and $\bar{\text{Conf}}(X)$-bundle to $\pi : E \to B$.

A chain of $S_p(E\bar{\text{Conf}}_2(\pi)) \otimes_{\mathbb{C}[\pi_2]} A^\otimes 2$ can be seen as that of $S_p(E\bar{\text{Conf}}(\pi)) \otimes_{\mathbb{C}} A^\otimes 2$ as follows.

The local coefficient system on $E$ induces that on $E \times_B E$ with coefficients in $A^\otimes 2$. Its structure can be interpreted as follows. A point of $E \times_B E$ can be given by data $(x, y)$, where $x, y \in \pi^{-1}(s)$.

A path $\eta$ in $E \times_B E$ is projected to a pair of two paths $(\eta_1, \eta_2)$ in $E$ by the fiberwise projections $\text{pr}_i : E \times_B E \to E_i$, ($i = 1, 2$). By taking the holonomy of each path, we obtain an element of $\pi \times \pi$ (see Figure 2). This gives a homomorphism

$$\text{Hol}_{E \times_B E} : \Pi(E \times_B E) \to \text{End}_\mathbb{C}(A^\otimes 2)$$

and gives a local coefficient system on $E \times_B E$ with coefficients in $A^\otimes 2$.

The local coefficient system on $E \times_B E$ given here induces that on $E\bar{\text{Conf}}_2(\pi)$. A chain of the $\pi \times \pi$-covering $E\bar{\text{Conf}}(\pi)$ of $E\bar{\text{Conf}}_2(\pi)$ has the following expression

$$\sum_{(g, h) \in \pi_2} \sigma_{g, h} \cdot (g, h),$$

where $\sigma_{g, h}$ is a $\mathbb{C}$-chain of $E\bar{\text{Conf}}_2(\pi)$. By forwarding the action of $(g, h)$ to that on $A^\otimes 2$, a chain of $S_p(E\bar{\text{Conf}}_2(\pi)) \otimes_{\mathbb{C}} A^\otimes 2$ is obtained.
The twisted boundary on $S_p(E\text{Conf}_2(\pi)) \otimes \mathbb{C} A^\otimes 2$ can be defined by considering the $\pi \times \pi$-holonomy of the segment $\eta$ between barycenters as in the case of $E$.

### 4.4 Invariant intersections of chains with twisted coefficients

In [Les1], equivariant intersection form was used for intersections of chains. On the other hand, on the twisted de Rham complex of a compact manifold, the pairing

$$(\alpha, \beta) = \int_M \text{Tr}(\alpha \wedge \beta)$$

is often used and it gives the Poincaré duality $H^*(M; A) \cong H^{m-*}(M; A)$. In the following, we use an analogue of these pairings on singular chains of $\text{Conf}_2(X)$ with coefficients in $A^\otimes 2$.

Suppose that two chains $C_1, C_2$ of $S^* \text{Conf}_2(X) \otimes \mathbb{C}[\pi_2] \otimes C A^\otimes 2 = S^* \text{Conf}_2(X) \otimes \mathbb{C}[\pi_2] \otimes C A^\otimes 2$ are transversal and satisfy the condition $\dim C_1 + \dim C_2 = \dim \text{Conf}_2(X) = 2n + 2$. If $C_i = \sum_{k,\ell} C_{i\ell} \otimes x_k \otimes x_\ell$, we define $(C_1, C_2) \in A^\otimes 4$ by the formula

$$(C_1, C_2) = \sum_{k,\ell} (C_{1\ell}^k C_{2\ell}^k) \text{Tr}(x_{k1} x_{\ell1} x_{k2} x_{\ell2}).$$

1. When $A$ is finite dimensional over $\mathbb{C}$, we define $\text{Tr} : (A^\otimes 2)^\otimes 2 \to \mathbb{C}$ by

$$\text{Tr}((x_1 \otimes y_1) \otimes (x_2 \otimes y_2)) = B(x_1, x_2^*) B(y_1, y_2^*),$$

where $x^*$ is the dual of $x$ with respect to $B$. This is $\rho_A(\pi)$-invariant in the sense of the following identities.

$$\text{Tr}((x_1 \otimes \rho_A(g)y_1) \otimes (x_2 \otimes \rho_A(g)y_2)) = \text{Tr}((x_1 \otimes y_1) \otimes (x_2 \otimes y_2))$$

$$\text{Tr}((\rho_A(g)x_1 \otimes y_1) \otimes (\rho_A(g)x_2 \otimes y_2)) = \text{Tr}((x_1 \otimes y_1) \otimes (x_2 \otimes y_2))$$

2. When $A = \Lambda = \mathbb{C}[\epsilon^{\pm 1}]$, we define $\text{Tr} : \Lambda^\otimes 2 \to \Lambda$ by

$$\text{Tr}(t^a \otimes t^b) = t^{a-b}.$$ 

This is $\mathbb{Z}$-invariant.

3. When $A = g \otimes \mathbb{C} \Lambda$, we define $\text{Tr} : (g^\otimes 2[\epsilon^{\pm 1}])^\otimes 2 \to \Lambda$ by

$$\text{Tr}((x_1 \otimes y_1) t^a \otimes (x_2 \otimes y_2) t^b) = B(x_1, x_2^*) B(y_1, y_2^*) t^{a-b}.$$ 

This is $\text{Ad}(G) \times \mathbb{Z}$-invariant and can be considered as obtained from a sesquilinear form on $g^\otimes 2$ by $\mathbb{C}[\epsilon^{\pm 1}]$-sesquilinear extension.
Theorem 4.7. \( \text{Tr}(\cdot , \cdot) \) is a coboundary. Namely, when \( \dim C_1 + \dim C_2 = 2n+3 \), we have
\[
\text{Tr}(\partial_A C_1, C_2) + (-1)^{2n+2-\dim C_1} \text{Tr}(C_1, \partial_A C_2) = 0.
\] (4.1)
Hence \( \text{Tr}(\cdot , \cdot) \) induces a well-defined sesquilinear pairings
\[
H_p(\text{Conf}_2(X), \partial\text{Conf}_2(X); A^{\otimes 2}) \otimes \mathbb{C} H_q(\text{Conf}_2(X); A^{\otimes 2}) \to \mathbb{C},
H_p(\text{Conf}_2(X), \partial\text{Conf}_2(X); \Lambda) \otimes \Lambda H_q(\text{Conf}_2(X); \Lambda) \to \Lambda,
H_p(\text{Conf}_2(X), \partial\text{Conf}_2(X); g^{\otimes 2}[t^\pm]) \otimes \Lambda H_q(\text{Conf}_2(X); g^{\otimes 2}[t^\pm]) \to \Lambda.
\]
We call \( \text{Tr}(\cdot , \cdot) \) an invariant intersection.

Proof. If we extend the definition of \( \langle \cdot , \cdot \rangle \) for \( \dim C_1 + \dim C_2 = 2n+3 \) similarly, then \( \langle C_1, C_2 \rangle \) gives a chain of \( S_1(\text{Conf}_2(X)) \otimes \mathbb{C} (A^{\otimes 2})^{\otimes 2} \). To prove the identity (4.1), we assume that \( C_i \) is an \( A^{\otimes 2} \)-linear combination of smooth simplices in \( S_n(\text{Conf}_2(X)) \):
\[
C_i = \sum_{\lambda} \sigma_\lambda \otimes m_\lambda,
\] (4.2)
where \( \sigma_\lambda : \Delta^p \to \text{Conf}_2(X) \) is a smooth simplex and \( m_\lambda \in A^{\otimes 2} \). Let \( \tau \) be a face of \( \sigma_{\lambda_0} \) for some \( \lambda_0 \) that is independent of \( \partial_A C_i \). Then some simplices \( \sigma_\lambda \) \( (\lambda \in \Lambda) \) in the sum in (4.2) are glued together at \( \tau \) (Figure 3-left). If we let \( \eta_\lambda \) be the segment between the barycenters of \( \sigma_\lambda \) and \( \tau \), and if the projections of \( \eta_\lambda \) in \( X \times X \) do not intersect walls of holonomy, then the reason that the face \( \tau \) does not contribute to \( \partial_A C_i \) is some linear relation \( \sum_{\lambda} m_\lambda = 0 \) satisfied in \( A^{\otimes 2} \). If the projections of the segments \( \eta_\lambda \) for \( \lambda \in \Lambda_1 \) in \( X \times X \) cross a wall of holonomy, then the term of \( \tau \) in the twisted boundary of \( C_i \) is one of the following.
\[
\sum_{\lambda \in \Lambda_1} m_\lambda + (\rho(g) \otimes 1) \sum_{\lambda \in \Lambda_1} m_\lambda \quad \text{or}
\sum_{\lambda \in \Lambda_1} m_\lambda + (1 \otimes \rho(g)) \sum_{\lambda \in \Lambda_1} m_\lambda
\]
Here, \( g \in \pi \) is determined by the wall \( S \) a path hits in \( X \times X \) (see Figure 3-right). Since \( \tau \) does not contribute to \( \partial_A C_i \), the lifts of the simplices \( \sigma_\lambda \) in \( \tilde{E} \times_B \tilde{E} \) with \( A^{\otimes 2} \)-coefficients are glued together at the lift of \( \tau \). Hence we must have the relation
\[
\sum_{\lambda \in \Lambda} m_\lambda = \sum_{\lambda \in \Lambda_1} m_\lambda + \sum_{\lambda \in \Lambda_1} m_\lambda = 0.
\]
In the case of a double intersection, the gap in the coefficients due to the action of a wall is \( (\rho(g) \otimes 1)^{\otimes 2} \) or \((1 \otimes \rho(g))^{\otimes 2} \), which will be eliminated after applying \( \text{Tr} \). Hence after applying \( \text{Tr} \), the boundary of a double intersection only contribute to \( \partial_A C_1 \) or \( \partial_A C_2 \). This completes the proof. \( \square \)
4.5 Propagator in a family for \( n \) odd

**Lemma 4.8.** We assume that \( n \) is odd, \( n \geq 3 \). Let \( X \) be a parallelizable \( \mathbb{Z} \)-homology \( S^n \times S^1 \) equipped with a local coefficient system \( A \), let \( \pi : E \to B \) be as in §4.1, and we assume that the base of \( B \) is an \((n-2)\)-dimensional closed oriented manifold. Under Assumptions 3.1 and 3.5, we have

\[
H_2^n(E_{\text{Conf}_2}(\pi); A^\otimes 2) = H_0(B) \otimes_c H_2(\text{Conf}_2(X); A^\otimes 2)
\]

and the natural map

\[
H_2^n(E_{\text{Conf}_2}(\pi); A^\otimes 2) \to H_2^n(E_{\text{Conf}_2}(\pi), \partial E_{\text{Conf}_2}(\pi); A^\otimes 2)
\]

is zero.

**Proof.** By Lemma 3.6, the \( E^2 \)-terms of the Leray–Serre spectral sequences for the \( \text{Conf}_2(X) \)-bundle and the \( (\text{Conf}_2(X), \partial \text{Conf}_2(X)) \)-bundle over \( B \) that converge to \( H_2^n \) are isomorphic to \( E_{2n}^2 \) and \( E_{n-2,n+2}^2 \), respectively, which agree with \( H_2^n \). Also, the map induced between these terms is obviously 0. \( \square \)

**Definition 4.9** (Propagator in family). Let \( n, X, A, \pi \) be as in Lemma 4.8. Let \( s_\tau : E \to ST^n(E) \) be the section that is given by the first vector of the framing \( \tau \) and let \( f : E \to X \) be a fiberwise pointed degree 1 map, which is transversal to \( K = \{\ast\} \times S^1 \). Let \( \tilde{K} = f^{-1}(K) \), which is a framed codimension \( n \) submanifold of \( E \). A propagator in family is a \( 2n \)-chain \( P \) of \( E_{\text{Conf}_2}(\pi) \) with coefficients in \( A^\otimes 2 \) that is transversal to the boundary and that satisfies

\[
\partial_A P = s_\tau(E) \otimes c_A - \partial(X, A) ST(\tilde{K}) \otimes c_A. \tag{4.3}
\]

**Lemma 4.10.** Let \( n, X, A, \pi \) be as in Lemma 4.8. Then there exists a propagator in family. Moreover, two propagators \( P, P' \) in family with \( \partial_A P = \partial_A P' \) (as chains) are related by

\[
P' - P = \lambda ST(\Sigma^n) \otimes c_A + \partial_A Q
\]

for some \( \lambda \in \mathbb{R} \) and a \((2n+1)\)-chain \( Q \) of \( E_{\text{Conf}_2}(\pi) \).
Proof. By the homology exact sequence for pair and by Lemma 4.8, we have the following exact sequence.

\[ 0 \to H_{2n}(E\text{Conf}_2(\pi), \partial E\text{Conf}_2(\pi); A^\otimes 2) \]
\[ \xrightarrow{\tilde{i}} H_{2n-1}(\partial E\text{Conf}_2(\pi); A^\otimes 2) \xrightarrow{\tilde{i}} H_{2n-1}(E\text{Conf}_2(\pi); A^\otimes 2) \]

Here, it follows from Lemma 3.6 and \( \dim B = n - 2 \) that both

\[ H_{2n-1}(\partial E\text{Conf}_2(\pi); A^\otimes 2) \quad \text{and} \quad H_{2n-1}(E\text{Conf}_2(\pi); A^\otimes 2) \]

are \( E_{n-2,n+1}^2 = E_{n-2,n+1}^2 \) in the Leray–Serre spectral sequence of bundles over \( B \). The map induced between the \( E^2 \)-terms is induced by the natural map

\[ i : H_{n+1}(E\text{Conf}_2(X); A^\otimes 2) \to H_{n+1}(E\text{Conf}_2(X); A^\otimes 2), \]

by the naturality of the Leray–Serre spectral sequence (e.g., [Hat2]). Hence \( \tilde{i} \) can be described explicitly by using the result for a single fiber (Corollary 3.8), and the existence of a chain \( P \) with (4.3) follows. Note that \( E_{n-2,n+2}^\infty \) of (4.4) are spanned by the cycles \( s_i(E) \otimes c_A, ST(K) \otimes c_A \), and by \( ST(K) \otimes c_A \), respectively. Indeed, since \( \partial E\text{Conf}_2(\pi) \cong S^n \times E \) and \( E\text{Conf}_2(\pi) \) is homologically equivalent to \( (D^{n+1}, \partial D^{n+1})[-1] \times E \), it suffices to see that \( [E] \) and \( [\tilde{K}] \) span \( H_{2n-1}(E; R) \) and \( H_{n-1}(E; R) \), respectively. The former is obvious and for the latter, we recall that \( H_{n-1}(\pi^{-1}(B_{n-2}), \pi^{-1}(B_{n-3}); R) \) is \( E_{n-2,1}^2 \) in the Leray–Serre spectral sequence for \( \pi \), where \( B_i \) is the \( i \)-skeleton of a handle or cell decomposition of \( B \). On the other hand, one may see that \( E_{n-2,1}^2 = E_{n-2,1}^\infty = H_{n-1}(E; R) = H_{n-1}(\pi^{-1}(B_{n-2}); R) \), which agrees with the image of the natural map \( H_{n-1}(\pi^{-1}(B_{n-2}); R) \to E_{n-2,1}^1 \). This shows that \( \tilde{K} \) represents an element of \( E_{n-2,1}^2 \). By the duality between \( E_{n-2,1}^2 \) and \( E_{0,n}^0 \) given by the intersection, we see that the class of \( \tilde{K} \) is the dual of \( [\Sigma \times \ast] \), which is the same as \( [B] \otimes [K] \), the generator of \( E_{n-2,1}^2 \).

For two propagators \( P, P' \) in family with common boundary \( \partial_A P = \partial_A P' \), the chain \( P' - P \) is a cycle of \( E\text{Conf}_2(\pi) \). By Lemmas 4.8 and 3.6, \( P' - P \) is homologous to \( \lambda ST(\Sigma^n) \otimes c_A \) for some \( \lambda \in R \). This completes the proof. \( \blacksquare \)

### 4.6 Propagator in a family for \( n \) even, \( X = \Sigma^n \times S^1 \)

**Lemma 4.11.** We assume that \( n \) is even, \( n \geq 4 \). Let \( \Sigma^{n-1} \) be a parallelizable \( \mathbb{Z} \) homology \( S^{n-1} \), let \( X = \Sigma^n \times S^1 \), where \( \Sigma^n = \Sigma^{n-1} \times S^1 \), equipped with a local coefficient system \( A \), and let \( \pi : E \to B \) be as in §4.1. We assume that the base of \( B \) is an \((n-2)\)-dimensional closed oriented manifold. Under Assumptions 3.1
and 3.5, we have

\[ H_{2n}(E_{Conf}^2(\pi); A^{\otimes 2}) \cong H_0(B) \otimes_C H_{2n}(\overline{Conf}_2(X); A^{\otimes 2}) \]
\[ \oplus H_1(B) \otimes_C H_{2n-1}(\overline{Conf}_2(X); A^{\otimes 2}) \]
\[ \oplus H_{n-2}(B) \otimes_C H_{n+2}(\overline{Conf}_2(X); A^{\otimes 2}) \]
\[ H_{2n}(E_{Conf}^2(\pi), \partial E_{Conf}^2(\pi); A^{\otimes 2}) \]
\[ \cong H_{n-2}(B) \otimes_C H_{n+2}(\overline{Conf}_2(X), \partial \overline{Conf}_2(X); A^{\otimes 2}). \]

The natural map
\[ H_{2n}(E_{Conf}^2(\pi); A^{\otimes 2}) \to H_{2n}(E_{Conf}^2(\pi), \partial E_{Conf}^2(\pi); A^{\otimes 2}) \]
is zero.

**Proof.** By Lemma 3.14, the \( E^2 \)-terms of the Leray–Serre spectral sequences for the \( \overline{Conf}_2(X) \)-bundle and the \( (\overline{Conf}_2(X), \partial \overline{Conf}_2(X)) \)-bundle over \( B \) that converge to \( H_{2n} \) are isomorphic to \( E^2_{0,2n} \oplus E^2_{1,2n-1} \oplus E^2_{n-2,n+2} \) and \( E^2_{n-2,n+2} \), respectively. The induced map on the \( E^2 \)-terms can be written explicitly with respect to the explicit bases of Lemma 3.14, which is zero. Hence the map induced between \( H_{2n} \) is zero. \( \Box \)

**Definition 4.12** (Propagator in family). Let \( n, X, A, \pi \) be as in Lemma 4.11. Let \( s_\tau : E \to ST^v(E) \) be the section that is given by the first vector of the framing \( \tau \). A propagator in family is a \( 2n \)-chain \( P \) of \( E_{Conf}^2(\pi) \) with coefficients in \( A^{\otimes 2} \) that is transversal to the boundary and that satisfies

\[ \partial_A P = s_\tau(E) \otimes c_A. \]  

**Lemma 4.13.** Let \( n, X, A, \pi \) be as in Lemma 4.11. Then there exists a propagator in family. Moreover, two propagators \( P, P' \) in family with \( \partial_A P = \partial_A P' \) (as chains) are related by

\[ P' - P = ST(\sigma) \otimes c_A + \partial_A Q \]

for some \( n \)-chain \( \sigma \) of \( E \) and a \((2n + 1)\)-chain \( Q \) of \( E_{Conf}^2(\pi) \).

**Proof.** By the homology exact sequence for pair and by Lemma 4.11, we have the following exact sequence.

\[ 0 \to H_{2n}(E_{Conf}^2(\pi), \partial E_{Conf}^2(\pi); A^{\otimes 2}) \]
\[ \to H_{2n-1}(\partial E_{Conf}^2(\pi); A^{\otimes 2}) \to H_{2n-1}(E_{Conf}^2(\pi); A^{\otimes 2}) \]

Here, it follows from Lemma 3.14 and \text{dim} \( B = n - 2 \) that both the \( E^2 \)-terms of

\[ H_{2n-1}(\partial E_{Conf}^2(\pi); A^{\otimes 2}) \quad \text{and} \quad H_{2n-1}(E_{Conf}^2(\pi); A^{\otimes 2}) \]

are zero. Hence the map induced between \( H_{2n-1} \) is zero. \( \Box \)
in the Leray–Serre spectral sequence of bundles over $B$ are $E_{n-2, n+1}^2 \oplus E_{n-3, n+2}^2 \oplus E_{0, 2n-1}^2$. The map induced between these terms is induced by the natural maps

$$i_r : H_r(\overline{\text{Conf}}_2(X); A^\otimes 2) \rightarrow H_r(\overline{\text{Conf}}_2(X); A^\otimes 2)$$

for $r = n + 1, n + 2, 2n - 1$. For $r = n + 2, 2n - 1$, $i_r$ is induced by the identity between rank 1 $R$-modules, and we have seen that $i_{n+1} = 0$ before Definition 3.16. Hence $i$ can be described explicitly by using the result for a single fiber (Corollary 3.8), and the existence of a chain $P$ with (4.5) follows.

For two propagators $P, P'$ in family with common boundary $\partial_A P = \partial_A P'$, the chain $P' - P$ is a cycle of $E\overline{\text{Conf}}_2(\pi)$. By Lemmas 4.11 and 3.14, $P' - P$ is homologous to $ST(\sigma) \otimes c_A$ for some $\sigma$. This completes the proof. \hfill $\square$

### 4.7 Direct product of cycles

There may not be a canonical way to see a $C$-chain of $X$ as an $A$-chain, unless $A$ is a $C$-algebra with 1. Namely, there may not be a canonical $C$-linear map $S_*(X; C) \rightarrow S_*(X; A)$. Instead, we will use the following interpretation later (in §6).

**Lemma 4.14.** For a subspace $V$ of $X$, $R = C$ or $C[t^{\pm 1}]$, we define $C$-linear maps

$$\eta : S_*(V; C) \rightarrow \text{Hom}_R(A, S_*(X; A)),$$
$$\eta_A : S_*(V; A) \rightarrow \text{Hom}_R(A, S_*(X; A))$$

by $\eta(\sigma) = (a \mapsto \sigma \otimes a)$ and $\eta_A(\gamma) = (a \mapsto (1 \otimes a^*)(a)(\gamma))$, respectively. Then $\eta_A$ is a chain map. Also, if the restriction of $A$ on $V$ is trivial, then $\eta$ is a chain map, too. In that case, by K"unneth formula, the maps between homologies $\eta_\ast : H_\ast(V; C) \rightarrow \text{Hom}_R(A, H_\ast(X; A))$, $\eta_A \ast : H_\ast(V; A) \rightarrow \text{Hom}_R(A, H_\ast(X; A))$ are induced.

**Proof.** For $\gamma = \sum_\mu \gamma_\mu e_\mu \in S_*(\overline{X}) \otimes_{C[\pi]} A$, we have

$$\eta_A(\partial A \gamma) = (a \mapsto \sum_\mu (\partial \gamma_\mu) \otimes a^*(e_\mu) a)$$
$$= (a \mapsto (\partial \otimes 1) \sum_\mu \gamma_\mu \otimes a^*(e_\mu) a) = \partial_A(\eta_A \gamma).$$

This completes the proof. \hfill $\square$

Note that the map $\eta$ may depend on the identification $S_*(X; A) = S_*(X) \otimes_C A$. When the restrictions of $A$ on $V, W \subset X$ are both trivial, $\eta$ induces

$$\eta_\ast^\otimes 2 : H_\ast(V; C) \otimes_{C} H_\ast(W; C) \rightarrow \text{Hom}_R(A^\otimes 2, H_\ast(X^\otimes 2; A^\otimes 2)).$$
Lemma 4.15.

1. When the restrictions of $\partial$ when $\gamma \rightarrow \gamma \times e_{\lambda}$ and $\delta \rightarrow \delta \times e_{\mu}$. Then we have

$$\mathrm{Tr}(\alpha \times e_{\lambda}, \gamma \times \delta) = \sum_{\lambda, \mu} (\alpha \times e_{\lambda}) (e_{\lambda} \otimes e_{\mu}) = \sum_{\lambda} (\alpha \times e_{\lambda}) (e_{\lambda} \otimes e_{\lambda}) = \mathrm{Tr}(\alpha \times e_{\lambda}, \gamma \times e_{\lambda} \delta).$$

The assertion 2 follows immediately from 1.

2. Suppose that a submanifold $U$ of $X^2$ is such that $H_*(U, \mathbb{A}^2)$ is freely spanned over $\mathbb{A}^2$ by cycles of the forms $\alpha_i \times e_{\lambda} \beta_i$ ($i = 1, \ldots, r$, $\alpha_i, \beta_i$ are $\mathbb{A}$-cycles of $X$). Suppose moreover that $\mathrm{Tr}(-, -)$ gives a duality between $H_*(U, \mathbb{A}^2)$ and $H_*(V, \partial U, \mathbb{A}^2)$, and that the dual $\mathbb{A}^2$-basis of $H_*(U, \partial U, \mathbb{A}^2)$ to $\{\alpha_i \times e_{\lambda} \beta_i\}$ is $\gamma_i \times \delta_i$ ($i = 1, \ldots, r$, $\gamma_i, \delta_i$ are $A$-cycles of $X$). Then the dual $\mathbb{A}^2$-basis can be represented by the cycles $\gamma_i \times e_{\lambda} \delta_i$ ($i = 1, \ldots, r$). In other words, $\gamma_i \times \delta_i$ is homologous to $\gamma_i \times e_{\lambda} \delta_i$.

Proof. Let $\gamma = \sum_\lambda \gamma_\lambda e_{\lambda}, \delta = \sum_\mu \delta_\mu e_{\mu}$. Then we have

$$\mathrm{Tr}(\alpha \times e_{\lambda}, \gamma \times \delta) = \sum_\lambda (\alpha \times e_{\lambda}) (e_{\lambda} \otimes e_{\lambda}) = \sum_\lambda (\alpha \times e_{\lambda}) (e_{\lambda} \otimes e_{\lambda}) = \mathrm{Tr}(\alpha \times e_{\lambda}, \gamma \times e_{\lambda} \delta).$$

The following identity for chains holds since $\partial A \alpha \times e_{\lambda}$ and $\partial A \beta$ are given by $\mathbb{A}$-cycles, for example.
Example 4.16. We shall consider the case $R = A = \mathbb{C}[t^{\pm 1}]$. When the restriction of $A$ on $V, W \subset X$ are both trivial, let $\alpha, \beta$ be $\mathbb{C}$-chains of $V, W$, respectively, and let $F$ be a chain of $X^{x^2}$ with coefficients in $A \otimes_R A = \mathbb{C}[t^{\pm 1}]$. In this case, the product $\alpha \times \beta$ can be considered as the mapping $1 \mapsto \alpha \times c_A \beta$ in

$$\text{Hom}_R(R, S_*(X^{x^2}; R)).$$

Then the intersection $\text{Tr} \langle F, \alpha \times \beta \rangle$ as an element of $\text{Hom}_R(R, R) = R$ is

$$\text{Tr}(F, \alpha \times c_A \beta).$$

If $F = \sum_m F_m t^m$, where $F_m$ is a $\mathbb{C}$-chain of $X$, we have

$$\text{Tr} \langle F, \alpha \times c_A \beta \rangle = \sum_m \langle F_m, \alpha \times \beta \rangle_{X \times X} t^m = \sum_m \langle F, t^m (\alpha \times \beta) \rangle_{X \times X} t^m$$

This agrees with the definition in [Les1] of the equivariant intersection.

4.8 Linking number

Let $\alpha, \beta$ be two disjoint nullhomotopic $\mathbb{Z}$-cycles of $X$ such that $\dim \alpha + \dim \beta = n$. Let $D$ be a $\mathbb{Z}$-chain of $X$ such that $\partial D = \beta$. As seen in the previous subsection, $\alpha, \beta, D$ can be considered as elements of $\text{Hom}_R(A, S_*(X; A))$. For each element $e_\lambda \otimes e_\mu$ of $A^{\otimes 2}$, $\text{Tr}(\alpha(e_\lambda), D(e_\mu))$ gives an element of $R$, and hence

$$\text{Tr}(\alpha, D) \in \text{Hom}_R(A^{\otimes 2}, R).$$

Moreover, under the identification $\text{Hom}_R(A^{\otimes 2}, R) = A^{\otimes 2}$; $\phi \mapsto (1 \otimes \phi \otimes 1) \circ (c_A \otimes c_A)$, this can be considered as an element of $A^{\otimes 2}$. We define the linking number of $\alpha, \beta$ by this element of $A^{\otimes 2}$:

$$\ell_{k_A}(\alpha, \beta) = \text{Tr}(\alpha, D) \in A^{\otimes 2}.$$  

This is determined only up to multiplication with invertible elements of $A^{\otimes 2}$, and satisfies the following properties, whose proofs are exactly the same as those in [Les1, Proposition 3.1].

Lemma 4.17.  1. $\ell_{k_A}(\alpha, \beta) = (-1)^{\text{codim} \alpha} \ell_{k_A}(\beta, \alpha)^*$

2. $\ell_{k_A}(\alpha, \beta) = \text{Tr}(\alpha \times \beta, P) \in A^{\otimes 2}$ ($P$: propagator)
5 Perturbative invariant

For compact oriented submanifolds $F_1, F_2, F_3$ of codimension $n$ of $EConf_2(\pi)$ with corners, we define their intersection by

$$\langle F_1, F_2, F_3 \rangle_\emptyset = F_1 \cap F_2 \cap F_3.$$  \hspace{1cm}(5.1)

This is defined only when $F_1, F_2, F_3$ are in a general position. This can be extended to generic $\mathbb{C}$-chain of $EConf_2(\pi)$ by $\mathbb{C}$-linearity.

When $\dim B = n - 2$, we take codimension $n$ chains $F_1, F_2, F_3$ from

$$S_{2n}(EConf_2(\pi)) \otimes_{\mathbb{C}[\pi]} A^{\otimes 2} = S_{2n}(EConf_2(\pi)) \otimes_{\mathbb{C}} A^{\otimes 2},$$

and define

$$\langle F_1, F_2, F_3 \rangle_\emptyset = F_1 \cap F_2 \cap F_3 \in S_0(EConf_2(\pi)) \otimes_{\mathbb{C}} (A^{\otimes 2})^{\otimes 3}$$

by extending (5.1) by $A^{\otimes 2}$-linearity.

5.1 Definition of the invariant when $\mathcal{O}(X,A) = 0$, $n$ odd

Let $n, X = \Sigma^n \times S^1$, $A, \pi$ be as in Lemma 4.8. We take propagators $P_1, P_2, P_3$ in family $EConf_2(\pi)$ so that they are parallel on the boundary. Each $P_i$ can be considered as an element of $S_{2n}(EConf_2(\pi); \mathbb{C}) \otimes_{\mathbb{C}} A^{\otimes 2}$. When $n$ is odd, we define $Z^0(\mathcal{O})_\emptyset(P_1, P_2, P_3) \in \mathcal{O}_\emptyset(A^{\otimes 2}; \rho_A(\pi) \times \mathbb{Z})$ by

$$Z^0(\mathcal{O})_\emptyset(P_1, P_2, P_3) = \frac{1}{6} \text{Tr}_\emptyset \langle P_1, P_2, P_3 \rangle_\emptyset,$$

where $\text{Tr}_\emptyset : (A^{\otimes 2})^{\otimes 3} \to \mathcal{O}_\emptyset(A^{\otimes 2}; \rho_A(\pi') \times \mathbb{Z})$ is the projection. Since we apply $\text{Tr}_\emptyset$, $Z^0(\mathcal{O})_\emptyset(P_1, P_2, P_3)$ does not depend on the way to represent $P_i$ an element of $S_{2n}(EConf_2(\pi); \mathbb{C}) \otimes_{\mathbb{C}} A^{\otimes 2}$, for a similar reason as in Lemma 4.7.

**Theorem 5.1.** Let $n, X = \Sigma^n \times S^1$, $A, \pi$ be as in Lemma 4.8. Then $Z^0(\mathcal{O})_\emptyset(P_1, P_2, P_3)$ does not depend on the choice of $P_i$, and gives a homomorphism

$$Z^0(\mathcal{O})_\emptyset : \Omega_{n-2}(\text{BDiff} \, \text{deg}(X^*, \partial)) \to \mathcal{O}_\emptyset(A^{\otimes 2}; \rho_A(\pi') \times \mathbb{Z}).$$

**Proof.** Let $\pi^+ : E^+ \to B^+$, $\pi^- : E^- \to B^-$ be framed $(X, U_{x_0})$-bundles over closed oriented manifolds $B^+$, $B^-$ with fiberwise degree 1 maps $f^\pm : E^\pm \to X$ that are bundle bordant as framed $(X, U_{x_0})$-bundles with fiberwise degree 1 maps. Namely, there is a framed $(X, U_{x_0})$-bundle $\pi : E \to B$ over a connected compact oriented cobordism $B$ and a smooth map $\tilde{f} : \tilde{E} \to X$ that restricts to $\pi^+, \pi^-$ and $f^+, f^-$ on the ends $\partial \tilde{B} = B^+ \bigsqcup (-B^-)$. Let $\tau_+, \tau_-$ be the vertical framings of $\pi^+, \pi^-$, respectively. Let $\tilde{\tau}$ be a vertical framing of $\tilde{E}$ extending $\tau_{\pm}$. The local coefficient systems $A$ on $E^\pm$ extends over $\tilde{E}$ by the pullback by $\tilde{f}$.  

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We take triples of propagators \((P_1^+, P_2^+, P_3^+)\) and \((P_1^-, P_2^-, P_3^-)\) that define \(Z_{0}^{\partial}\) on the end \(\partial B\). We first assume that there is a triple of propagators \((\tilde{P}_1, \tilde{P}_2, \tilde{P}_3)\) in \(E\text{Conf}_2(\tilde{\pi})\) that extends those on \(\partial B\). Namely, \(\tilde{P}_i\) is a chain of \(S_i(E\text{Conf}_2(\tilde{\pi}); \mathbb{C}) \otimes_{\mathbb{C}} A^{\otimes 2}\), satisfying the following identity:

\[
\partial A \tilde{P}_i = P_1^+ - P_1^- + s^{(i)}_\pi (\tilde{E}) \otimes c_A,
\]

where \(s^{(i)}_\pi\) is a slight perturbation of \(s_\pi\) obtained by applying an \(SO_{n+1}\)-rotation.

The intersection \(\langle \tilde{P}_1, \tilde{P}_2, \tilde{P}_3 \rangle_\Theta\) gives a chain of \(S_1(E\text{Conf}_2(\tilde{\pi})) \otimes_{\mathbb{C}} (A^{\otimes 2})^{\otimes 3}\). Then the boundary of \(\text{Tr}_\Theta(\tilde{P}_1, \tilde{P}_2, \tilde{P}_3)_\Theta\) corresponds to the intersection of the chains in

\[
\partial E\text{Conf}_2(\tilde{\pi}) = E\text{Conf}_2(\pi^+) - E\text{Conf}_2(\pi^-) + \partial^{ib} E\text{Conf}_2(\tilde{\pi}).
\]

Since \(\tilde{E}\) has a vertical framing and the propagators \(\tilde{P}_1, \tilde{P}_2, \tilde{P}_3\) are parallel on \(\partial^{ib} E\text{Conf}_2(\tilde{\pi})\), they do not have intersection there. Hence the boundary only survives in \(E\text{Conf}_2(\pi^+) - E\text{Conf}_2(\pi^-)\) and we have

\[
\text{Tr}_\Theta(P_1^+, P_2^+, P_3^+)_\Theta - \text{Tr}_\Theta(P_1^-, P_2^-, P_3^-)_\Theta = 0.
\]

This completes the proof under the assumption of the existence of the triple \((\tilde{P}_1, \tilde{P}_2, \tilde{P}_3)\).

In Lemma 5.2 below, we shall see that if we replace \(P_i^-\) with \(P_i^- + \lambda ST(\Sigma^n)' \otimes c_A\) for some \(\lambda \in \mathbb{R}\), where \(ST(\Sigma^n)'\) is a parallel copy of \(ST(\Sigma^n)\) obtained by pushing slightly toward the inward normal vector field on the boundary of the base fiber \(\text{Conf}_2(X)\), then a propagator \(P_i\) as above exists. Then we need to show that the value of \(\text{Tr}_\Theta(P_1^-, P_2^-, P_3^-)_\Theta\) does not change by additions of \(\lambda ST(\Sigma^n)' \otimes c_A\). Namely, we consider the triple intersection

\[
\text{Tr}_\Theta(P_1^+ + \lambda_1 ST(\Sigma^n)' \otimes c_A, P_2^- + \lambda_2 ST(\Sigma^n)'' \otimes c_A, P_3^- + \lambda_3 ST(\Sigma^n)''' \otimes c_A)_\Theta,
\]

where \(ST(\Sigma^n)''\) and \(ST(\Sigma^n)'''\) are parallel copies of \(ST(\Sigma^n)'\) in the interior of \(\text{Conf}_2(X)\). Since \(ST(\Sigma^n)'\), \(ST(\Sigma^n)''\), \(ST(\Sigma^n)'''\) are disjoint, we need only to show that the terms of triple intersections among two propagators \(P_i^-, P_j^-\) and one copy of \(ST(\Sigma^n)\) vanish. By the boundary transversality assumption for propagator, the values of these are the same as the intersections with \(ST(\Sigma^n)\) on the boundary. By the explicit form of the boundary of \(P_i^-\), we see that the intersection among \(P_i^-, P_j^-\) and \(ST(\Sigma^n)\) is empty. This completes the proof that we may change \(P_i^-\) as above without changing the value of \(\text{Tr}_\Theta(P_1^-, P_2^-, P_3^-)_\Theta\). \(\square\)

**Lemma 5.2.** Let \(P_1^+, s^{(i)}_\pi (\tilde{E})\) be as in the proof of Theorem 5.1. Then for some \(\lambda \in \mathbb{R}\), there exists a \((2n + 1)\)-chain \(\tilde{P}_i\) of \(E\text{Conf}_2(\tilde{\pi})\) such that

\[
\partial A \tilde{P}_i = P_1^+ - (P_i^- + \lambda ST(\Sigma^n)' \otimes c_A) + s^{(i)}_\pi (\tilde{E}) \otimes c_A.
\]
Proof. Since \( \tilde{B} \) does not have a closed component, its \( \mathbb{Z} \) homology is like that of an \((n - 2)\)-dimensional complex and the computation of \( H_*(\text{EConf}_2(\pi); A^{\otimes 2}) \) is similar to that of Lemma 4.8. The 2n-cycle \( P^+_i - P^-_i + s_i(\tilde{E}) \otimes c_A \) represents a class in
\[
H_{2n}(\text{EConf}_2(\pi); A^{\otimes 2}) = R[ST(\Sigma^n) \otimes c_A],
\]
where we consider \( \Sigma^n \) is in the base fiber of \( E^- \). Thus for some \( \lambda \in R \), the 2n-cycle \( P^+_i - P^-_i + s_i(\tilde{E}) \otimes c_A \) is homologous to \( \lambda ST(\Sigma^n) \otimes c_A \). This completes the proof. \( \square \)

5.2 Definition of the invariant when \( \mathcal{O}(X, A) = 0 \), \( n \) even
Let \( n, X = \Sigma^n \times S^1, A, \pi \) be as in Lemma 4.11. We take propagators \( P_1, P_2, P_3 \) in family \( \text{EConf}_2(\pi) \) so that they are parallel on the boundary. When \( n \) is even, we define \( Z^1_{\Theta}(P_1, P_2, P_3) \in \mathcal{A}^1_{\Theta}(A^{\otimes 2}; \rho_A(\pi) \times \mathbb{Z}) \) by
\[
Z^1_{\Theta}(P_1, P_2, P_3) = \frac{1}{6} \text{Tr}_{\Theta}(P_1, P_2, P_3) \Theta,
\]
where \( \text{Tr}_{\Theta} : (A^{\otimes 2})^{\otimes 3} \to \mathcal{A}^1_{\Theta}(A^{\otimes 2}; \rho_A(\pi') \times \mathbb{Z}) \) is the projection.

Theorem 5.3. Let \( n, X = \Sigma^n \times S^1, A, \pi \) be as in Lemma 4.11. Then \( Z^1_{\Theta}(P_1, P_2, P_3) \) does not depend on the choice of \( P_i \), and gives a homomorphism
\[
Z^1_{\Theta} : \Omega_{n-2}(\text{BDiff}_{\text{deg}}(X^\bullet, \partial)) \to \mathcal{A}^1_{\Theta}(A^{\otimes 2}; \rho_A(\pi') \times \mathbb{Z}).
\]

Proof is almost parallel to that of Theorem 5.1. In this case, we use a bordism analogue of Lemma 4.13 instead of that of Lemma 4.10.

6 Surgery formula

6.1 \( R \)-decorated graphs
Let \( R \) be an algebra over \( \mathbb{C} \) having 1, and let \( H \) be a subset of the set of units of \( R \). We assume that \( R \) has a \( \mathbb{C} \)-linear involution \( R \to R \), which maps \( p \) to its “adjoint” \( p^* \), such that \( p^* = p^{-1} \) for \( p \in H \). We call a pair \((\Gamma, \phi)\) of the following objects an \( R \)-decorated graph.

1. \( \Gamma \): abstract, labeled, edge-oriented trivalent graph, where a label is the pair of bijections \( \alpha : \{1, 2, \ldots, m\} \to \text{Edges}(\Gamma) \) and \( \beta : \{1, 2, \ldots, n\} \to \text{Vertices}(\Gamma) \).

2. \( \phi : \text{Edges}(\Gamma) \to R \): a map.

We also write an \( R \)-decorated graph \((\Gamma, \phi)\) as \( \Gamma(x_1, x_2, \ldots, x_{3k}) (x_i = \phi(\alpha(i))) \).
Figure 4: Decomposition of embedded trivalent graph into Y-shaped pieces.

6.2 Graph surgery

We take an embedding $\Gamma \to X$ of a labeled, edge-oriented graph $\Gamma$. The homotopy class of this embedding can be represented as $\Gamma(g_1, g_2, \ldots, g_k)$ ($g_i \in \pi$) for the $\mathbb{C}[\pi]$-decoration given by the $\pi$-valued holonomy for each edge. We consider that this expression also represents an embedded graph.

We put an $(n+1)$-dimensional Hopf link at the middle of each edge, as in Figure 4. For some choice of dimensions of spheres on edges, the embedded graph is decomposed into $2^k$ Y-shaped components (Type I and II).

We take small closed tubular neighborhoods of these objects and denote them by $V^{(1)}, V^{(2)}, \ldots, V^{(2k)}$. They form a disjoint union of handlebodies embedded in $X$. A Type I handlebody is diffeomorphic to the handlebody obtained from an $(n+1)$-ball by attaching two 1-handles and one $(n-1)$-handle in a standard way. A Type II handlebody is diffeomorphic to the handlebody obtained from an $(n+1)$-ball by attaching one 1-handle and two $(n-1)$-handles in a standard way.

Let $\alpha_I : S^0 \to \text{Diff}(\partial V)$, $S^0 = \{-1, 1\}$, be the map defined by $\alpha_I(-1) = -1$, and by setting $\alpha_I(1)$ as the “Borromean twist” corresponding to the Borromean string link $D^{n-1} \cup D^{n-1} \cup D^1 \to D^{n+1}$. The detailed definition of $\alpha_I$ can be found in [Wa1, §4.5].

Let $\alpha_{II} : S^{n-2} \to \text{Diff}(\partial V)$ be the map defined by comparing the relative
isomorphism class of the family of complements of an $S^{n-2}$-family of embeddings $D^{n-1} \cup D^1 \cup D^1 \to D^{n+1}$ obtained by parametrizing the second component in the Borromean string link $D^{n-1} \cup D^{n-1} \cup D^1 \to D^{n+1}$ with the trivial family. The detailed definition of $\alpha_{11}$ can be found in [Wa1, §4.6].

Let $B = K_1 \times \cdots \times K_{2k}$ ($K_i = S^0$ or $S^{n-2}$). Let $K_i$ be $S^0$ or $S^{n-2}$ depending on whether the $i$-th vertex is of Type I or II. Accordingly, let $\alpha_i : K_i \to \text{Diff}(\partial V^{(i)})$ be $\alpha_1$ or $\alpha_{11}$. By using the families of twists above, we define

$$E^\Gamma = ((X - \text{Int} (V^{(1)} \cup \cdots \cup V^{(2k)})) \times B) \cup_\partial ((V^{(1)} \cup \cdots \cup V^{(2k)})) \times B),$$

where the gluing map is given by

$$\psi : (\partial V^{(1)} \cup \cdots \cup \partial V^{(2k)}) \times B \to (\partial V^{(1)} \cup \cdots \cup \partial V^{(2k)}) \times B$$

$$\psi(x, t_1, \ldots, t_{2k}) = (\alpha_i(t_i)(x), t_1, \ldots, t_{2k}) \quad \text{(for } x \in \partial V^{(i)}).$$

**Proposition 6.1** (Proposition 9.6). The natural projection $\pi^\Gamma : E^\Gamma \to B$ is an $X$-bundle, and it admits a vertical framing that is compatible with the surgery, and it gives an element of

$$\Omega_{(n-2)k}(B \text{Diff}_{\deg}(X^\bullet, \partial)).$$

We denote this element by $\Psi_k(\Gamma(g_1, g_2, \ldots, g_{3k})).$

**Theorem 6.2.** Let $n \geq 3$ and let $\Theta(g_1, g_2, g_3)$ be a $\mathbb{C}[\pi]$-decorated $\Theta$-graph. Let $\varepsilon = n + 1 \mod 2$. The element $[\Psi_1(\Theta(g_1, g_2, g_3))] \in \Omega_{n-2}(B \text{Diff}_{\deg}(X^\bullet, \partial))$ belongs to the image from $\pi_{n-2} \text{Diff}_{\deg}(X^\bullet, \partial)$, and the following identity holds.

$$Z_\Theta^\varepsilon(\Psi_1(\Theta(g_1, g_2, g_3))) = 2[\Theta(\rho_A(g_1), \rho_A(g_2), \rho_A(g_3))].$$

We will give a proof of Theorem 6.2 in the next subsection.

### 6.3 Proof of Theorem 6.2

The outline of the proof is similar to [KT]. Namely, proof of Theorem 6.2 boils down to one lemma (Proposition 6.3), which guarantees the existence of conveniently normalized propagator. Proof of Proposition 6.3 is almost the same as that of [Les1, Proposition 11.1].

Let $W = X - \text{Int} (V^{(1)} \cup V^{(2)})$, $\widetilde{W} = W \times B_\Theta$. Then $E^\Theta - \text{Int} \widetilde{W}$ is the disjoint union of the parts corresponding to $V^{(1)}, V^{(2)} \subset X$:

$$E^\Theta - \text{Int} \widetilde{W} = E^\Theta(V^{(1)}) \bigsqcup E^\Theta(V^{(2)}),$$

where $E^\Theta(V^{(1)}) \cong \widetilde{V}^{(1)} \times K_2$ and $E^\Theta(V^{(2)}) \cong \widetilde{V}^{(2)} \times K_1$.

By the relation

$$\int_{(s_i, s_j) \in K_i \times K_j} (V^{(i)}(s_i) \times \{s_j\}) \times (\{s_i\} \times V^{(j)}(s_j))$$

$$= \int_{s_i \in K_i} (V^{(i)}(s_i) \times \{s_i\}) \times \int_{s_j \in K_j} (\{s_j\} \times V^{(j)}(s_j))$$

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among fiber products, where \( \int = \bigcup \), we have the following identification:

\[
(\widetilde{V}^{(i)} \times K_j) \times_{K_i \times K_j} (K_i \times \widetilde{V}^{(j)}) = \widetilde{V}^{(i)} \times \widetilde{V}^{(j)}.
\]

Similarly, \( \widetilde{V}^{(i)} \times K_i \widetilde{W} = \widetilde{V}^{(i)} \times W, \widetilde{W} \times_{B_\Theta} \widetilde{W} = (W \times W) \times B_\Theta, \) and

\[
\int_{(s_1, s_2) \in K_1 \times K_2} E^{\Theta}(V^{(1)})(s_1, s_2) \times E^{\Theta}(V^{(2)})(s_1, s_2) = \widetilde{V}^{(1)} \times \widetilde{V}^{(2)}.
\]

1. Let \( V^{(i)} \) be a handlebody of Type I or II, and let \( z_1^{(i)}, z_2^{(i)}, z_3^{(i)} \) be the cycles in \( V^{(i)} \) that are given by the cores of the three handles of positive indices. If \( V^{(i)} \) is of Type I, two of them are circles and one is \((n-1)\)-dimensional sphere. If \( V^{(i)} \) is of Type II, one of them is a circle and two of them are \((n-1)\)-dimensional spheres. Let \( a_1^{(i)}, a_2^{(i)}, a_3^{(i)} \) be the dual spheres to \( z_1^{(i)}, z_2^{(i)}, z_3^{(i)} \) with respect to the intersection in \( \partial V^{(i)} \). They are the boundaries of the cocores of the three handles in \( V^{(i)} \) of positive indices. We take a base point \( p_{V^{(i)}} \) of \( \partial V^{(i)} \) that is disjoint from the cycles \( z_j^{(i)}, a_j^{(i)} \).

2. Let \( \Sigma(a_k^{(i)}) \) be a submanifold of \( V^{(i)} \) that is bounded by \( a_k^{(i)} \). Let \( \Sigma(z_k^{(i)}) \) be a submanifold of \( X - \text{Int} V^{(i)} \) that is bounded by \( z_k^{(i)} \). Let \( \gamma_{V^{(i)}} \) be a 1-chain of \( W \) with coefficients in \( A \) that is bounded by \( p_{V^{(i)}} \), which exists by \( H_0(X; A) = 0 \).

3. \( \Sigma(z_k^{(i)}) \) may intersect a handle of \( V^{(j)} \) \((j \neq i)\) transversally. We assume that the intersection agrees with \( \Sigma(a_m^{(j)}) \) for some \( m \). This is possible by the way of linking among the handlebodies in graph surgery.

4. The boundary of \( \widetilde{V}^{(i)} \) may be canonically identified with \( \partial V^{(i)} \times K_i \). Let \( \widetilde{z}_k^{(i)} = z_k^{(i)} \times K_i \) and \( \widetilde{a}_k^{(i)} = a_k^{(i)} \times K_i \). The cycle \( \widetilde{a}_k^{(i)} \) bounds a submanifold \( \Sigma(\widetilde{a}_k^{(i)}) \) of \( \widetilde{V}^{(i)} \). This corresponds to a Seifert surface of one component in the Borromean rings, and can be chosen so that its normal bundle is trivial.

5. We identify a small tubular neighborhood of \( \partial V^{(i)} \) in \( X \) with \( \partial V^{(i)} \times [-4, 4] \) so that \( \partial V^{(i)} \times \{0\} = \partial V^{(i)} \). For a chain \( x \) of \( \partial V^{(i)} \), let \( x[h] = x \times \{h\} \).
Also, let

\[
V_h^{(i)} = \begin{cases} 
V^{(i)} \cup (\partial V^{(i)} \times [0,h]) & (h \geq 0) \\
V^{(i)} - (\partial V^{(i)} \times (h,0]) & (h < 0)
\end{cases}
\]

\[
\Sigma_h(z^{(i)}_\ell) = \begin{cases} 
\Sigma(z^{(i)}_\ell) \cap (X \setminus V_h^{(i)}) & (h \geq 0) \\
\Sigma(z^{(i)}_\ell) \cup (z^{(i)}_\ell \times [h,0]) & (h < 0)
\end{cases}
\]

\[
\Sigma_h(a^{(i)}_\ell) = \begin{cases} 
\Sigma(a^{(i)}_\ell) \cup (a^{(i)}_\ell \times [0,h]) & (h > 0) \\
\Sigma(a^{(i)}_\ell) \cap V_h^{(i)} & (h \leq 0)
\end{cases}
\]

\[
W_h = \begin{cases} 
X - \text{Int}(V_h^{(1)} \cup V_h^{(2)}) & (h \geq 0) \\
X \cup ((\partial V^{(1)} \cup \partial V^{(2)}) \times [h,0]) & (h < 0)
\end{cases}
\]

\[\widetilde{V}_h^{(i)}, \widetilde{W}_h, \Sigma_h(\tilde{a}^{(i)}_\ell)\text{ etc. can be defined in a similar way.}\]

**Proposition 6.3.** There exists a propagator \(P\) in family \(E\text{Conf}_2(\pi^\Theta)\) that satisfies the following conditions.

1. On \(\widetilde{W} \times_{B_\Theta} \widetilde{W} = (W \times W) \times B_\Theta\), \(P\) agrees with \(P_0 \times B_\Theta\), where \(P_0\) is a propagator in \(\text{Conf}_2(X)\).

2. On \(E^\Theta(V^{(j)}) \times_{B_\Theta} \widetilde{V}_3 = (\widetilde{V}^{(j)} \times V_3) \times K_i\) \((i \neq j)\), \(P\) agrees with the direct product of the following chain of \(\widetilde{V}^{(j)} \times V_3\) and \(K_i\).

3. On \(\widetilde{W}_3 \times_{B_\Theta} E^\Theta(V^{(j)}) = (W_3 \times \widetilde{V}^{(j)}) \times K_i\) \((i \neq j)\), \(P\) agrees with the direct product of the following chain of \(W_3 \times \widetilde{V}^{(j)}\) and \(K_i\).

4. On \(E^\Theta(V^{(j)}_3) \times_{B_\Theta} E^\Theta(V^{(j)}_3) = (\widetilde{V}^{(j)}_3 \times K_j) \widetilde{V}^{(j)}_3 \times K_j\) \((i \neq j)\), \(P\) agrees with the direct product of some chain of \(\widetilde{V}^{(j)}_3 \times K_j\) \(\widetilde{V}^{(j)}_3\) and \(K_j\).

5. On \(E^\Theta(V^{(i)}) \times_{B_\Theta} E^\Theta(V^{(j)}) = \widetilde{V}^{(i)} \times \widetilde{V}^{(j)}\), \(P\) agrees with the following chain of \(\widetilde{V}^{(i)} \times \widetilde{V}^{(j)}\).

\[
\sum_{\ell, m} \ell k_A(z^{(i)}_\ell, z^{(j)}_m) + \sum_{\ell, m} \ell k_A(z^{(i)}_\ell, a^{(j)}_m) \Sigma(\tilde{a}^{(j)}_m)\]

Proof of Proposition 6.3 is given in §6.4–6.6.

**Proof of Theorem 6.2.** We choose propagators \(P_1, P_2, P_3\) as in Proposition 6.3. On the parts 1–4 in Proposition 6.3, the triple intersection does not contribute to
$\text{Tr}_\Theta(P_1, P_2, P_3)_{\Theta}$ since the triple intersection on each part is the direct product of that in strictly lower dimensional subspace and positive dimensional space, or the same value is counted on each point of $S^0$ with opposite orientation and is cancelled. Thus it follows that $\text{Tr}_\Theta(P_1, P_2, P_3)_{\Theta}$ agrees with that on the following subspace of $E\text{Conf}_2(\pi^0)$.

$$\langle \tilde{V}(1) \times \tilde{V}(2) \rangle \prod \langle \tilde{V}(2) \times \tilde{V}(1) \rangle$$

The restriction to one component $\tilde{V}^{(n_1)} \times \tilde{V}^{(n_2)}$ corresponds to counting configurations of type $\Theta$ such that each edge $e$ of $\Theta$ is mapped to $\tilde{V}^{(n_1)} \times \tilde{V}^{(n_2)}$.

Let us compute the value of $\text{Tr}_\Theta(P_1, P_2, P_3)_{\Theta}$ on the component $\tilde{V}^{(1)} \times \tilde{V}^{(2)}$. According to Proposition 6.3, the restriction of the propagator for the edge $e$ on $\tilde{V}^{(1)} \times \tilde{V}^{(2)}$ is of the form

$$L_1^* \Sigma(a_1^{(1)}) \times_{\varepsilon_1} \Sigma(a_2^{(2)}) + L_2^* \Sigma(a_2^{(1)}) \times_{\varepsilon_2} \Sigma(a_3^{(2)}) + L_3^* \Sigma(a_3^{(1)}) \times_{\varepsilon_3} \Sigma(a_3^{(2)}).$$

$(L_i \in A^{\otimes 2})$ We abbreviate this as $L_1^* S_1 + L_2^* S_2 + L_3^* S_3$. The value of the intersection $\text{Tr}_\Theta(P_1, P_2, P_3)_{\Theta}$ on $\tilde{V}^{(1)} \times \tilde{V}^{(2)}$ is

$$\sum_{\sigma \in \Theta_3} \langle S_{\sigma(1)}, S_{\sigma(2)}, S_{\sigma(3)} \rangle_{\tilde{V}^{(1)} \times \tilde{V}^{(2)}} \text{Tr}_\Theta(L_1^* \otimes L_2^* \otimes L_3^*).$$

When $n$ is odd, the terms of the sum are all equal by the $\Theta_3$-antisymmetry of the triple intersection form and the $\Theta_3$-antisymmetry of $\text{Tr}_\Theta$. When $n$ is even, the terms of the sum are all equal by the $\Theta_3$-symmetry of the triple intersection form and the $\Theta_3$-symmetry of $\text{Tr}_\Theta$. In both cases, the value of the sum is equal to

$$6 \langle S_1, S_2, S_3 \rangle_{\tilde{V}^{(1)} \times \tilde{V}^{(2)}} \text{Tr}_\Theta(L_1 \otimes L_2 \otimes L_3).$$

We may consider similarly for the value of $\text{Tr}_\Theta(P_1, P_2, P_3)_{\Theta}$ on the component $\tilde{V}^{(2)} \times \tilde{V}^{(1)}$. For some sign $\varepsilon = \pm 1$, it can be written as

$$6 \langle S_1^*, S_2^*, S_3^* \rangle_{\tilde{V}^{(2)} \times \tilde{V}^{(1)}} \text{Tr}_\Theta(L_1 \otimes L_2 \otimes L_3)
= 6 \varepsilon \langle S_1, S_2, S_3 \rangle_{\tilde{V}^{(1)} \times \tilde{V}^{(2)}} \text{Tr}_\Theta(L_1 \otimes L_2 \otimes L_3).$$

Here the orientation of $\Sigma(a_1^{(2)}) \times \Sigma(a_1^{(1)}) \subset \tilde{V}^{(2)} \times \tilde{V}^{(1)}$ differs from that of $\Sigma(a_1^{(1)}) \times \Sigma(a_2^{(2)}) \subset \tilde{V}^{(1)} \times \tilde{V}^{(2)}$ by $(-1)^{n-1}$. Also, the orientation of $\tilde{V}^{(2)} \times \tilde{V}^{(1)}$ and of $\tilde{V}^{(1)} \times \tilde{V}^{(2)}$ induced from that of $E\text{Conf}_2(\pi)$ differs by $(-1)^{(n+1)(n+1)+(n-2)}$. Hence we have

$$\varepsilon = \{(-1)^{n-1}\}^3 \times (-1)^{(n+1)(n+1)+(n-2)} = 1.$$ 

Now we obtain the following.

$$Z_{\Theta}(\Psi_1(\Gamma(g_1, g_2, g_3))) = \frac{1}{6} \cdot 12 \cdot [\Theta(\rho_A(g_1), \rho_A(g_2), \rho_A(g_3)]
= 2 [\Theta(\rho_A(g_1), \rho_A(g_2), \rho_A(g_3)].$$

This completes the proof. \qed

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6.4 Normalization of propagator with respect to one handlebody

We put \( V = V^{(j)} \) and abbreviate \( a_i^{(j)}, z_i^{(j)} \) etc. as \( a_i, z_i \) etc. for simplicity.

**Proposition 6.4.** There exists a propagator \( P \) that satisfies the following.

1. The intersection of \( P \) with \( V \times (X \setminus \hat{V}_3) \) is of the form.
   \[
   V \times_{c_A} \gamma_{i \in V}^3 + \sum_{i, \ell} \ell k_A(z_i, a_i^{+})^* \Sigma(a_i) \times_{c_A} \Sigma_3(z_\ell)
   \]

2. The intersection of \( P \) with \( (X \setminus \hat{V}_3) \times V \) is of the form.
   \[
   (-1)^{n+1} \gamma_{i \in V}^3 \times_{c_A} V + \sum_{i, \ell} \ell k_A(z_i, a_i^{+}) \Sigma_3(z_\ell) \times_{c_A} \Sigma(a_i)
   \]

3. \( \text{Tr}(P, \Sigma_3(a_i) \times_{c_A} p_V) = 0, \text{Tr}(P, p_V \times_{c_A} \Sigma_3(a_i)) = 0. \)

Let \( H = V \times (X \setminus \hat{V}_3). \) Each term in the formula of Proposition 6.4-1 represents an element of \( H_{n+2}(H, \partial H; A^{\otimes 2}). \) We start with any propagator \( P_0 \) and check that its restriction on \( H \) gives the same class in \( H_{n+2}(H, \partial H; A^{\otimes 2}) \) as the formula of Proposition 6.4-1. Then it follows that by adding the boundary of some \((n+3)\)-chain of \( H \) to \( P_0 \) we obtain a propagator with Proposition 6.4-1.

To do so, we compare the values of the invariant intersections with a basis of the dual \( H_n(H; A^{\otimes 2}). \) The condition 2 is similar.

**Lemma 6.5.** \( H_i(X \setminus V; A) = H_{i+1}(V, \partial V; \mathbb{C}) \otimes_A A \) for \( i \geq 0. \) Namely, for \( i \geq 0, \)

\[
H_i(X \setminus V; A) = \left[ \mathbb{C}[a_1] \oplus \mathbb{C}[a_2] \oplus \mathbb{C}[a_3] \oplus \mathbb{C}[\partial V] \right] _i \otimes_A A
\]

**Proof.** In the homology long exact sequence for the pair \( (X, X \setminus V), \) we have \( H_* (X; A) = 0. \) Also, by excision, we have \( H_{i+1}(X, X \setminus V; A) = H_{i+1}(V, \partial V; A) = H_{i+1}(V, \partial V; \mathbb{C}) \otimes_A A. \) The result follows.

The following is obtained by the Künneth formula.

**Lemma 6.6.**

\[
H_i(H; A^{\otimes 2}) = \left[ H_* (V) \otimes_A (\mathbb{C}[a_1] \oplus \mathbb{C}[a_2] \oplus \mathbb{C}[a_3] \oplus \mathbb{C}[\partial V]) \right] _i \otimes_A A^{\otimes 2}
\]

\[
H_i(\partial V \times (X \setminus \hat{V}_3); A^{\otimes 2}) = \left[ H_* (\partial V) \otimes_A (\mathbb{C}[a_1] \oplus \mathbb{C}[a_2] \oplus \mathbb{C}[a_3] \oplus \mathbb{C}[\partial V]) \right] _i \otimes_A A^{\otimes 2}
\]

**Lemma 6.7.** \( H_{n+2}(H, \partial H; A^{\otimes 2}) \) is generated by the following elements over \( A^{\otimes 2}. \)

\[
[\Sigma(a_i) \times_{c_A} \Sigma_3(z_\ell)], \quad [V \times_{c_A} \gamma_{i \in V}^3]
\]

**Hence** \( \text{Tr} \langle \cdot, \cdot \rangle \) gives a nondegenerate pairing

\[
H_n(H; A^{\otimes 2}) \otimes_R H_{n+2}(H, \partial H; A^{\otimes 2}) \rightarrow R.
\]

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Proof. We only prove the lemma for Type I since the case of Type II is similar. We use the homology long exact sequence for the pair $(H, \partial H)$. We know that $H_{n+2}(H; A^\otimes 2) = \mathbb{C}[z_3 \times \partial V_3] \otimes A^\otimes 2$, and that its image in $H_{n+2}(H, \partial H; A^\otimes 2)$ is 0. Hence we have

$$H_{n+2}(H, \partial H; A^\otimes 2) \cong \ker \left[ H_{n+1}(\partial H; A^\otimes 2) \to H_{n+1}(H; A^\otimes 2) \right].$$

We apply the Mayer–Vietoris exact sequence for $\partial H = (V \times \partial V_3) \cup (\partial V \times (X \setminus \hat{V}_3))$:

$$\to H_{n+1}(\partial V \times \partial V_3) \xrightarrow{i_{n+1}} H_{n+1}(V \times \partial V_3) \oplus H_{n+1}(\partial V \times (X \setminus \hat{V}_3)) \to H_{n+1}(\partial H) \xrightarrow{\partial V} H_n(\partial V \times \partial V_3) \oplus H_n(\partial V \times (X \setminus \hat{V}_3))$$

(coefficients are in $A^\otimes 2$) to determine $H_{n+1}(\partial H; A^\otimes 2)$. $	ext{Coker} i_{n+1}$ is isomorphic to

$$H_{n+1}(H; A^\otimes 2) = \left[ \mathbb{C}\{[s], [z_1], [z_2], [z_3]\} \otimes \mathbb{C}\{[a_1], [a_2], [a_3], [\partial A]\}\right]_{n+1} \otimes \mathbb{C} A^\otimes 2.$$  

Ker $i_n$ is isomorphic to

$$\left[ \mathbb{C}\{[a_1], [a_2], [a_3], [\partial A]\} \otimes \mathbb{C}\{[s], [z_1], [z_2], [z_3]\} \right]_n \otimes \mathbb{C} A^\otimes 2.$$  

The right hand side is generated over $A^\otimes 2$ by the images of $A_\ell = \partial_A(\Sigma(a_i) \times_{c_A} \Sigma_3(z_\ell))$, $A_{00} = \partial_A(V \times_{c_A} \gamma_V[3])$ under the Mayer–Vietoris boundary map $\partial_M$. Thus we have

$$H_{n+1}(\partial H; A^\otimes 2) = H_{n+1}(H; A^\otimes 2) \oplus \left( \mathbb{C}[A_{00}] \oplus \bigoplus_{i,\ell} \mathbb{C}[A_\ell] \right) \otimes \mathbb{C} A^\otimes 2.$$  

Then the result follows. 

Proof of Proposition 6.4. This proof is similar to [Les1, Proposition 11.2, 11.6, 11.7]. The invariant intersection between the basis $z_i \times_{c_A} a_\ell[3]$, $p_V \times_{c_A} \partial V_3$ of $H_n(H; A^\otimes 2)$ and $P$ is

$$\text{Tr}(z_i \times_{c_A} a_\ell[3], P) = \ell k_A(z_i, a_\ell^+), \quad \text{Tr}(p_V \times_{c_A} \partial V_3, P) = 1.$$  

From the identities

$$\text{Tr}(z_i \times_{c_A} a_\ell[3], \Sigma(a_i) \times_{c_A} \Sigma_3(z_\ell)) = 1, \quad \text{Tr}(p_V \times_{c_A} \partial V_3, V \times_{c_A} \gamma_V[3]) = 1$$

and the sesquilinearity of the invariant intersection, it follows that the chain of Proposition 6.4-1 and the restriction of $P$ gives the same element of $H_{n+2}(H, \partial H; A^\otimes 2)$.

The condition 3 can be satisfied by adding multiples of $\partial_A(\Sigma_2(z_i) \times_{c_A} V_1)$, $\partial_A(V_1 \times_{c_A} \Sigma_2(z_i))$ to $P$, without affecting the conditions 1,2. 

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6.5 Normalization of propagator with respect to a finite set of handlebodies

Let \( V^{(1)}, \ldots, V^{(2k)} \) be the disjoint handlebodies in \( X \) that defines \( \Psi_k(\Gamma(g_1, \ldots, g_{3k})) \). We normalize propagator with respect to this set of handlebodies. Although we need only the case \( k = 1 \) for the purpose of this paper, we give a result for \( k \geq 1 \).

**Proposition 6.8.** There exists a propagator \( P \) that satisfies the following conditions.

1. For each \( j = 1, 2, \ldots, k \), the intersection of \( P \) with \( V^{(j)} \times (X \setminus \hat{V}^{(j)}_3) \) is of the form
   \[
   V^{(j)} \times_{cA} \gamma_{V^{(j)}} [3] + \sum_{i,\ell} \ell k_A(z_i^{(j)}, a_{\ell}^{(j)})^* \Gamma(a_i) \times_{cA} \Sigma(z_{\ell}^{(j)}).
   \]

2. For each \( j = 1, 2, \ldots, k \), the intersection of \( P \) with \( (X \setminus \hat{V}^{(j)}_3) \times V^{(j)} \) is of the form
   \[
   (-1)^n \gamma_{V^{(j)}} [3] \times_{cA} V^{(j)} + \sum_{i,\ell} \ell k_A(z_i^{(j)}, a_{\ell}^{(j)})^* \Sigma(z_{\ell}^{(j)}) \times_{cA} \Gamma(a_i).\]

3. \( \text{Tr}(P, \Sigma_{3}(a_i^{(j)}) \times_{cA} p_{V^{(j)}}) = 0 \), \( \text{Tr}(P, p_{V^{(j)}} \times_{cA} \Sigma_{3}(a_i^{(j)})) = 0 \) (\( j = 1, 2, \ldots, k \)).

We prove Proposition 6.8 by induction on \( k \). Suppose that \( P_{k-1} \) satisfies the conditions of Proposition 6.8 for \( j < k \). We write \( V^{(1, \ldots, k-1)} = V^{(1)} \cup \ldots \cup V^{(k-1)} \). Then \( P_{k-1} \) satisfies the conditions on \( V^{(k)} \times V^{(1, \ldots, k-1)} \). We put

\[
D = V^{(k)} \times (X \setminus (\hat{V}^{(2)}_3 \cup V^{(1, \ldots, k-1)})),
\]

\[
H_1 = V^{(k)} \times (X \setminus \hat{V}^{(k)}).
\]

Let \( PD \) be the chain of \( D \) defined by The condition 1. The boundary of \( D \) is

\[
(V^{(k)}_1 \times \partial V^{(1, \ldots, k-1)}_1) \cup (D \cap \partial H_1).
\]

By assumption on \( P_{k-1} \) and the explicit form of \( PD \), they coincide on the part \( V^{(k)}_1 \times \partial V^{(1, \ldots, k-1)}_1 \). Hence if

\[
[P_{k-1}] = [PD] \in H_{n+2}(D, D \cap \partial H_1; A^\otimes 2)
\]

is proved, then we will see that \( P_{k-1} \) can be modified so that it agrees with \( PD \) on \( D \) by adding the boundary of an \((n+2)\)-chain of \( D \). To see this, we compare the values of the invariant intersections between \( P_{k-1}, PD \) and a basis of \( H_n(D, V^{(k)}_1 \times \partial V^{(1, \ldots, k-1)}_1; A^\otimes 2) \), which is dual to \( H_{n+2}(D, D \cap \partial H_1; A^\otimes 2) \) with respect to the invariant intersection.
Lemma 6.9. \( H_n(D, V_1^{(k)} \times \partial V_1^{(1,\ldots,k-1)}; A^{\otimes 2}) \) is generated over \( A^{\otimes 2} \) by the following cycles.

\[
z_i^{(k)} \times_{c_A} \gamma_{V_1^{(j)}} , \quad p_{V_1^{(k)}} \times_{c_A} \Sigma(z_i^{(j)}) , \quad p_{V_1^{(k)}} \times_{c_A} \partial V_1^{(k)} , \quad z_i^{(k)} \times_{c_A} a_{\ell}^{(k)[3]}
\]

Proof. By the Künneth formula, we have

\[
H_n(D, V_1^{(k)} \times \partial V_1^{(1,\ldots,k-1)}; A^{\otimes 2}) = \bigg[ H_*(V_1^{(k)}; A) \otimes_R H_*(X \setminus (\tilde{V}_2^{(k)} \cup \tilde{V}_1^{(1,\ldots,k-1)}), \partial V_1^{(1,\ldots,k-1)}; A) \bigg]_n
\]

\((R = \mathbb{C} \text{ or } \mathbb{C}[t^{\pm 1}])\), where \( S_* = H_*(X \setminus (\tilde{V}_2^{(k)} \cup \tilde{V}_1^{(1,\ldots,k-1)}), \partial V_1^{(1,\ldots,k-1)}; A) \)
\((* = 1, n - 1, n)\) can be determined by the homology exact sequence for the pair

\[
\begin{align*}
&\rightarrow H_n(\partial V_1^{(1,\ldots,k-1)}; A) \xrightarrow{i} H_n(X \setminus (\tilde{V}_2^{(k)} \cup \tilde{V}_1^{(1,\ldots,k-1)}); A) \rightarrow S_n \\
&\rightarrow H_{n-1}(\partial V_1^{(1,\ldots,k-1)}; A) \xrightarrow{i} H_{n-1}(X \setminus (\tilde{V}_2^{(k)} \cup \tilde{V}_1^{(1,\ldots,k-1)}); A) \rightarrow S_{n-1} \\
&\rightarrow H_{n-2}(\partial V_1^{(1,\ldots,k-1)}; A) \rightarrow \cdots \\
&\rightarrow H_1(\partial V_1^{(1,\ldots,k-1)}; A) \xrightarrow{i} H_1(X \setminus (\tilde{V}_2^{(k)} \cup \tilde{V}_1^{(1,\ldots,k-1)}); A) \rightarrow S_1 \\
&\rightarrow H_0(\partial V_1^{(1,\ldots,k-1)}; A) \xrightarrow{i} H_0(X \setminus (\tilde{V}_2^{(k)} \cup \tilde{V}_1^{(1,\ldots,k-1)}); A) = 0
\end{align*}
\]

as follows:

\[
S_n \cong (\mathbb{C}[\partial V_2^{(k)}] \otimes \mathbb{C}) \oplus \text{Ker } i_{n-1}, \quad S_{n-1} \cong \text{Coker } i_{n-1}, \\
S_1 \cong H_0(\partial V_1^{(1,\ldots,k-1)}; A) \oplus \text{Coker } i_1.
\]

\(\square\)

Lemma 6.10. 1. \( \text{Tr} \langle P_D, z_i^{(k)} \times_{c_A} \gamma_{V_1^{(j)}} \rangle = \text{Tr} \langle P_{k-1}, z_i^{(k)} \times_{c_A} \gamma_{V_1^{(j)}} \rangle = 0. \)

2. \( \text{Tr} \langle P_D, p_{V_1^{(k)}} \times_{c_A} \Sigma(z_i^{(j)}) \rangle = \text{Tr} \langle P_{k-1}, p_{V_1^{(k)}} \times_{c_A} \Sigma(z_i^{(j)}) \rangle = 0. \)

3. \( \text{Tr} \langle P_D, p_{V_1^{(k)}} \times_{c_A} \partial V_1^{(k)} \rangle = \text{Tr} \langle P_{k-1}, p_{V_1^{(k)}} \times_{c_A} \partial V_1^{(k)} \rangle. \)

4. \( \text{Tr} \langle P_D, z_i^{(k)} \times_{c_A} a_{\ell}^{(k)[3]} \rangle = \text{Tr} \langle P_{k-1}, z_i^{(k)} \times_{c_A} a_{\ell}^{(k)[3]} \rangle. \)

Proof. The third and the fourth assertion follow from the facts that \( p_{V_1^{(k)}} \times_{c_A} \partial V_1^{(k)}, z_i^{(k)} \times_{c_A} a_{\ell}^{(k)[3]} \) are cycles, and that the relative homology classes of \( P_D \) and \( P_{k-1} \) agree.

For the first assertion, the explicit form of \( P_D \) on \( V_1^{(k)} \times (X \setminus \tilde{V}_2^{(k)}) \) shows that \( \text{Tr} \langle P_D, z_i^{(k)} \times_{c_A} \gamma_{V_1^{(j)}} \rangle = 0. \) On the other hand, by

\[
z_i^{(k)} \times_{c_A} \gamma_{V_1^{(j)}} = \Sigma(z_i^{(k)}) \times_{c_A} p_{V_1^{(k)}} + \partial_A(\Sigma(z_i^{(k)}) \times_{c_A} \gamma_{V_1^{(j)})},
\]

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we have
\[ \text{Tr}(P_{k-1}, z_i^{(k)} \times_{c_A} \gamma_{V_1^{(j)}}) = \text{Tr}(P_{k-1}, \Sigma(z_i^{(k)}) \times_{c_A} p_{V_1^{(j)}}). \]

If \( \Sigma(z_i^{(k)}) \cap V_3^{(j)} = \emptyset \), we have \( \text{Tr}(P_{k-1}, \Sigma(z_i^{(k)}) \times_{c_A} p_{V_1^{(j)}}) = 0 \) by the explicit form of \( P_{k-1} \) on \( (X \setminus V_3^{(j)}) \times V^{(j)} \). If \( \Sigma(z_i^{(k)}) \cap V_3^{(j)} \neq \emptyset \), we have \( \text{Tr}(P_{k-1}, \Sigma(a_i^{(j)}) \times_{c_A} p_{V_1^{(j)}}) = 0 \) by assumption on \( P_{k-1} \) again. Hence we have \( \text{Tr}(P_{k-1}, \Sigma(z_i^{(k)}) \times_{c_A} p_{V_1^{(j)}}) = 0. \)

For the second assertion, the explicit form of \( P_D \) on \( V_1^{(k)} \times (X \setminus V_2^{(k)}) \) shows that \( \text{Tr}(P_D, p_{V_1^{(k)}} \times_{c_A} \Sigma(z_i^{(j)})) = 0. \) On the other hand, by
\[ p_{V_1^{(k)}} \times_{c_A} \Sigma(z_i^{(j)}) = -\gamma_{V_1^{(k)}} \times_{c_A} z_i^{(j)} - \partial_A(\gamma_{V_1^{(k)}} \times_{c_A} \Sigma(z_i^{(j)})) \]
we have
\[ \text{Tr}(P_{k-1}, p_{V_1^{(k)}} \times_{c_A} \partial V_2^{(k)}) = -\text{Tr}(P_{k-1}, \gamma_{V_1^{(k)}} \times_{c_A} z_i^{(j)}). \]

According to the explicit form of \( F_{k-1} \) on \( (X \setminus V_2^{(j)}) \times V_1^{(j)} \), the right hand side is 0.

**Proof of Proposition 6.8.** It suffices to show that we can assume moreover that
\[ \text{Tr}(P_k, \Sigma(a_i^{(k)}) \times_{c_A} p_{V_1^{(k)})} = 0, \]
\[ \text{Tr}(P_k, p_{V_1^{(k)}} \times_{c_A} \Sigma(a_i^{(k)})) = 0. \]

Since
\[
\begin{align*}
\text{Tr}(\partial_A(\Sigma_2(z_i^{(k)}) \times_{c_A} V_1^{(k)}), \Sigma_3(a_i^{(k)}) \times_{c_A} p_{V_1^{(k)})}) \\
= \text{Tr}(z_i^{(k)} \times_{c_A} V_1^{(k)}, \Sigma_3(a_i^{(k)}) \times_{c_A} p_{V_1^{(k)})} \pm \text{Tr}(\Sigma_2(z_i^{(k)}) \times_{c_A} \partial V_1^{(k)}, \Sigma_3(a_i^{(k)}) \times_{c_A} p_{V_1^{(k)})}) \\
= \text{Tr}(z_i^{(k)} \times_{c_A} V_1^{(k)}, \Sigma_3(a_i^{(k)}) \times_{c_A} p_{V_1^{(k)})} = \pm 1,
\end{align*}
\]
the first identity is fulfilled after adding to \( P_k \) a linear combination \( C \) of
\[ \partial_A(\Sigma_2(z_i^{(k)}) \times_{c_A} V_1^{(k)}) \subset \partial((X \setminus V_2^{(k)}) \times V_1^{(k)}). \]

Here, this modification should not intersect \( V^{(j)} \times (X \setminus V_3^{(j)}), \ j < k, \) not to break the induction hypothesis. Under our assumption on the handlebodies, this is equivalent to requiring that
\[ \text{Tr}(C, z_r^{(j)} \times_{c_A} \gamma_{V_1^{(k)})} = 0 \]
for all \( r \) such that \( \dim z_r^{(j)} = n - 1 \), and for all \( j < k \). Proof of this condition is exactly the same as [Les1, Lemma 11.10].
6.6 Extension over $\overline{\text{Conf}}_2(V)$ – concluding the proof of Proposition 6.3

In the following, we shall prove the existence of a normalized propagator with respect to the handlebodies $V^{(1)}, V^{(2)}$ that defines $\Psi_1(\Theta(g_1, g_2, g_3))$. The following proof also is a proof for surgery along a trivalent graph with $2k$ vertices that does not have self-loop.

We consider the following three parts $T, N, S$ of $\overline{\text{Conf}}_2(X)$. See Figure 5.

1. Let $T = \overline{\text{Conf}}_2(X \setminus (\hat{V}^{(1)}_1 \cup \hat{V}^{(2)}_2))$. The restriction of the bundle $\pi^\Theta$ on $T$ is trivial. We define a chain $\tilde{P}_T$ on the restriction of the bundle on $\overline{\text{Conf}}_2(X \setminus (\hat{V}^{(1)}_1 \cup \hat{V}^{(2)}_2))$ by taking the direct product of the propagator of Proposition 6.8 on the base fiber and $B^\Theta$.

2. Let $\pi^\Theta(j) : E^\Theta(j) \to K_j$ be the restriction of $\pi^\Theta$ on $K_j \subset B^\Theta$. On the subbundle

$$\tilde{V}^{(j)} \times_{K_j} (E^\Theta(j) \setminus \text{Int} \tilde{V}^{(j)}_3) + (E^\Theta(j) \setminus \text{Int} \tilde{V}^{(j)}_3) \times_{K_j} \tilde{V}^{(j)} \to K_j \quad (6.1)$$

of $E^\Theta(j)$, an explicit chain $P(j)$ is defined by replacing $V^{(j)}, \Sigma(a^{(j)}_i)$ in the description of Proposition 6.8 with $\tilde{V}^{(j)}, \Sigma(\tilde{a}^{(j)}_i)$, respectively. Let $\tilde{N}^{(j)} \to B^\Theta$ be the bundle obtained by pulling back the bundle (6.1) by the projection $B^\Theta \to K_j$. By the pullback of $P(j)$, we obtain a chain $\tilde{P}(j)$ of $\tilde{N}^{(j)}$. Let $N = \tilde{N}^{(1)} \cup \tilde{N}^{(2)}$ and $\tilde{P}_N = \tilde{P}(1) \cup \tilde{P}(2)$.

3. On $S = \partial E\overline{\text{Conf}}_2(\pi^\Theta)$, we put

$$\tilde{P}_S = s_\tau(E^\Theta) \otimes c_A$$

by using the framing $\tau$ that is compatible with surgery.

The three chains $\tilde{P}_T, \tilde{P}_N, \tilde{P}_S$ agree on the overlaps between $T, N, S$. We denote by

$$\tilde{P}_T \cup \tilde{P}_N \cup \tilde{P}_S$$

the chain obtained from $\tilde{P}_T$ by extension by $\tilde{P}_N$ and $\tilde{P}_S$. This fulfills the conditions 1–3, 5 of Proposition 6.3. For the condition 4, it suffices to prove the following lemma.

**Lemma 6.11.** The chain $\tilde{P} = \tilde{P}_T \cup \tilde{P}_N \cup \tilde{P}_S$ of $T \cup N \cup S$ extends to a propagator in family $E\overline{\text{Conf}}_2(\pi^\Theta)$ with coefficients in $A^\otimes 2$.

This concludes the proof of Proposition 6.3. In the following, we shall prove Lemma 6.11. We put $V = V^{(i)}_3$ and prove that the restriction of $\tilde{P}$ on $\partial \overline{\text{Conf}}_2(V)$ extends to the whole of $\overline{\text{Conf}}_2(V)$.

**Lemma 6.12** (Type I, II). *Let $T_d = \mathbb{C}\{[z_j \times z_\ell] \mid \dim z_j + \dim z_\ell = d}\).*
We have by excision, and

\[ H_n(V^2; \mathbb{C}) = T_n, \]

\[ H_{n+1}(V^2; \mathbb{C}) = \begin{cases} T_4 & (n = 3) \\ 0 & (\text{otherwise}) \end{cases} \]

\[ H_{n+2}(V^2; \mathbb{C}) = \begin{cases} T_6 & (n = 4) \\ 0 & (\text{otherwise}) \end{cases} \]

\[ H_{n-1}(\text{Conf}_2(V); \mathbb{C}) = H_{n-1}(V^2; \mathbb{C}), \]

\[ H_n(\text{Conf}_2(V); \mathbb{C}) = H_n(V^2; \mathbb{C}) \oplus \mathbb{C}[ST(*)], \]

\[ H_{n+1}(\text{Conf}_2(V); \mathbb{C}) = H_{n+1}(V^2; \mathbb{C}) \oplus \mathbb{C}[[ST(z_i) \mid \dim z_i = 1]], \]

\[ H_{2n-1}(\text{Conf}_2(V); \mathbb{C}) = \mathbb{C}[[ST(z_i) \mid \dim z_i = n - 1]]. \]

**Proof.** The assertions 1, 2, 3 follows from the Künneth formula. In the homology exact sequence for the pair

\[ \rightarrow H_{p+1}(\hat{V}^2) \rightarrow H_{p+1}(\hat{V}^2, \hat{V}^2 \setminus \Delta_{\hat{V}}) \rightarrow H_p(\text{Conf}_2(\hat{V})) \rightarrow \]

(coefficients are in \( \mathbb{C} \)), we see that the map \( H_{p+1}(\hat{V}^2) \rightarrow H_{p+1}(\hat{V}^2, \hat{V}^2 \setminus \Delta_{\hat{V}}) \) is zero by the explicit basis of \( H_{\ast}(V^2) \). Hence we have the isomorphism

\[ H_p(\text{Conf}_2(\hat{V})) \cong H_{p+1}(\hat{V}^2, \hat{V}^2 \setminus \Delta_{\hat{V}}) \oplus H_p(\hat{V}^2). \]

We have \( H_i(\hat{V}^2, \hat{V}^2 \setminus \Delta_{\hat{V}}) = H_{n+1}(D^{n+1}, \partial D^{n+1}) \otimes H_{i-n-1}(\Delta_{\hat{V}}) \cong H_{i-n-1}(\hat{V}) \) by excision, and

\[ H_{n+1+r}(\hat{V}^2, \hat{V}^2 \setminus \Delta_{\hat{V}}) = \begin{cases} \mathbb{C}[D^{n+1}, \partial D^{n+1}] \otimes H_r(V; \mathbb{C}) & (r \geq 0) \\ 0 & (r < 0) \end{cases} \]

The assertion 4 follows from this.
Let \( a \) be \( a_i \subset \partial V \) that is \((n - 1)\)-dimensional. Let \( \Sigma = \Sigma(a) \). If \( V \) is of Type I, we can assume \( \Sigma \) is given by a framed embedding from \( S^1 \times S^{n-1} - \) (open disk) since \( \Sigma \) is a Seifert surface of one component in the Borromean rings that is disjoint from other components. In the following, we take such \( \Sigma \) and let \( c_1, c_2 \) be the cycles that form a basis of the reduced homology of \( \Sigma \). Let \( c_1^*, c_2^* \) be the basis of \( H_*(\Sigma; \mathbb{Z}) \) dual to \( c_1, c_2 \) with respect to the intersection on \( \Sigma \).

**Lemma 6.13** (Type I, II).

1. The \( n \)-cycle \( c_1 \times c_1^* + c_2 \times c_2^* \) is homologous to \( \sum_{j,\ell} \lambda_{j\ell} z_j \times z_\ell \) in \( V^2 \) for some \( \lambda_{j\ell} \in \mathbb{C} \).

2. The \( n \)-cycle \( c_1 \times c_1^{*+} + c_2 \times c_2^{*+} \) is homologous to \( \sum_{j,\ell} \lambda_{j\ell} z_j \times z_\ell + ST(*) \) in \( \text{Conf}_2(V) \).

**Proof.** The assertion 1 follows from Lemma 6.12. For 2, the component of \( z_j \times z_\ell \) in the homology class of \( c_1 \times c_1^{*+} + c_2 \times c_2^{*+} \) agrees with that of 1. The coefficient of \( ST(*) \) in the homology class is \( \ell k(c_1, c_1^{*+}) + \ell k(c_2, c_2^{*+}) = 1 \). \( \square \)

### 6.6.1 Extension over Type I handlebody

We consider an analogue of Lescop’s chain \( F^2(a) \) of \([\text{Les}1, \text{Lemma} 11.13]\).

- We identify a small tubular neighborhood of \( a \) in \( \partial V \) with \( a \times [-1, 1] \) so that \( a \times \{0\} = a \).
- Let \( \Sigma^+ = (\Sigma \cap V_{-1}) \cup \{(a(v), t, t - 1) \ | \ v \in S^{n-1}, t \in [0, 1]\} \).
- By \( S^{n-1} = S(S^{n-2}) = (S^{n-2} \times [-1, 1])/(S^{n-2} \times \{-1, 1\} \cup \{\infty\} \times [-1, 1]) \), we equip \( a \) with a coordinate. Let \( p(a) \) be the base point of \( a \) that corresponds to \( \infty \in S^{n-1} \). Let \( p(a)^+ = (p(a), 1) \in a \times [-1, 1] \subset \partial V \).
- Let \( A(a) \) be the closure of \( \{(a(v), 0), (a(w), t) \ | \ t \in (0, 1], v \in S^{n-2} \times [-1, 1] \} \) in \( \text{Conf}_2(X) \), that is a manifold with corners.
- Let \( T(a) = \{(a(v', y), 0), (a(w', z), 1) \ | \ v', w' \in S^{n-2}, y, z \in [-1, 1], y \geq z \} \).
- Let \( \text{diag}(n)\Sigma \) be the chain given by the section of \( ST(V)|_\Sigma \) by the normalization of a positive normal vector field on \( \Sigma \).

**Lemma 6.14** (Type I). The \( n \)-cycle

\[
F^n(a) = A(a) + T(a) - p(a) \times \Sigma^+ - \Sigma \times p(a)^+ + \text{diag}(n)\Sigma
- \left\{ \sum_{j,\ell} \lambda_{j\ell} z_j \times z_\ell + ST(*) \right\}
\]

in \( \partial\text{Conf}_2(V) \) is null-homologous in \( \text{Conf}_2(V) \) over \( \mathbb{C} \).
Proof. The first line of the formula of \( F^n(a) \) is obtained from an analogue of \( C^*_\Sigma, \Sigma^+ \) in [Les1, Lemma 8.11] by homotopy. Namely, if we let

\[
J \times_{\ast, \Sigma} J^+ = \{(J(v', y), J(w', z)) \mid v', w' \in S^{n-2}, y, z \in [-1, 1], y \geq z\}
\]

\[
diag(\Sigma \times \Sigma^+) = \{(x, x^+) \mid x \in \Sigma\},
\]

the chain

\[
C_{\ast, \Sigma} = \text{diag}(\Sigma \times \Sigma^+) - \ast \times \Sigma^+ - \Sigma \times \ast^+ + J \times_{\ast, \Sigma} J^+
\]

of \( \Sigma \times \Sigma^+ \) is a cycle, and the following holds in \( H_n(\Sigma \times \Sigma^+; \mathbb{Z}) \).

\[
[C_{\ast, \Sigma}] = [c_1 \times e_{\ast, \Sigma}^+] + [c_2 \times e_{\ast, \Sigma}^+]
\]

Since the first line of the formula of \( F^n(a) \) is homotopic to \( C^*_\Sigma, \Sigma^+ \), the result follows from Lemma 6.13.

Lemma 6.15 (Type I). For the chain \( \tilde{P} \) of Lemma 6.11, the chain

\[
\tilde{P}_{n+1} = \langle \partial \text{Conf}_2(V^{(j)'}) \rangle
\]

bounds a \((n+2)\)-chain \( \tilde{P}(V^{(j)'}) \) in \( \text{Conf}_2(V^{(j)'}) \) with untwisted coefficients in \( A^{{\otimes} 2} \).

Proof. We consider the following commutative diagram.

\[
\begin{array}{ccc}
H_n(\text{Conf}_2(V)) & \xrightarrow{\cong} & H_n(\partial \text{Conf}_2(V)) \\
\downarrow & & \downarrow \\
\tilde{P}_{n+1} \in H_{n+1}(\partial \text{Conf}_2(V)) & \xleftarrow{\cong} & H_{n+1}(\text{Conf}_2(V), \partial \text{Conf}_2(V)) \\
\downarrow & & \downarrow \\
\iota[\tilde{P}_{n+1}] \in H_{n+1}(\text{Conf}_2(V)) & \xleftarrow{\cong} & H_{n+1}(\text{Conf}_2(V), \partial \text{Conf}_2(V)) \\
\downarrow & & \downarrow \\
H_{n+1}(\text{Conf}_2(V), \partial \text{Conf}_2(V)) & \xleftarrow{\cong} & H_{n+1}(\text{Conf}_2(V))
\end{array}
\]

(coefficients are in \( A^{{\otimes} 2} \), untwisted) We would like to prove \( \iota[\tilde{P}_{n+1}] = 0 \). Here, \( H_n(\partial \text{Conf}_2(V); \mathbb{C}) \) includes the following subspace

\[
\mathbb{C}[F^3(a_1)] \oplus \mathbb{C}[F^3(a_2)] \oplus \mathbb{C}[\partial(\Sigma(a_3) \times \Sigma(a_3))] \quad \text{(if } n = 3) \\
\mathbb{C}[F^n(a_1)] \oplus \mathbb{C}[F^n(a_2)] \quad \text{(if } n > 3)
\]

By Lemma 6.14, there exist \((n + 1)\)-chains \( G_{n+1}(a_1), G_{n+1}(a_2) \) of \( \text{Conf}_2(V) \) with coefficients in \( \mathbb{C} \) such that \( \partial G_{n+1}(a_i) = F^n(a_i) \) \((i = 1, 2)\). This together

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with Lemma 6.12 shows that $H_{n+1}(\text{Conf}_2(V), \partial\text{Conf}_2(V); \mathbb{C})$ is of the following form.

$$
\begin{align*}
&\mathbb{C}[G^4(a_1)] \oplus \mathbb{C}[G^4(a_2)] \oplus \mathbb{C}[[a_3] \times \Sigma(a_3)] \quad (\text{if } n = 3) \\
&\mathbb{C}[G^{n+1}(a_1)] \oplus \mathbb{C}[G^{n+1}(a_2)] \\
&\quad (\text{if } n > 3)
\end{align*}
$$

Hence it suffices to show that the intersections of $i\tilde{P}_{n+1}$ with this basis vanish.

Here we use the intersection pairing

$$
H_{n+1}(\text{Conf}_2(V); A^\otimes 2) \otimes_\mathbb{C} H_{n+1}(\text{Conf}_2(V), \partial\text{Conf}_2(V); \mathbb{C})
\rightarrow A^\otimes 2 \otimes \mathbb{C} = A^\otimes 2.
$$

Since we have

$$
\langle i\tilde{P}_{n+1}, G^{n+1}(a_i) \rangle = \langle \tilde{P}_{n+1}, F^n(a_i) \rangle \quad (i = 1, 2),
\langle i\tilde{P}_4, \Sigma(a_3) \times \Sigma(a_3) \rangle = \langle \tilde{P}_4, \partial(\Sigma(a_3) \times \Sigma(a_3)) \rangle
$$

it suffices to check that the right hand side vanishes.

On $\partial\text{Conf}_2(V)$, we have

$$
\langle \tilde{P}_{n+1}, F^n(a_i) \rangle = \langle P, F^n(a_i) \rangle = 0 \quad (i = 1, 2)
$$

by the same argument as [Les1, Proof of Lemma 11.11] for 3-manifolds. For $n = 3$,

$$
\langle \tilde{P}_4, \partial(\Sigma(a_3) \times \Sigma(a_3)) \rangle = 0
$$

follows from the explicit form of the normalization.

\[\square\]

### 6.6.2 Extension over family of Type II handlebodies

- Let $\tilde{a}$ be $\tilde{a}_i \subset \partial\tilde{V}$ that is $1 + (n - 2) = n - 1$-dimensional.
- Let $p(\tilde{a}) = p(a) \times S^{n-2}$, $p(\tilde{a})^+ = p(a)^+ \times S^{n-2}$, $A(\tilde{a}) = A(a) \times S^{n-2}$, $T(\tilde{a}) = T(a) \times S^{n-2}$, and $\tilde{\Sigma} = \Sigma(\tilde{a})$.
- Let $\tilde{\Sigma}^+ = (\Sigma(\tilde{a}) \cap \tilde{V}_{-1}) \cup \{(\tilde{v}, s, t, t-1) \mid (v, s) \in S^1 \times S^{n-2}, t \in [0, 1]\}$.
- Let $\text{diag}(\Sigma)\tilde{\Sigma}$ be the chain given by the section of $ST(\tilde{V})|_{\tilde{\Sigma}}$ by the normalization of a positive normal vector field on $\tilde{\Sigma}$.
- Let $V'$ be a Type I handlebody included in the Type II handlebody, corresponding to the inclusion of an $S^1$ leaf into an $S^{n-1}$ leaf of Y-graphs. Let $z_1, z_2, z_3$ be the cycles in $V'$ that are given by the cores of the three handles of positive indices, as before.

**Lemma 6.16 (Type II).** The n-cycle

$$
F^n(\tilde{a}) = A(\tilde{a}) + T(\tilde{a}) - p(\tilde{a}) \times_{S^{n-2}} \tilde{\Sigma}^+ - \tilde{\Sigma} \times_{S^{n-2}} p(\tilde{a})^+ + \text{diag}(\Sigma)\tilde{\Sigma}
\rightarrow \sum_{j, \ell} \lambda_{j, \ell} z_j \times z_\ell + ST(*)
$$

in $\partial\text{Conf}_2(V)$ is null-homologous in $E\text{Conf}_2(\tilde{V})$ over $\mathbb{C}$.

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defines \( \sim \) nullhomologous. For \( F \) are defined similarly as above, and the following holds. We take a fiberwise nonsingular vector field \( n \) orthogonal to \( U \), where \( \tilde{\Sigma}_1 \) is shrunk into a small neighborhood of a single fiber.

**Proof.** Roughly, the \( S^{n-2} \)-family of embeddings \( D^{n-1} \cup D^1 \cup D^1 \to D^{n+1} \) that defines \( \tilde{V} \) is such that the first and third components are constant families, and the locus of the second component with the first and third components forms a Borromean string link of dimensions \((n - 1, n - 1, 1)\) in \( D^{n+1} \). By deforming the parameter space \( S^{n-2} \) by isotopy, we may assume that most of the locus of the second component is collapsed into the preimage of a small neighborhood \( U_s \) of a single parameter \( s \in S^{n-2} \). Then \( \tilde{\Sigma} \) can be decomposed as the sum of the following submanifolds with corners \( \tilde{\Sigma}_0 \) and \( \tilde{\Sigma}_1 \).

1. \( \tilde{\Sigma}_0 \cap \tilde{\Sigma}_1 \) is an \((n - 1)\)-disk whose boundary is in \( \partial \tilde{V} \). Let \( \delta \) be this \((n - 1)\)-disk.
2. \( \tilde{\Sigma}_1 \) is diffeomorphic to \( S^1 \times S^{n-1} \) (open disk) and included in \( \pi_{V}^{-1}(U_s) \).
3. \( \tilde{\Sigma}_0 \) is diffeomorphic to \( D^2 \times S^{n-2} \). On \( S^{n-2} - U_s \), the bundle structure on \( \tilde{V} \) induces a product structure \( D^2 \times (S^{n-2} - U_s) \).
4. Let \( \tilde{a}_0 = \partial \tilde{\Sigma}_0 \) and \( \tilde{a}_1 = \partial \tilde{\Sigma}_1 \). We have \( \tilde{a}_0 \cong S^1 \times S^{n-2} \) and \( \tilde{a}_1 \cong S^{n-1} \). We have \( \tilde{a}_0 + \tilde{a}_1 = \tilde{a} \) as a chain.

We take a fiberwise nonsingular vector field \( n \in \pi(ST^n \tilde{V}|_{\tilde{\Sigma}}) \) on \( \tilde{\Sigma} \) such that it is orthogonal to \( \tilde{\Sigma} \) on \( \partial \tilde{\Sigma} \) and orthogonal to \( \tilde{\Sigma}_1 \) on \( \delta \). By pushing \( \tilde{a} \) slightly in a direction of \( n \), we obtain parallels \( \tilde{a}_0^+, \tilde{a}_1^+ \) of \( \tilde{a} \). Then the chains \( F^n(\tilde{a}_0), F^n(\tilde{a}_1) \) are defined similarly as above, and the following holds.

\[
[F^n(\tilde{a})] = [F^n(\tilde{a}_0)] + [F^n(\tilde{a}_1)]
\]

\([F^n(\tilde{a}_0)] = 0 \) follows since \( C_{s, \geq}(D^2, (D^2)^+) \) in the proof of Lemma 6.14 is nullhomologous. For \([F^n(\tilde{a}_1)]\), if the radius of \( U_s \) is sufficiently small, then \( \tilde{\Sigma}_1 \)

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is close to $\Sigma(a')$ for an $(n-1)$-cycle $a'$ of the boundary of a Type I handlebody included in a single fiber of $\overline{V}$.

and $F^n(a_1)$ is homologous to $F^n(a')$ in $\overline{EConf}_2(\overline{V})$. By Lemma 6.14 for the single fiber, we have $[F^n(a_1)] = 0$. Hence we have $[F^n(a)] = 0$.

**Lemma 6.17 (Type II).** $H_{2n-1}((\overline{EConf}_2(\overline{V}); \mathbb{C}) = E^2_{0,2n-1} \oplus E^2_{n-2,n+1}$, where

$$E^2_{0,2n-1} = \mathbb{C}[ST(z_2)] \oplus \mathbb{C}[ST(z_3)],$$

$$E^2_{n-2,n+1} = \mathbb{C}[ST(z_1)] \oplus (\mathbb{C}[S^{n-2}] \otimes \mathbb{C} H_{n+1}(V^2; \mathbb{C})).$$

**Proof.** This follows from the Leray–Serre spectral sequence for the fibration $\overline{EConf}_2(V) \to \overline{EConf}_2(\overline{V}) \to S^{n-2}$ and Lemma 6.12. Since the basis of $E^2_{0,2n-1} \oplus E^2_{n-2,n+1}$ can be given by explicit cycles in $\overline{EConf}_2(\overline{V})$ that are linearly independent in homology, the spectral sequence collapses at $E^2$. \qed

**Lemma 6.18 (Type II).** For the chain $\overline{P}$ of Lemma 6.11, the chain

$$\overline{P}_{2n-1} = \langle \partial \overline{EConf}_2(\overline{V}^{(k)}), \overline{P} \rangle$$

bounds a 2n-chain $\overline{P}(\overline{V}^{(k)'})$ in $\overline{EConf}_2(\overline{V}^{(k)'})$ with untwisted coefficients in $A^\otimes 2$.

**Proof.** We consider the following commutative diagram.

$$\begin{array}{ccc}
[\overline{P}_{2n-1}] \in H_{2n-1}(\partial \overline{EConf}_2(\overline{V})) & \xrightarrow{\cong} & H_n(\partial \overline{EConf}_2(\overline{V})) \\
\iota \downarrow & & \uparrow \\
\iota[\overline{P}_{2n-1}] \in H_{2n-1}(EConf_2(\overline{V})) & \xrightarrow{\cong} & H_{n+1}(EConf_2(\overline{V}), \partial EConf_2(\overline{V}))
\end{array}$$

(the coefficients are in $A^\otimes 2$) We would like to show that $\iota[\overline{P}_{2n-1}] = 0$. Here, $H_n(\partial \overline{EConf}_2(\overline{V}); \mathbb{C})$ includes the following subspace.

$$\mathbb{C}[F^3(a_1)] \oplus \mathbb{C}[F^3(a_2)] \oplus \mathbb{C}[F^3(a_3)]$$

$$\oplus \mathbb{C}[(\partial(\Sigma(a_j) \times \Sigma(a_\ell))] \mid \dim a_j + \dim a_\ell = 4 \} \text{ (if } n = 3)$$

$$\mathbb{C}[F^n(a_1)] \oplus \mathbb{C}[F^n(a_2)] \oplus \mathbb{C}[F^n(a_3)] \text{ (if } n > 3)$$

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By Lemma 6.16, there exist \((n+1)\)-chains \(G^{n+1}(\bar{a}_2), G^{n+1}(\bar{a}_3)\) of \(E \overline{\text{Conf}}_2(\tilde{V})\) with coefficients in \(\mathbb{C}\) such that \(\partial G^{n+1}(\bar{a}_i) = F^n(\bar{a}_i)\) \((i = 2, 3)\). This together with Lemma 6.17 shows that

\[
H_{n+1}(E \overline{\text{Conf}}_2(\tilde{V}), \partial E \overline{\text{Conf}}_2(\tilde{V}); \mathbb{C}) = \mathbb{C}[G^{n+1}(a_1)] \oplus \mathbb{C}[G^{n+1}(\bar{a}_2)] \oplus \mathbb{C}[G^{n+1}(\bar{a}_3)]
\]

\[\oplus \left\{ \begin{array}{ll} \mathbb{C}\{[\Sigma(a_j) \times \Sigma(a_\ell)] | \dim a_j + \dim a_\ell = 4\} & (n = 3) \\ 0 & (n > 3) \end{array} \right. \]

Hence it suffices to show that the intersections of \(\iota \tilde{P}_{2n-1}\) with this basis vanish. As before, we need only to compute \(\langle \tilde{P}_{2n-1}, F^n(a_1) \rangle, \langle \tilde{P}_{2n-1}, F^n(\bar{a}_i) \rangle\) \((i = 2, 3)\), and \(\langle \tilde{P}_5, \partial(\Sigma(a_j) \times \Sigma(a_\ell)) \rangle\) (if \(n = 3\)). The computation is the same as before. \(\square\)
7 \( \Sigma^n \times S^1 \)-bundles supported on \( \Sigma^n \times I \)

**Proposition 7.1.** Let \( \Sigma^3 = \Sigma(2, 3, 5) \). The image of the composition of the natural map

\[
i_\ast : H_1(B\text{Diff}_{\text{deg}}(\Sigma^3 \times I, \partial)) \to H_1(B\text{Diff}_{\text{deg}}((\Sigma^3 \times S^1)^\bullet, \partial))
\]

and \( Z^0_\ast : H_1(B\text{Diff}_{\text{deg}}((\Sigma^3 \times S^1)^\bullet, \partial)) \to \mathcal{A}_G^0(g^\otimes [t^{\pm 1}]; \rho(\pi') \times \mathbb{Z}) \) is included in the image from \( \mathcal{A}_G^0(g^\otimes [t^{\pm 1}]; \rho(\pi')) \).

**Proof.** Let \( J = S^1 - \text{Int} \ I \) and \( B = S^1 \). Let \( \pi : E \to B \) be a framed \( \Sigma^3 \times S^1 \)-bundle that has support in \( \Sigma^3 \times I \). Namely, we assume that \( E \) can be obtained by gluing the trivial framed \( \Sigma^3 \times J \)-bundle \( (\Sigma^3 \times J) \times B \) and some framed \( (\Sigma^3 \times I, \partial) \)-bundle \( \pi_I : E_I \to B \) together along the boundaries. Moreover, we assume that a fiberwise degree 1 map \( E \to \Sigma^3 \times S^1 \) is given so that its restriction on \( (\Sigma^3 \times J) \times B \) agrees with the projection \( (\Sigma^3 \times J) \times B \to \Sigma^3 \times J \). The class of such a framed bundle belongs to the image of \( i_\ast \).

The product \( P_{\Sigma^3} \times J \) is a propagator in \( \text{Conf}_2(\Sigma^3 \times J) \) = \( B\ell((\Sigma^3 \times J)^\times, \Delta_{\Sigma^3 \times J}) \), where \( P_{\Sigma^3} \) is the \( (n+1) \)-chain of \( \text{Conf}_2(\Sigma^3) \) considered in the proof of Proposition 3.12. By extending \( P_{\Sigma^3} \times J \) by \( s_\ast(E) \) for the vertical framing \( \tau \) on \( E \), we obtain a cycle in \( \partial E\text{Conf}_2(\pi_I) = \partial \text{Conf}_2(\Sigma^3 \times I) \times B \). As in the proof of the existence of propagator in family (Lemma 4.10), there is no homological obstruction to extending this cycle to a propagator in family \( E\text{Conf}_2(\pi_I) \). The sum of this extension and \( (P_{\Sigma^3} \times J) \times B \) gives a propagator \( \tilde{P}_I \) in family \( E\text{Conf}_2(\pi) \).

We take \( \tilde{P}_I \) as above and its parallel copies \( \tilde{P}_I' \) and \( \tilde{P}_I'' \) each of which has similar product structure as \( \tilde{P}_I \) on \( J \). We consider the triple intersection \( \langle \tilde{P}_I, \tilde{P}_I', \tilde{P}_I'' \rangle_\Theta \) in \( E\text{Conf}_2(\pi) \), which gives \( 6Z_\Theta \). Here, for the transversality of the intersection \( \langle \tilde{P}_I, \tilde{P}_I', \tilde{P}_I'' \rangle_\Theta \), we must modify slightly the product structure of \( \Sigma^3 \times J \) in \( \Sigma^3 \times S^1 \) to define the parallels \( \tilde{P}_I' \) and \( \tilde{P}_I'' \). Namely, if we merely took fiberwise parallel copies of \( \tilde{P}_I \) in the same product structure on \( \Sigma^3 \times J \), then they would have finitely many intersection points in each fiber \( \Sigma^3 \times \{ z \} \). We perturb \( \Sigma^3 \times J \) to define \( \tilde{P}_I' \), as follows. First, we take the triple intersection of three propagators defined as above by the same product structure of \( \Sigma^3 \times J \). Then in each fiber \( \Sigma^3 \times \{ z \}, z \in J \), there may be finitely many configurations of two points corresponding to the triple intersection of the propagators, and the triple intersection in \( \text{Conf}_2(\Sigma^3 \times J) \) is a product of a finite set \( S \subset \text{Conf}_2(\Sigma^3 \times \{ z \}) \) and \( J \). We may assume that such two-point configurations are mutually disjoint in each fiber \( \Sigma^3 \times \{ z \} \). Let \( U^+_\delta \) be the union of the (disjoint) balls of small radius \( \delta \) around the points in the image of the first projection of \( S \) in \( \Sigma^3 \times \{ z \} \), and let \( U^-_\delta \) be defined similarly for the second projection of \( S \) in \( \Sigma^3 \times \{ z \} \). By assumption, we have \( U^+_\delta \cap U^-_\delta = \emptyset \). Then we perturb \( \Sigma^3 \times \{ z \} \) in \( \Sigma^3 \times S^1 \) into
the following embedding $\Sigma^3 \times \{z\} \to \Sigma^3 \times S^1$:

$$(x, z) \mapsto \begin{cases} 
(x, z) & (x \notin U^+_\delta \cup U^-_\delta) \\
(x, z + \lambda(x)) & (x \in U^+_\delta) \\
(x, z - \lambda(x)) & (x \in U^-_\delta)
\end{cases}$$

where $\lambda : \Sigma^3 \to [0, \varepsilon]$ (for $\varepsilon > 0$ small) is a bump function supported on $U^+_\delta \cup U^-_\delta$ that takes the value $\varepsilon$ at the center of each ball. After doing the same perturbation of $\Sigma^3 \times \{z\}$ for all $z \in J$, we obtain a slightly perturbed cylinder $\Sigma^3_3 \times J$ in $\Sigma^3 \times S^1$. We define $\tilde{P}_I$ by using the product structure of this perturbed cylinder $\Sigma^3_3 \times J$. Since the perturbation does not change the projection $\Sigma^3 \times J \to \Sigma^3$, the triple intersection in $\overline{\text{Conf}}_2(\Sigma^3 \times J)$ must be in the preimage of the triple intersection in $\overline{\text{Conf}}_2(\Sigma^3)$. By the choices of $\tilde{P}_I$ and $\tilde{P}_I'$ on $\overline{\text{Conf}}_2(\Sigma^3 \times J)$, we see that the triple intersection in $\overline{\text{Conf}}_2(\Sigma^3 \times J)$ is empty.

Since each configuration in the image of $\tilde{P}_I$ is such that two points are both in $\Sigma^3 \times I$ or both in $\Sigma^3 \times J$, the triple intersection in question is the sum of those on $\Sigma^3 \times I$ and $\Sigma^3 \times J$. Namely, we do not need to consider configuration of two points such that one point is on $\Sigma^3 \times I$ and the other point is on $\Sigma^3 \times J$.

On $\Sigma^3 \times J$, namely, on $\overline{\text{Conf}}_2(\Sigma^3 \times J) \times B$, we have seen above that the triple intersection is empty.

On $\Sigma^3 \times I$, namely, on $E\overline{\text{Conf}}_2(\pi_I)$, we may assume that the $Z$-component of the holonomy in $\pi' \times Z$ is trivial, we may take the triple intersection so that no term has power of $t$ in the coefficient. Hence its class in $\mathcal{A}^0_{\Theta}(g^\otimes [t^\pm 1]; \rho(\pi') \times Z)$ is included in $\mathcal{A}^0_{\Theta}(g^\otimes; \rho(\pi'))$. This completes the proof. \[\square\]

Proof of the following proposition is the same as that of Proposition 7.1.

**Proposition 7.2.** Let $\Sigma^4 = \Sigma(2, 3, 5) \times S^1$. The image of the composition of the natural map

$$\pi_2 \tilde{\text{BDiff}}_{\text{deg}}(\Sigma^4 \times I, \partial) \to \pi_2 \tilde{\text{BDiff}}_{\text{deg}}((\Sigma^4 \times S^1)^*, \partial)$$

and $Z^4_{\Theta} : \pi_2 \tilde{\text{BDiff}}_{\text{deg}}((\Sigma^4 \times S^1)^*, \partial) \to \mathcal{A}^0_{\Theta}(g^\otimes [t^\pm 1]; \rho(\pi') \times Z)$ is included in the image from $\mathcal{A}^0_{\Theta}(g^\otimes; \rho(\pi'))$.

**Proposition 7.3.** Let $p_0 = 3$ and $p_1 = 1$. For $\varepsilon = 0$ or 1, the set $\{\Theta(1, t, t^p) \mid p \geq p_\varepsilon\}$ is linearly independent in the cokernel of the natural embedding

$$\mathcal{A}^0_{\Theta}(g^\otimes; \rho(\pi')) \to \mathcal{A}^0_{\Theta}(g^\otimes [t^\pm 1]; \rho(\pi') \times Z).$$

Proof. The result follows immediately from Propositions 2.4, 2.8 and Example 3.4. \[\square\]

**Theorem 7.4.** Let $\Sigma^3 = \Sigma(2, 3, 5)$ and $\Sigma^4 = \Sigma(2, 3, 5) \times S^1$.

1. The quotient set $\pi_0 \text{Diff}_0(\Sigma^3 \times S^1)/\pi_0 \text{Diff}_0(\Sigma^3 \times I, \partial)$ is at least countable.

Thus the set $\pi_0 \text{Emb}_0(\Sigma^3, \Sigma^3 \times S^1)$ is at least countable. 

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2. The abelianization of the group \( \pi_1 \text{Diff}_0(\Sigma^4 \times S^1) / i_* \pi_1 \text{Diff}_0(\Sigma^4 \times I, \partial) \) has at least countable infinite rank. Thus the abelianization of the group \( \pi_1 \text{Emb}_0(\Sigma^3, \Sigma^4 \times S^1) \) has at least countable infinite rank.

**Proof.** For 1, we have the following commutative diagram.

\[
\begin{array}{ccc}
\pi_1 \text{B} \text{Diff}_{\text{deg}}(\Sigma^3 \times I, \partial) & \xrightarrow{i_*} & \pi_1 \text{B} \text{Diff}_{\text{deg}}((\Sigma^3 \times S^1)^*, \partial) \\
\downarrow \downarrow & & \downarrow \downarrow \\
\pi_1 \text{B} \text{Diff}_0(\Sigma^3 \times I, \partial) & \xrightarrow{i_*} & \pi_1 \text{B} \text{Diff}_0(\Sigma^3 \times S^1)
\end{array}
\]

The vertical maps are both surjective, since they are the maps induced from projections of fibrations with path-connected fiber (§4.1).

It follows from Corollary 4.4 and Proposition 4.5 that the set \( \Phi \) of the infinitely many nontrivial elements of \( \pi_1 \text{B} \text{Diff}_{\text{deg}}((\Sigma^3 \times S^1)^*, \partial) \) detected in Theorem 1.1 is mapped to a subset of infinitely many nontrivial elements of \( \pi_1 \text{B} \text{Diff}_0(\Sigma^3 \times S^1) \). Moreover, it turns out that \( \Phi \) is independent in the abelianization of the image of \( i_* \) from \( \pi_1 \text{B} \text{Diff}_{\text{deg}}(\Sigma^3 \times I, \partial) \) by Propositions 7.1, 7.3. By the commutativity of the diagram above, it follows that

\[
\pi_1 \text{B} \text{Diff}_0(\Sigma^3 \times S^1) / i_* \pi_1 \text{B} \text{Diff}_0(\Sigma^3 \times I, \partial)
\]

is an infinite set. The assertion 2 follows by a similar argument as above by using Proposition 7.2. \( \square \)

### 8 Lifting to a finite cyclic cover

It follows from the proof of Theorem 7.4 that \( \Psi_1(\Theta(1,gt^p,ht^q)) \) gives a sequence of nontrivial elements of \( \Omega_1(\text{B} \text{Diff}_0(\Sigma^3 \times S^1)) \). As shown in Theorem 7.4, these loops induce a sequence of elements of \( \pi_0 \text{Emb}_0(\Sigma^3, \Sigma^3 \times S^1) \).

**Proposition 8.1.** Let \( \Sigma^3 = \Sigma(2,3,5) \). The embedding \( \Sigma^3 \to \Sigma^3 \times S^1 \) induced by \( \Psi_1(\Theta(1,gt^p,ht^q)) \) can be trivialized after taking the lift in a finite cyclic covering of \( \Sigma^3 \times S^1 \) in the \( S^1 \) direction.

**Proof.** We shall see how \( \Sigma^3[1] = \Sigma^3 \times \{1\} \subset \Sigma^3 \times S^1 \) will be deformed in a cyclic covering after passing through the 1-parameter family of \( \Sigma^3 \times S^1 \)’s given by surgery along a \( \Theta \)-graph.

Recall that surgery along an embedded \( \Theta \)-graph was defined by replacing two disjoint handlebodies \( V^{(1)}, V^{(2)} \) with some families of them. By deforming the embedded \( \Theta \)-graph by isotopy in \( \Sigma^3 \times S^1 \), we may assume that \( V^{(1)}, V^{(2)} \) and \( \Sigma^3[1] \) are put in \( \Sigma^3 \times S^1 \) as follows: \( \Sigma^3[1] \) does not intersect the terminal 1-handle and 2-handles ("leaves") of \( V^{(2)} \), and also does not intersect the terminal
2-handle of $V^{(1)}$, whereas does intersect the terminal 1-handles of $V^{(1)}$ transversally at 3-disks. Moreover, $\Sigma^3[1]$ may intersect the edges of the Y-shaped part of $V^{(2)}$ transversally several times (Figure 7-left).

Let $V'^{(1)}, V'^{(2)}, \Sigma^3[1]'$ be the lifts of $V^{(1)}, V^{(2)}, \Sigma^3[1]$ in the infinite cyclic covering $\Sigma^3 \times \mathbb{R}$ of $\Sigma^3 \times S^1$, where the lifts are such that the 2-handle of $V'^{(1)}$ and the 1-handle of $V'^{(2)}$ are linked together in the form of the Hopf link, and $\Sigma^3[1]'$ intersect the two 1-handles of $V^{(1)}$ transversally. Then $V'^{(1)} \cap \Sigma^3[1]'$ is a disjoint union of four 3-disks $\delta_1, \delta_1', \delta_2, \delta_2'$ (Figure 7-right), and the two 2-handle leaves of $V'^{(2)}$ are null-leaves, namely, they have spanning 3-disks in $\Sigma^3 \times \mathbb{R}$ that do not intersect $V'^{(1)} \cup \Sigma^3[1]'$.

Let $\Sigma^3[1]''$ be the submanifold of $\Sigma^3 \times \mathbb{R}$ that is obtained by letting $\Sigma^3[1]'$ pass through the 1-parameter family of $\Sigma^3 \times \mathbb{R}$’s, which is obtained by fiberwise infinite cyclic covering of the 1-parameter family of $\Sigma^3 \times S^1$'s constructed by surgery along a $\Theta$-graph.

First, we see that $\delta_1, \delta_1', \delta_2, \delta_2'$ are sent to four disjoint disks $\delta_1'', \delta_1''', \delta_2'', \delta_2'''$ in the handlebody $V''^{(1)}$ obtained from $V'^{(1)}$ by one Borromean twist. The boundaries of the four disks are the same as those of the original ones $\delta_1, \delta_1', \delta_2, \delta_2'$. By a property of the Borromean rings, the change of the surgery on $V'^{(1)}$ alone can be realized by isotopy in $\Sigma^3 \times \mathbb{R}$ ([Wa1, Lemma 4.13]).

Next, we construct a 1-parameter family of surgeries of $\Sigma^3 \times \mathbb{R}$ by the parametrized Borromean twist along $V'^{(2)}$. This gives a 1-parameter family of deformed $V''^{(1)}$’s. Now $\Sigma^3[1]'''$ is the union of the four disks that are obtained by letting $\delta_1'', \delta_1''', \delta_2'', \delta_2'''$ pass through the 1-parameter family and $\Sigma^3[1]'' \cap (\Sigma^3 \times \mathbb{R} - (V'^{(1)} \cup V'^{(2)}))$.

Here, since the two leaves of $V'^{(2)}$ are null in $\Sigma^3 \times \mathbb{R}$, the 1-parameter family of deformed $V''^{(1)}$’s can be deformed into the original constant family of $V'^{(1)}$’s by fiberwise isotopy, by a property of the Borromean rings again. Thus
the relative isotopy classes of \( \delta'_d, \delta''_2, \delta''_1 \) in \( V''(1) \) do not change before and after the surgery along \( V''(2) \). This together with the fact that the deformation \( V''(1) \to V''(1) \) can be realized by isotopy shows that \( \Sigma^3[1]'' \) is isotopic to \( \Sigma^3[1]' \) in \( \Sigma^3 \times \mathbb{R} \). Since \( \Sigma^3[1]' \) is compact, this isotopy can be taken in some finite cyclic covering.

**Proposition 8.2.** Let \( \Sigma^3 = \Sigma(2, 3, 5) \). For every positive integer \( r \), there exists an embedding \( \varphi : \Sigma^3 \to \Sigma^3 \times S^1 \) satisfying the following conditions.

1. \( \varphi \) is homotopic to the inclusion \( \iota : \Sigma^3 = \Sigma^3 \times \{1\} \to \Sigma^3 \times S^1 \).

2. The lift of \( \varphi \) in the \( r \)-fold cyclic covering of \( \Sigma^3 \times S^1 \) in the \( S^1 \) direction is not isotopic to \( \iota \).

**Proof.** If we project the embedding \( \varphi(1, gt^p, ht^q) : \Theta \to \Sigma^3 \times S^1 \) corresponding to the decorated \( \Theta \)-graph \( \Theta(1, gt^p, ht^q) \) by the \( r \)-fold covering map \( \mathbb{I} \times \rho^r : \Sigma^3 \times S^1 \to \Sigma^3 \times S^1 \), we obtain an embedding \( \varphi(1, gt^p, ht^q) : \Theta \to \Sigma^3 \times S^1 \) corresponding to \( \Theta(1, gt^p, ht^q) \). The preimage of \( \varphi(1, gt^p, ht^q) \)(\( \Theta \)) under \( \mathbb{I} \times \rho^r \) is the sum of \( \mathbb{Z}_r \)-symmetric \( r \) disjoint copies of \( \Theta(1, gt^p, ht^q) \):

\[
\Theta(1, gt^p, ht^q) + \zeta \Theta(1, gt^p, ht^q) + \zeta^2 \Theta(1, gt^p, ht^q) + \cdots + \zeta^{r-1}\Theta(1, gt^p, ht^q).
\]

(\( \zeta = t^{1/r} \)) If we do the same computation in the proof of Theorem 6.2 after replacing \( V^{(i)} \) with \( V^{(i)} + \zeta V^{(i)} + \cdots + \zeta^{r-1}V^{(i)} \), the resulting value of \( \mathbb{Z}_\Theta \) will be

\[
2r[\Theta(1, gt^p, ht^q)].
\]

If \( g, h, p, q \) are such that \([\Theta(1, gt^p, ht^q)] \in \mathcal{A}_2(\mathbb{Z}_r^2[t^{\pm 1}]; \rho(\pi'))/\mathcal{A}_2(\mathbb{Z}_r^2; \rho(\pi')) \) is nontrivial, \( \Psi_1(\Theta(1, gt^p, ht^q)) \) gives the embedding that satisfies the conditions 1 and 2. \( \square \)

9 Extension to pseudo-isotopies

9.1 Framed link for Type I surgery

Let \( d = \dim X \geq 4 \). Let \( K_1, K_2, K_3 \) be the unknotted spheres that are parallel to the cores of the handles of Type I handlebody \( V \) of indices 1, 1, \( d-1 \), respectively. Let \( c_i \) be a small unknotted sphere that links with \( K_i \) with the linking number 1. Let \( L'_1 \cup L'_2 \cup L'_3 \) be a Borromean rings of dimensions \( d-2, d-2, 1 \) embedded in a small ball in \( \text{Int} \ V \) that is disjoint from \( K_1 \cup K_2 \cup K_3 \). For each \( i = 1, 2, 3 \), let \( L_i \) be a knotted sphere in \( \text{Int} \ V \) obtained by connect summing \( c_i \) and \( L'_i \) along an embedded arc that is disjoint from the cocores of the 1-handles and from other components, so that \( L_i \)'s are mutually disjoint. Then \( K_i \cup L_i \) is a Hopf link in \( d \)-dimension. Each component of the six component link \( \bigcup_{i=1}^3 (K_i \cup L_i) \) is an unknot in \( X \), and we may consider it as a framed link by canonical framings
The following lemma can be proved by a straightforward analogue of [GGP, Lemma 2.1], which describes the effect of a Y-surgery.

**Lemma 9.1.** The relative diffeomorphism type of the pair \((V', \partial V)\) of the handlebody \(V'\) obtained by surgery along a Type I handlebody \(V\) and its boundary \(\partial V' = \partial V\) is isomorphic to the pair obtained from \((V, \partial V)\) by surgery along the framed six component link \(\bigcup_{i=1}^{3} (K_i \cup L_i)\).

### 9.2 Family of framed links for Type II surgery

As above, let \(K_1, K_2, K_3\) be the unknotted spheres that are parallel to the cores of the handles of Type II handlebody \(V\) of indices \(1,d-1,d-1\), respectively. Let \(c_i\) be a small unknotted sphere that links with \(K_i\) with the linking number 1. Let \(L_{i,s} \cup L'_{i,s} \cup L''_{i,s} (s \in S^{d-3})\) be a \((d-3)\)-parameter family of three component links of dimensions \(d-2, 1, 1\) with unknotted components embedded in a small ball in \(\text{Int} V\) disjoint from \(K_1 \cup K_2 \cup K_3\), such that \(L'_{i,s}, L''_{i,s}\) are unknotted components in \(V\) that do not depend on \(s\), and the union of the locus of \(L'_{i,s}\) and \(L''_{i,s}\) forms a Borromean rings of dimensions \(d-2, d-2, 1\).

For each \(i = 1, 2, 3\), let \(L_{i,s}\) be a knotted sphere in \(\text{Int} V\) obtained by connect summing \(c_i\) and \(L'_{i,s}\) along an embedded arc that is disjoint from the cocores of the 1-handles and from other components, so that \(L_{i,s}\)'s are mutually disjoint. Then \(K_i \cup L_{i,s}\) is a Hopf link in \(d\)-dimension. Each component of the six component link \(\bigcup_{i=1}^{3} (K_i \cup L_{i,s})\) is fiberwise isotopic to a constant family of an unknotted sphere in \(X\), and we may consider it as a family of framed links by canonical framings (Figure 8, \(V^{(2)}\)). The following lemma is analogous to Lemma 9.1.

**Lemma 9.2.** The relative bundle isomorphism type of the pair \((\tilde{V}, \partial \tilde{V})\) of the family of handlebodies \(\tilde{V}\) obtained by surgery along a Type II handlebody \(V\) and its boundary \(\partial \tilde{V} = \partial V \times S^{d-3}\) is isomorphic to the pair obtained from \((V, \partial V) \times S^{d-3}\) by surgery along the family of framed six component links \(\bigcup_{i=1}^{3} (K_i \cup L_{i,s})\).

**Theorem 9.3** (Teichner). Let \(d = \dim X \geq 4\). The element of \(\Omega_{(d-3)k} (\text{BDiff}(X))\) given by \(\pi^\Gamma : E^\Gamma \to B\Gamma\) belongs to the image of the restriction induced map \(\Omega_{(d-3)k} (\text{B\mathcal{E}}(X)) \to \Omega_{(d-3)k} (\text{BDiff}(X)).\)

**Proof.** Since the restriction of the X-bundle \(\pi^\Gamma : E^\Gamma \to B\Gamma\) on the components of \(B\Gamma\) with some \(S^0\) component being -1 is trivial, we may exclude such components in advance.

By Lemmas 9.1, 9.2, \(\pi^\Gamma : E^\Gamma \to B\Gamma\) can be obtained by surgery along a \((S^{d-3}) \times k\)-parameter family of framed links with \(6 \times 2k = 12k\) components (Figure 8). Also, by the choice of the framed links, the \(S^{d-2}\)-components in the links form a parameter independent \(3 \times 2k = 6k\)-component unlink in \(X\). In particular, they span mutually disjoint \((d-1)\)-disks in \(X\). The spanning \((d-1)\)-disks may intersect \(S^1\)-components transversally.
By attaching 2-handles and \((d-1)\)-handles along framed links in the top face \(X \times \{1\}\) of \(X \times I\), we obtain a \((S^{d-3})^k\)-parameter family of \((d+1)\)-dimensional cobordisms whose boundary is \(E^T \coprod (X \times (S^{d-3})^k)\).

Since the \(S^{d-2}\)-components in the framed links form a constant family of unlinks that span mutually disjoint \((d-1)\)-disks in \(X\), we may exchange \((d-1)\)-handles attached to \(X \times I\) with reversed \((d-1)\)-handles removed from \(X \times I\) obtained by pushing the spanning \((d-1)\)-disks below \(X \times \{1\}\). This replacement does not change the boundary. Then we may attach the several 2-handles on the resulting (constant) family of cobordisms to get a family of cobordisms whose boundary is \(E^T \coprod (X \times (S^{d-3})^k)\).

Moreover, removing a reversed \((d-1)\)-handle from \(X \times I\) and attaching a 1-handle to \(X \times I\) yield the same cobordism. Here, the boundary circle of the cocore of a reversed \((d-1)\)-handle corresponds to a 1-cycle of the top boundary that is parallel to the core of a 1-handle. The \((d-1)\)-sphere obtained by gluing the \((d-1)\)-disk on the top boundary that is parallel to the core of a reversed \((d-1)\)-handle and a spanning \((d-1)\)-disk in \(X \times \{1\}\) of the attaching \((d-2)\)-sphere corresponds to the boundary \(S^{d-1}\) of the cocore of a 1-handle (Figure 9).

The fiber of the fiber bundle obtained after exchanging all the reversed \((d-1)\)-handles with 1-handles is diffeomorphic to \(X \times I\). To see this, we need only to check this for the fiber on the base point of \((S^{d-3})^k\). On the base fiber, the \(12k\)-component framed link consists of \(6k\) Hopf links, and in a single Type II handlebody \(V\), there are three disjoint trivial Hopf links out of \(12k\)-components. If a Hopf link in \(V\) has a null \(S^{d-2}\)-component, e.g., \(K_1^{(2)} \cup L_1^{(2)}\) in Figure 8,
it corresponds to a cancelling pair of 1-handle and 2-handle, whose pair of attaching spheres can be pushed into a small ball $U$ in $V$ by sliding handles, without affecting other components (Figure 9-right). Then the cancelling pair can be removed without changing the diffeomorphism type of the cobordism. After this cancellation, there will be a Hopf link in other handlebody with a null $S^{d-2}$-component, e.g., $K^{(1)}_3 \cup L^{(1)}_3$ in Figure 8, and we may continue the cancellation. By repeating this process, all the components can be removed. This shows that the base fiber is diffeomorphic to $X \times I$.

Finally, the $X \times I$-bundle now obtained gives an element of $\Omega_{(d-3)k}(B\mathcal{C}(X))$ since surgery along framed links does not change the bottom face $X \times \{0\}$. Also, the restriction of the $X \times I$-bundle to the top face, giving an element of $\Omega_{(d-3)k}(B\text{Diff}(X))$, is the original bundle $\pi^T : E^T \to B_T$.

**Remark 9.4 (Teichner).** For $d = 4$, the proof of Theorem 9.3 implies that the class of $\pi^\Theta : E^\Theta \to B_0$ in $\Omega_1(B\text{Diff}(X))$ can be represented by an $h$-cobordism bundle over $S^1$ which has a fiberwise handle decomposition consisting of six pairs of cancelling $(1, 2)$-handles. It follows from an argument similar to the proof above that five of the cancelling pairs can be removed in pairs and the other one pair, which corresponds to the Hopf link $K_2^{(2)} \cup L_2^{(2)}$ in Figure 9, can be so near the base point of $S^1$. The resulting family of handle structures has the graphic (in the sense of [Ce]) as in Figure 10, where the attaching 1-sphere of the 2-handle moves around in the middle level surface $X \# (S^1 \times S^3)$ in a quite complicated way, though can be described explicitly.

**Proposition 9.5.** Let $d = \dim X \geq 4$. The element of $\Omega_{k(d-3)}(B\text{Diff}(X))$ given by $\pi^T : E^T \to B_T$ belongs to $\Omega_{k(d-3)}(B\text{Diff}_0(X))$.

**Proof.** This follows since an element of $\mathcal{C}(X)$ gives a homotopy of a diffeomorphism of $X$ to the identity. The element of $\Omega_{k(d-3)}(B\text{Diff}(X))$ given by $\pi^T : E^T \to B_T$ is mapped from $\Omega_{k(d-3)}(B\mathcal{C}(X))$ and the result follows. □
Proposition 9.6. Let $d = \dim X \geq 4$. The element of $\Omega_{(d-3)k}(\text{BDiff}_0(X))$ given by $\Psi_k(\Gamma(g_1, \ldots, g_{3k}))$ ($g_1, \ldots, g_{3k} \in \pi_1(X)$) belongs to the image of the projection induced map

$$
\Omega_{(d-3)k}(\text{BDiff}_{\deg}(X^\ast, \partial)) \to \Omega_{(d-3)k}(\text{BDiff}_0(X^\ast, \partial)).
$$

Proof. The existence of vertical framing was shown in [Wa1, Lemma 4.12]. Thus we need only to show that a fiberwise degree 1 map $E^f \to X$ exists. Each component of the family of $12k$-component framed links that gives the surgery along $V^{(1)} \cup \cdots \cup V^{(2k)}$ is fiberwise nullhomotopic in $X \times B_{\Gamma}$. By attaching handles along the family of $12k$-component framed links in the top face of $(X \times I) \times B_{\Gamma}$, we obtain a fiberwise cobordism $E$ between $E^f$ and $- (X \times B_{\Gamma})$. Since the families of attaching spheres of handles are fiberwise nullhomotopic, the projection $X \times B_{\Gamma} \to X$ extends over families of handles continuously and we obtain a continuous extension $E \to X$. That its restriction to $E^f$ is fiberwise degree 1 map follows since the preimage of a point of $X - (V^{(1)} \cup \cdots \cup V^{(2k)})$ is one point in each fiber.

Proposition 9.7. There exists an element of $\pi_1\text{Emb}_0(\Sigma^4, \Sigma^4 \times S^1)/\text{Diff}_0(\Sigma^4)$, $\Sigma^4 = \Sigma(2,3,5) \times S^1$, that cannot be trivialized after taking the lift in any finite cyclic covering of $\Sigma^4 \times S^1$ in the $S^1$ direction.

Proof. According to the proof of Theorem 9.3, the class of $\pi^\Theta : E^\Theta \to B_\Theta$ for an embedded $\Theta$-graph can be obtained by surgery along an $S^1$-family of 12 component framed links in $\Sigma^4$, consisting of six Hopf links of 2- and 1-spheres. Also, we have seen that the 3-handles attached to the 2-spheres can be pushed into $\Sigma^4 \times (-\varepsilon, 0]$. The roles of 2-spheres and 1-spheres in the family of six Hopf links can be exchanged so that the 1-spheres form a constant family of unlinks. Hence the 2-handles attached to the 1-spheres can be pushed into $\Sigma^4 \times [0, \varepsilon)$.

This shows that the nontrivial family of $\Sigma^4$ obtained by the family of six Hopf links can be embedded into $\Sigma^4 \times (-\varepsilon, \varepsilon) \subset \Sigma^4 \times S^1$. Such family is obviously nontrivial in any finite cyclic covering of $\Sigma^4 \times S^1$ in the $S^1$ direction.

Figure 10: The graphic for the $h$-cobordism bundle between $\pi^\Theta$ and the trivial $X$-bundle, $\dim X = 4$. 

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10 Nontrivial $D^n \times S^1$-bundles

Theorem 10.1. For $n \geq 3$, $\varepsilon = n + 1 \mod 2$, a homomorphism

$$Z_{\Theta} : \pi_{n-2} \widetilde{BDiff}(D^n \times S^1, \partial) \to \mathcal{A}_{\Theta}(\mathbb{C}[t^{\pm 1}]; \mathbb{Z})$$

is defined, and the image of $Z_{\Theta}$ has at least countable infinite rank.

Proof (sketch). We replace $\Sigma^n \times S^1$ with $\mathbb{R}^n \times S^1$. In this case, a propagator in $\overline{\text{Conf}_2}(\mathbb{R}^n \times S^1)$ with coefficients in $\mathbb{C}[t^{\pm 1}]$ can be given by direct product of the standard propagator in $\overline{\text{Conf}_2}(\mathbb{R}^n)$ and $S^1$, as in the proof of Proposition 3.12. Here, the compactification $\overline{\text{Conf}_2}(\mathbb{R}^n \times S^1)$ of $\text{Conf}_2(\mathbb{R}^n \times S^1)$ is defined by blowing-up $(S^n \times S^1) \times 2$ along the following sequence of strata:

$$(\infty \times S^1) \times S^1, (\infty \times S^1) \cong \infty \times S^1 \times \Delta S^1,$$

$$(\infty \times S^1) \times (\infty \times S^1) \cong \infty \times (S^1 \times S^1),$$

$$\Delta_{\mathbb{R}^n \times S^1},$$

$$(\infty \times S^1) \times (\mathbb{R}^n \times S^1) \cup (\mathbb{R}^n \times S^1) \times (\infty \times S^1).$$

If we require that propagator is standard near infinity, we may prove the existence of a propagator in family with coefficients in $\mathbb{C}[t^{\pm 1}]$, for a similar reason as Lemma 4.10. An explicit propagator near the boundary can be given by an analogue of the explicit construction given in [Les1, §12].

As before, it suffices to check that the explicit propagator constructed on the boundary lies in the kernel of the natural map

$$H_{2n-1}(\partial E\overline{\text{Conf}}(\pi); \Lambda) \to H_{2n-1}(E\overline{\text{Conf}}(\pi); \Lambda).$$

To see this, it is enough to consider the same problem for $H_{n+1}$ in the base fiber, which is obvious by the existence of a standard propagator in the base fiber.

Then the rest is similar to the case of $\Sigma^n \times S^1$. \qed

References


