GAROUFALIDIS–LEVINE’S FINITE TYPE INVARIANTS FOR
$\mathbb{Z}_\pi$-HOMOLOGY EQUIVALENCES FROM 3-MANIFOLDS TO THE
3-TORUS

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Abstract. Garoufalidis and Levine defined a filtration for 3-manifolds equipped
with some degree 1 map ($\mathbb{Z}_\pi$-homology equivalence) to a fixed 3-manifold $N$
and showed that there is a natural surjection from a module of $\pi = \pi_1N$-
decorated graphs to the graded quotient of the filtration over $\mathbb{Z}[\frac{1}{2}]$. In this pa-
per, we show that in the case of $N = T^3$ the surjection of Garoufalidis–Levine
is actually an isomorphism over $\mathbb{Q}$. For the proof, we construct a perturbative
invariant by applying Fukaya’s Morse homotopy theoretic construction to a
local coefficient system of the quotient field of $\mathbb{Q}_\pi$. The first invariant is an
extension of the Casson invariant to $\mathbb{Z}_\pi$-homology equivalences to the 3-torus.
The results of this paper suggest that there is a highly nontrivial equivari-
ant quantum invariants for 3-manifolds with $b_1 = 3$. We also discuss some
generalizations of the perturbative invariant for other target spaces $N$.

1. Introduction

In [Wa2], we constructed a perturbative invariant of 3-manifolds with $b_1 = 1$ by
modifying Fukaya’s Morse homotopy theoretic invariant ([Fu]). In this paper, we
extend the invariant of [Wa2] to 3-manifolds with higher first Betti numbers and
see that it is effectively utilized in the study of finite type invariants of 3-manifolds
defined by Garoufalidis and Levine ([GL]).

A theory of finite type invariant of 3-manifolds was first developed by Ohtsuki in
[Oh1] for integral homology 3-spheres. Ohtsuki’s finite type invariant was defined
by means of a filtration on the module spanned by diffeomorphism classes of integral
homology 3-spheres. Ohtsuki’s filtration was defined with respect to surgeries on
algebraically split framed links in $S^3$. It has been shown that the graded quotient
of his filtration is isomorphic over $\mathbb{Q}$ to certain space of trivalent graphs ([Oh1, GO,
Le]), with the help of the LMO invariant ([LMO]). Garoufalidis and Levine defined
in [GL] a filtration for 3-manifolds equipped with some degree 1 map ($\mathbb{Z}_\pi$-homology
equivalence) to a fixed 3-manifold. They gave an upper bound for the graded
quotient of their filtration by the space of $\pi$-decorated graphs. To see that this
correspondence is an isomorphism, it is important to find a perturbative invariant
that is a counterpart to the LMO invariant. Some perturbative invariants were
found for 3-manifolds with $b_1 = 1$ in [Oh2, Oh3, Les2, Les3, Wa2]. We construct

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a perturbative invariant for $\mathbb{Z}\pi$-homology equivalences to the 3-torus via Morse homotopy theory and by utilizing it we show that when the target manifold is the 3-torus $T^3$, the space of $\pi$-decorated graph is naturally isomorphic to the graded quotient in Garoufalidis–Levine’s filtration. In this paper, we mainly consider the filtration of [GL] defined by surgery on $Y$-links (or graph claspers, [GGP, Ha]).

We also give a formula relating the value of our first invariant $\tilde{Z}_2$ for the connected sum of $T^3$ with an integral homology sphere $S$ to the Casson invariant of $S$. We also discuss some generalizations of the perturbative invariant for other target spaces $N$ with abelian fundamental groups.

1.1. $\mathbb{Z}\pi$-homology equivalences. First, we recall the definitions from [GL]. We fix an oriented closed 3-manifold $N$ and let $\pi = \pi_1 N$. A degree 1 map $f : M \rightarrow N$ from another oriented closed 3-manifold $M$ is said to be a $\mathbb{Z}\pi$-homology equivalence if the induced map $f_* : H_*(M) \rightarrow H_*(N)$ is an isomorphism. Here, $M, N$ are the $\pi$-coverings of $M, N$. Two $\mathbb{Z}\pi$-homology equivalences $f : M \rightarrow N$ and $f' : M' \rightarrow N$ are diffeomorphically equivalent if there exists a diffeomorphism $g : M \rightarrow M'$ such that $f$ is homotopic to $f \circ g$. Let $\mathcal{H}(N)$ be the set of all diffeomorphism equivalence classes of $\mathbb{Z}\pi$-homology equivalences $f : M \rightarrow N$.

In the following, we assume for simplicity that $\pi_2 N = 0$. In [GL], it is shown that a degree 1 map $f : M \rightarrow N$ can be represented by a nullhomotopic framed link in $N$, and when $\pi_2 N = 0$ the class of $f$ is uniquely determined by a framed link. Moreover, in [GL], a “surgery obstruction map”

$$\Phi : \mathcal{H}(N) \rightarrow B(\pi)$$

to certain semi-group $B(\pi)$ was defined by assigning a matrix of $\mathbb{Z}\pi$-valued linking numbers (or $\pi$-equivariant linking number) of the framed link.

1.2. $\mathcal{H}(N)$ and finite type invariants. Let $\mathcal{H}(N) = \text{Ker} \Phi$. An element of $\mathcal{H}(N)$ is given by a $\pi$-algebraically split framed link, namely, a framed link such that its $\mathbb{Z}\pi$-linking matrix is a diagonal matrix all of whose diagonal entries are $\pm 1$ and that its components are nullhomotopic. Let $\mathcal{F}(N)$ be the vector space over $\mathbb{Q}$ spanned by the set $\mathcal{H}(N)$. In [GL], it was shown that any element of $\mathcal{H}(N)$ can be obtained from $N$ by surgery on a $Y$-link with only nullhomotopic leaves. We consider the filtration of $\mathcal{F}(N)$ given by surgery on $Y$-links with only nullhomotopic leaves. For the terminology related to $Y$-links (or graph-claspers), we refer the reader to [GGP, Ha]. Let $G = \{G_1, G_2, \ldots, G_n\}$ be a $Y$-link consisting of disjoint $Y$-graphs in $M$ with only nullhomotopic leaves and let

$$[M, G] = \sum_{G' \subset G} (-1)^{|G'|} M^{G'},$$

where $M^{G'}$ denotes the map $M^{G'} \rightarrow N$ obtained by surgery on $G'$. Namely, by surgery a 4-cobordism $X$ between $M^{G'}$ and $M$ is obtained and the identity map on $M$ extends over it uniquely up to homotopy, and then compose with the $\mathbb{Z}\pi$-homology equivalence $f : M \rightarrow N$. Let $\mathcal{F}_n^Y(N)$ ($n = 0, 1, 2, \ldots$) be the subspace of $\mathcal{F}(N)$ spanned by $[M, G]$ for $Y$-link $G$ with $n$ disjoint $Y$-graphs. This defines a descending filtration on $\mathcal{F}(N)$: $\mathcal{F}(N) = \mathcal{F}^0_0(N) \supset \mathcal{F}^1_1(N) \supset \mathcal{F}^2_2(N) \supset \cdots$.

*In [GL], the indeterminacy for the case $\pi_2 N \neq 0$ is determined.
linear map \( \lambda : \mathcal{K}(N) \to A \) to a vector space \( A \) is said to be a finite type invariant of type \( n \) if \( \lambda(F_{n+1}(N)) = 0 \).

We call a pair \((\Gamma, \alpha)\) of an abstract, vertex-oriented, edge-oriented trivalent graph \( \Gamma \) and a map \( \alpha : \text{Edges}(\Gamma) \to \pi \) a \( \pi \)-decorated graph. We will denote \((\Gamma, \alpha)\) also by \( \Gamma(\alpha) \) or \( \Gamma(\alpha(1), \ldots, \alpha(m)) \) (\( m = |\text{Edges}(\Gamma)| \)). We define the degree of a \( \pi \)-decorated graph as the number of vertices. Let \( \mathcal{A}_n(\pi) \) be the vector space over \( \mathbb{Q} \) spanned by \( \Gamma(\alpha) \) of degree \( n \) quotiented by the relations AS, IHX, Orientation Reversal, Linearity and Holonomy (Figure 1). When \( \text{Im} \alpha \subset \pi \), we call \((\Gamma, \alpha)\) a monomial \( \pi \)-decorated graph. A monomial \( \pi \)-decorated graph \((\Gamma, \alpha)\) determines uniquely a homotopy class \( \bar{\alpha} \) of a map \( \Gamma \to N \). We choose a ribbon structure on \( \Gamma \) that is compatible with the vertex-orientation, choose an embedding of the ribbon graph into \( N \) that represents \( \bar{\alpha} \) and replace the image with a set \( G = \{G_1, \ldots, G_n\} \) of \( n \) disjoint Y-graphs. By assigning \([N, G]\) to \( \Gamma(\alpha) \), a well-defined linear map

\[
\psi_n : \mathcal{A}_n(\pi) \to \mathcal{F}_n^Y(N)/\mathcal{F}_{n+1}^Y(N)
\]

is obtained. In [GL], it was shown that this map is surjective\(^1\). When \( n \) is odd, \( \mathcal{A}_n(\pi) = 0 \) and hence \( \mathcal{F}_n^Y(N) = \mathcal{F}_{n+1}^Y(N) \). The main theorem of this paper is the following.

**Theorem 1.1.** When \( N = T^3 \), the map \( \psi_{2n} : \mathcal{A}_{2n}(\pi) \to \mathcal{F}_{2n}^Y(N)/\mathcal{F}_{2n+1}^Y(N) \) is an isomorphism for every positive integer \( n \).

For the results of other \( N \), see §6.

### 1.3. Plan of the paper and conventions

In the rest of this paper we prove Theorem 1.1.

In Section 2, we define a perturbative invariant \( \tilde{Z}_{2n} \) for \( \mathbb{Z}\pi \)-homology equivalence. The way of construction of the invariant is almost the same as [Wa2].

In Sections 3 and 4, we show that \( \tilde{Z}_{2n} \) is a finite type invariant and compute its values on \( \mathcal{F}_{2n}^Y(N)/\mathcal{F}_{2n+1}^Y(N) \). The main theorem follows as a corollary of this result and of the surjectivity of \( \psi_{2n} \) of [GL]. The outline of the proof is analogous to that of [KT]. Instead of modifying chains, we modify gradients of Morse functions to localize the counts of graphs in an alternating sum.

\(^1\)In [GL], this was proved for \( \mathbb{Z}[\frac{1}{2}] \)-coefficients.
In Section 5, we relate the value of $\tilde{Z}_2(N \# S)$ for an integral homology sphere $S$ to the Casson invariant of $S$, as a straightforward example.

In Section 6, we discuss some generalizations of the results for other target spaces $N$ with abelian fundamental groups.

Throughout this paper, manifolds and maps between them are assumed to be smooth. For a Morse–Smale gradient $\xi$ of a Morse function, we denote by $\mathcal{A}_p(\xi)$ (resp. $\mathcal{D}_p(\xi)$) the ascending manifold (resp. descending manifold) of a critical point $p$ of $\xi$. We denote by $\mathcal{A}_p(\xi)$ (resp. $\mathcal{D}_p(\xi)$) the usual compactification of $\mathcal{A}_p(\xi)$ (resp. $\mathcal{D}_p(\xi)$) into a smooth manifold with corners (e.g., [BH]), and consider these as chains by the natural maps to a manifold. We will often abbreviate $\mathcal{D}_p(\xi)$, $\mathcal{A}_p(\xi)$ etc. as $\mathcal{D}_p$, $\mathcal{A}_p$ etc. for simplicity. By a ribbon graph, we mean a compact surface equipped with a handle structure with only 0- and 1-handles.

2. Perturbative invariant $\tilde{Z}_{2n}$

2.1. Acyclic Morse complex. In the following, let $N = T^3$, $\pi = \mathbb{Z}$, and $\Lambda = \mathbb{Z}[t^1, t^2, t^3]$. Let $\Lambda$ be the field of fractions $Q(\Lambda)$ of $\Lambda$. We fix connected oriented 2-submanifolds $\Sigma_1, \Sigma_2, \Sigma_3$ of $N$ so that they form a basis of $\Omega_2(N) \cong \mathbb{Z}^3$. By means of these data, we define the holonomy of a path in $N$ as follows. Let $J$ be one of intervals $[a, b]$, $[a, b)$, $(a, b)$ ($-\infty \leq a < b \leq \infty$) and consider a piecewise smooth path $\gamma : J \to N$ that intersects each $\Sigma_i$ transversally in finitely many points. In this case, let $n_i = \langle \gamma, \Sigma_i \rangle$ ($i = 1, 2, 3$) and define

$$\text{Hol}(\gamma) = t_1^{n_1} t_2^{n_2} t_3^{n_3}.$$ 

For a $\mathbb{Z}[\pi]$-homology equivalence $f : M \to N$ and a Morse function $h : M \to \mathbb{R}$, a chain complex $C_\ast(f, h)$ is defined as follows. Take a Riemann metric $\rho$ on $M$ and consider the gradient $\xi$ of $h$. Here, we assume without loss of generality that the critical values of the critical points of $h$ are all different and critical points are disjoint from $\Sigma_1 \cup \Sigma_2 \cup \Sigma_3$. The gradient $\xi$ is Morse–Smale for generic metric $\rho$, namely, ascending manifolds and descending manifolds of $\xi$ are transversal. In this case, there are finitely many (unparametrized) flow-lines between a critical point of index $i$ and a critical point of index $i - 1$. For each such flow-line with parametrization $\gamma : \mathbb{R} \to M$, we assign its weight $\pm \text{Hol}(f \circ \gamma)$, where the sign is the same one as given in [Wa1, §2.5]. For a critical point $p$ of index $i$ and a critical point $q$ of index $i - 1$, we define the incidence coefficient $n(\xi; p, q) \in \Lambda$ as the sum of the weights of flow-lines of $-\xi$ between $p$ and $q$. Let $P_i = P_i(h)$ be the set of critical points of $h$ of index $i$ and let $C_i(f, h) = \Lambda P_i$. We define the boundary $\partial : C_i(f, h) \to C_{i-1}(f, h)$ by

$$\partial(p) = \sum_{q \in P_{i-1}} n(\xi; p, q) q.$$ 

$(C_\ast(f, h), \partial)$ forms a chain complex and its homology is identified with $H_\ast(\tilde{M}; \mathbb{Q})$ as a $\Lambda$-module.
Next, we put $C_i(f; h; \hat{\Lambda}) = C_i(f, h) \otimes_{\Lambda} \hat{\Lambda} = \hat{\Lambda}^P_i$. Since $\hat{\Lambda}$ is a flat $\Lambda$-module (e.g., [We, Theorem 3.2.2]), we have $\text{Tor}_1^\Lambda(\hat{\Lambda}, A) = 0$ for every $\Lambda$-module $A$. Hence by the universal coefficient theorem ([CE, Theorem VI.3.3]), we have

$$H_i(C_*(f; h; \hat{\Lambda})) \cong H_i(\tilde{M}; \mathbb{Q}) \otimes_{\Lambda} \hat{\Lambda}.$$ 

Since $f$ is a $\mathbb{Z}\pi$-homology equivalence, we have $H_i(\tilde{M}; \mathbb{Q}) \cong H_i(\tilde{N}; \mathbb{Q}) \cong H_i(\text{pt}; \mathbb{Q})$. Since the action of $\pi$ on this module is trivial, we have $H_0(M; \mathbb{Q}) \otimes_{\Lambda} \hat{\Lambda} = 0$. Namely, $(C_*(f; h; \hat{\Lambda}), \partial \otimes 1)$ is acyclic.

Since the chain complex $C = C_*(f; h; \hat{\Lambda})$ of based free $\hat{\Lambda}$-modules is acyclic, it follows that $\text{End}_\Lambda(C) = \text{Hom}_\Lambda(C, C)$ is acyclic too, by Künneth theorem ([CE, Theorem VI.3.1a]). Since $1 \in \text{End}_\Lambda(C)$ is a cycle of degree 0, there exists an element $g \in \text{End}_\Lambda(C)$ of degree 1 such that $\partial g + g \partial = 1$ ($g$ is called a combinatorial propagator).

### 2.2. Perturbation theory with holonomies in $\hat{\Lambda}$

This part is almost a copy from [Wa2].

Let $h_1, h_2, \ldots, h_{3n} : M \to \mathbb{R}$ be a sequence of Morse functions and let $\xi_i$ be the gradient of $h_i$. We consider an edge-oriented trivalent graph with its sets of vertices and edges labelled and with $2n$ vertices and $3n$ edges. By the labelling $\{1, 2, \ldots, 3n\} \to \text{Edges}(\Gamma)$ of $\Gamma$, we identify edges with numbers. Choose some of the edges and split each chosen edge into two arcs. We attach elements of $P_*(h_i)$ on the two 1-valent vertices (white-vertices) that appear after the splitting of the $i$-th edge. We call such obtained graph a $\mathcal{C}$-graph ($\mathcal{C} = (C_*(f, h_1; \hat{\Lambda}), \ldots, C_*(f, h_{3n}; \hat{\Lambda})$, see Figure 2). If a $\mathcal{C}$-graph is obtained from a connected trivalent graph, then we call it a primitive $\mathcal{C}$-graph. A $\mathcal{C}$-graph has two kinds of “edges”: a compact edge, which is connected, and a separated edge, which consists of two arcs. We call vertices that are not white vertices black vertices. If $p_i$ (resp. $q_i$) is the critical point attached on the input (resp. output) white vertex of a separated edge $i$, we define the degree of $i$ by $\text{deg}(i) = \text{ind} p_i - \text{ind} q_i$, where $\text{ind}$ denotes the Morse index. We define the degree of a compact edge $i$ by $\text{deg}(i) = 1$. We define the degree of a $\mathcal{C}$-graph by $\text{deg}(\Gamma) = (\text{deg}(1), \text{deg}(2), \ldots, \text{deg}(3n))$.

We say that a continuous map $I$ from a $\mathcal{C}$-graph $\Gamma$ to $M$ is a flow-graph for the sequence $\xi = (\xi_1, \xi_2, \ldots, \xi_{3n})$ if it satisfies the following conditions (see Figure 3).

1. Every critical point $p_i$ attached on the $i$-th edge is mapped by $I$ to $p_i$ in $M$.
2. The restriction of $I$ to each edge of $\Gamma$ is a smooth embedding and at each point $x$ of the $i$-th edge that is not on a white vertex, the tangent vector of $I$ at $x$ (chosen along the edge orientation) is a positive multiple of $(-\xi_i)_x$. 

![Figure 2. A trivalent graph (left) and a $\mathcal{C}$-graph (right)](image-url)
For a $\tilde{C}$-graph $\Gamma$, let $\mathcal{M}_T(\tilde{\xi})$ be the set of all flow-graphs for $\tilde{\xi}$ from $\Gamma$ to $M$. By extracting black vertices, a natural map from $\mathcal{M}_T(\tilde{\xi})$ to the configuration space $C_{2k}(M)$ of ordered tuples of $2k$ points is defined. It follows from a property of the gradient that this map is injective. This induces a topology on the set $\mathcal{M}_T(\tilde{\xi})$.

**Lemma 2.1** (Fukaya [Fu, Wa1]). If $\tilde{h} = (h_1, h_2, \ldots, h_{3n})$ and $\rho$ are generic, then for a $\tilde{C}$-graph $\Gamma$ with $2k$ black vertices and $\deg(\Gamma) = (1, 1, \ldots, 1)$, the space $\mathcal{M}_T(\tilde{\xi})$ is a compact $0$-dimensional manifold. Moreover, this property can be assumed for all $\tilde{C}$-graphs with $2k$ black vertices simultaneously.

When the assumption of Lemma 2.1 is satisfied, we may define an orientation of $\mathcal{M}_T(\tilde{\xi})$ by the same way as [Wa1, Wa2]. We define $\mathcal{M}_T(\tilde{\xi})$ as follows. For simplicity, we assume $\mathfrak{D}_2n(\tilde{\Lambda})$ the space of $\tilde{\Lambda}$-decorated graphs, defined by replacing $\mathbb{Q}\pi$ in the definition of $\tilde{\pi}$-decorated graph with $\tilde{\Lambda}$. Let $\mathfrak{g} = (g^{(1)}, g^{(2)}, \ldots, g^{(2n)})$ be a sequence of combinatorial propagators for $\tilde{C} = (C_\ast(f, h_1; \tilde{\Lambda}), \ldots, C_\ast(f, h_{3n}; \tilde{\Lambda}))$. Then we define $z_{2n}(f, \tilde{\xi}) \in \mathcal{M}_T(\tilde{\Lambda})$ by

$$z_{2n}(f, \tilde{\xi}) = \text{Tr}_{\mathfrak{g}} \left( \sum_{\Gamma} \mathcal{M}_T(\tilde{\xi}) \cdot \Gamma \right).$$

Here, the sum is taken over all primitive $\tilde{C}$-graphs $\Gamma$ with $2k$ black vertices and $\deg(\Gamma) = (1, 1, \ldots, 1)$, and $\text{Tr}_{\mathfrak{g}}$ is defined as follows. For simplicity, we assume that the labels for the separated edges in a $\tilde{\pi}$-decorated $\tilde{C}$-graph $\Gamma(u_1, u_2, \ldots, u_{3n})$ are $1, 2, \ldots, r$. Let $p_i, q_i$ be the critical points on the input and output of the $i$-th edge of $\Gamma$, respectively and let $g^{(i)}_{q_i p_i} \in \tilde{\Lambda}$ be the coefficient of $p_i$ in $g^{(i)}(q_i)$. Then $\text{Tr}_{\mathfrak{g}}(\Gamma(u_1, u_2, \ldots, u_{3n}))$ is the equivalence class in $\mathfrak{D}_2n(\tilde{\Lambda})$ of a $\tilde{\Lambda}$-decorated graph obtained by identifying each pair of the two white vertices of the separated edges in

$$\Gamma(-g^{(1)}_{q_1 p_1} u_1, \ldots, -g^{(r)}_{q_r p_r} u_r, u_{r+1}, \ldots, u_{3n}).$$
The definition of $\text{Tr}$ can be generalized to graphs with other degrees in the same manner. The following lemma is an analogue of [Wa2, Lemma 3.2].

**Lemma 2.2.** $z_{2n}(f, \xi)$ does not depend on the choices of $\vec{g}$ and of the hypersurfaces $\Omega_1, \Omega_2, \Omega_3 \subset N$ within their oriented bordism classes.

We shall define $\tilde{z}_{2n}$ by adding a correction term to $z_{2n}(f, \xi)$. Take a compact oriented 4-manifold $X$ with $\partial X = M$ and with $\chi(X) = 0$. By the condition $\chi(X) = 0$, the outward normal vector field to $M$ in $TX|_{\partial X}$ can be extended to a nonsingular vector field $\nu_X$ on $X$. Let $T^2X$ be the orthogonal complement of the span of $\nu_X$. Then $T^2X$ is a rank 3 subbundle of $TX$ that extends $TM$. Take a sequence $\vec{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_{3n})$ of generic sections of $T^2X$ so that $\gamma_i$ is an extension of $-\xi$. We define

$$z_{2n}^{\text{anomaly}}(\vec{\gamma}) = \sum_{\Gamma} \# \mathcal{M}_{\Gamma}^{\text{local}}(\vec{\gamma}) [\Gamma(1, 1, \ldots, 1)] \in \mathcal{A}_{2n}(\tilde{\Lambda}).$$

The sum is taken over all primitive $\vec{C}$-graphs with $2k$ vertices and with only compact edges. Here, $\mathcal{M}_{\Gamma}^{\text{local}}(\vec{\gamma})$ is the moduli space of affine graphs in the fibers of $T^2X$ whose $i$-th edge is a positive scalar multiple of $\gamma_i$. The number $\# \mathcal{M}_{\Gamma}^{\text{local}}(\vec{\gamma}) \in \mathbb{Z}$ is the count of the signs of the affine graphs that are determined by transversal intersections of some codimension 2 chains in a configuration space bundle over $X$. See [Sh, Wa1] for detail. By a similar argument as in [Sh, Wa1], it can be shown that $z_{2n}^{\text{anomaly}}(\vec{\gamma}) - \mu_{2n} \text{sign} X$, where $\mu_{2n} \in \mathcal{A}_{2n}(\tilde{\Lambda})$ is the constant given in [Wa1], does not depend on the choices of $X$, $\nu_X$ and the extension $\vec{\gamma}$ of $\xi$. We define $\tilde{z}_{2n}(f, \vec{\xi}) \in \mathcal{A}_{2n}(\tilde{\Lambda})$ by the following formula.

$$\tilde{z}_{2n}(f, \vec{\xi}) = z_{2n}(f, \vec{\xi}) - z_{2n}^{\text{anomaly}}(\vec{\gamma}) + \mu_{2n} \text{sign} X,$$

$$\tilde{z}_{2n}(f, \vec{\xi}) = \sum_{\epsilon_i = \pm 1} \tilde{z}_{2n}(f, (\epsilon_1 \xi_1, \ldots, \epsilon_{3n} \xi_{3n})).$$

**Theorem 2.3.** $\tilde{z}_{2n}(f, \vec{\xi})$ is an invariant of the diffeomorphism equivalence class of $Z\pi$-homology equivalence $f : M \to N$ and of oriented bordism classes of $\Omega_1, \Omega_2, \Omega_3$.

Proof of Theorem 2.3 can be done by using Cerf theory and is the same as [Wa2, Theorem 3.3]. The invariance under a homotopy of $f$ is analogous to [Wa2, Lemma 3.2] and follows by the graph relations.

Next, we extend the invariant to possibly disconnected graphs. For each $n \geq 1$, we fix a collection of rational numbers $\{a_{2n}^\Gamma\}_{\Gamma}$ ($\Gamma$ is a labelled edge-oriented connected trivalent graph of degree $2n$) so that $\sum_{\Gamma} a_{2n}^\Gamma[\Gamma] = \mu_{2n}$. Let $\Gamma$ be a not necessarily primitive $\vec{C}$-graph and let $\{\Gamma_1, \Gamma_2, \ldots, \Gamma_{r(\Gamma)}\}$ be the set of primitive components of $\Gamma$. For a sequence $\vec{\xi} = (\xi_1, \ldots, \xi_{2n})$ of gradient on $M$, let $\vec{\xi}_{\Gamma_1}$ be the subsequence of $\vec{\xi}$ restricted to the edge-labels for $\Gamma_1$. Let $\vec{\gamma}_{\Gamma_1}$ be the restriction of $\vec{\gamma}$. Let $n(\Gamma_i)$ be the number of black vertices of $\Gamma_i$. We define $I_{\Gamma_i}, \hat{I}_{\Gamma_i}$ by the following formula.

$$I_{\Gamma_i} = \# \mathcal{M}_{\Gamma_i}(\vec{\xi}_{\Gamma_i})$$

$$\hat{I}_{\Gamma_i} = \begin{cases} \# \mathcal{M}_{\Gamma_i}(\vec{\xi}_{\Gamma_i}) & \text{if } \Gamma_i \text{ has separated edge} \\ \# \mathcal{M}_{\Gamma_i}(\vec{\xi}_{\Gamma_i}) - \# \mathcal{M}_{\Gamma_i}^{\text{local}}(\vec{\gamma}_{\Gamma_i}) + a_{2n}^\Gamma(\vec{\Gamma}) \text{sign} X & \text{otherwise} \end{cases}$$
spectively that are compact manifolds with boundaries. Let
\[ Z_{2n}(f, \xi) = \frac{1}{2^{6n}(2n)!(3n)!} \text{Tr}_g \left( \sum_{\Gamma} (I_{\Gamma_1} \otimes \cdots \otimes I_{\Gamma_{r(\Gamma)}}) \cdot \Gamma \right), \]
\[ \tilde{Z}_{2n}(f, \xi) = \frac{1}{2^{6n}(2n)!(3n)!} \text{Tr}_g \left( \sum_{\Gamma} (\tilde{I}_{\Gamma_1} \otimes \cdots \otimes \tilde{I}_{\Gamma_{r(\Gamma)}}) \cdot \Gamma \right), \]
where the sums are taken for all \( \bar{G} \)-graphs \( \Gamma \) (not necessarily primitive) of degree
\( 2n \). We define
\[ \tilde{Z}_{2n}(f, \xi) = \sum_{\varepsilon_i = \pm 1} \tilde{Z}_{2n}(f, (\varepsilon_1 \xi_1, \ldots, \varepsilon_{3n} \xi_{3n})). \]

**Theorem 2.4.** \( \tilde{Z}_{2n}(f, \xi) \) is an invariant of the diffeomorphism equivalence class of \( \mathbb{Z}_2 \)-homology equivalence \( f : M \to N \) and of oriented bordism classes of \( \Sigma_1, \Sigma_2, \Sigma_3 \).

Since the bifurcations of \( \tilde{I}_r \), are exactly of the same kinds as those of \( \hat{z}_{2n}(f, \xi) \), the invariance of \( \tilde{Z}_{2n}(f, \xi) \) follows from the proof of invariance of \( \hat{z}_{2n}(f, \xi) \). In the following, we abbreviate \( \tilde{Z}_{2n}(f, \xi) \) as \( \tilde{Z}_{2n}(M) \), keeping in mind that \( \tilde{Z}_{2n}(M) \) depends not only on the diffeomorphism type of \( M \) but also of the choice of a degree 1 map \( f : M \to N \).

### 3. \( \tilde{Z}_{2n} \) as finite type invariant

In this and the next section, we prove the following theorem, which implies that \( \tilde{Z}_{2n} \) induces the inverse of \( \psi_{2n} \).

**Theorem 3.1.** Let \( n \geq 1 \).

1. \( \tilde{Z}_{2n} \) is a finite type invariant of type \( 2n \) in the sense of Garoufalidis–Levine’s filtration \( \mathcal{F}_Y \), namely, \( \tilde{Z}_{2n}(\mathcal{F}_{2n+1}(N)) = 0 \).
2. Let \( \Gamma(\alpha) \) be a monomial \( \pi \)-decorated graph of degree \( 2n \). The following identity holds.

\[ \tilde{Z}_{2n}(\psi_{2n}(\Gamma(\alpha))) = [\Gamma(\alpha)]. \]

Theorem 1.1 follows as a corollary to Theorem 3.1 and a result of [GL]. By considering similarly as the main theorem of [Le], it turns out that \( 1 + \sum_{n \geq 1} \tilde{Z}_{2n} \in \prod_{n \geq 0} \mathcal{A}_{2n}(\pi) \) is universal among \( \mathbb{Q} \)-valued finite type invariants.

The strategy for the proof of Theorem 3.1 is as follows. For both claims (1) and (2), it suffices to compute the value of \( \tilde{Z}_{2n}(\{M, G\}) \) for \( Y \)-link \( G \) in \( M \) with at least \( 2n \) disjoint \( Y \)-graphs. For this purpose, we perturb \( \xi \) and corresponding sequences of gradients for the results of surgeries, so that most of the terms in the alternating sum cancel each other. We will first show the corresponding claims (1) and (2) for \( \tilde{Z}_{2n}(f, \xi) \) for the perturbed gradients and then show that the results are the same for \( \tilde{Z}_{2n}(f, (\varepsilon_1 \xi_1, \ldots, \varepsilon_{3n} \xi_{3n})) \) for arbitrary \( \varepsilon_i = \pm 1 \).

#### 3.1. Morse function on \( M \) compatible with \( Y \)-link

For \( m \geq 2n \), let \( G = \{ G_1, G_2, \ldots, G_m \} \) be a \( Y \)-link consisting of \( Y \)-graphs with only nullhomotopic leaves. Let \( V_1, V_2, \ldots, V_m \) be closed regular neighborhoods of \( G_1, G_2, \ldots, G_m \) respectively that are compact manifolds with boundaries.

Take a Morse function \( \mu_i : V_i \to (-\infty, 0] \) on \( V_i \) satisfying the following conditions.
FINITE TYPE INVARIANTS FOR \( Z_{\pi} \)-HOMOLOGY EQUIVALENCE

Figure 4. Morse function on \( M \) compatible with \( Y \)-link

![Figure 4](image)

Figure 5. Resolutions of crossings with \( G_j \) by bordism modification of \( \Sigma_i' \)

![Figure 5](image)

- \( \mu_i^{-1}(0) = \partial V_i \).
- The set of critical points of \( \mu_i \) consists of one critical point of index 0 and three critical points of index 1.

We extend the Morse function \( \mu_1 \cup \cdots \cup \mu_m \) on \( V_1 \cup \cdots \cup V_m \) to a Morse function \( h : M \to \mathbb{R} \) (Figure 4). Here, we choose \( h \) so that \( h^{-1}(-\infty, 0] = V_1 \cup \cdots \cup V_m \). Moreover, let \( h_1, \ldots, h_{3n} : M \to \mathbb{R} \) be a sequence of Morse functions that are obtained from \( h \) by small perturbations so that they satisfy the genericity of Lemma 2.1, and let \( \xi_1, \ldots, \xi_{3n} \) be their gradients. We may also assume that \( h^{-1}_j(0) = \partial V_1 \cup \cdots \cup \partial V_m \) and \( h^{-1}_j(-\infty, 0] = V_1 \cup \cdots \cup V_m \).

**Lemma 3.2.** We may assume that \( f^{-1}(\Sigma_1), f^{-1}(\Sigma_2), f^{-1}(\Sigma_3) \) are all disjoint from \( V_1 \cup \cdots \cup V_m \) by modifying \( \Sigma_1, \Sigma_2, \Sigma_3 \) within their oriented bordism classes suitably.

**Proof.** Let \( \Sigma_i' \) \( (i = 1, 2, 3) \) be an oriented 2-submanifold of \( M \) that represents \( \text{PD} \circ f^* \circ \text{PD}([\Sigma_i]) \), where \( \text{PD} \) denotes the Poincaré dual. It suffices to show that \( \Sigma_i' \) \( (i = 1, 2, 3) \) can be modified within their oriented bordism classes so that they are disjoint from the \( Y \)-link \( G = \{ G_1, \ldots, G_m \} \) in \( M \). If \( \Sigma_i' \) intersects a component \( G_j \), then we may arrange that \( \Sigma_i' \) does not intersect \( G_j \) by bordism modifications (Figure 5), since the leaves are nullhomotopic in \( M \). The bordism may be taken to be disjoint from other \( Y \)-graphs. \( \square \)

In the following, we assume the condition of Lemma 3.2. Although the result of the bordism modification of \( \Sigma_i' \) may have self-intersections, such a modification of \( \Sigma_i' \) does not change the value of \( Z_{2n} \) and there is no problem.

### 3.2. \( Y \)-link surgery and coherent family of gradients

We see that after modifying the gradients for \( \mu_i \) and the ones induced on the result of surgery in \( V_i \), the types of the flow-graphs that contribute to the alternating sum will be restricted...
to special ones. As is well-known, the Y-surgery in $V_i$ can be realized by a Torelli surgery along $\partial V_i$. Namely, there is a diffeomorphism $\varphi_i : \partial V_i \to \partial V_i$ that represents an element of the Torelli group and the surgery is obtained by cutting $M$ along $\partial V_i$ and gluing $V_i$ back after applying $\varphi_i$ to $\partial V_i$. After this surgery, a Morse function $h^{G_i} : M^{G_i} \to \mathbb{R}$ is induced from $h$. A Morse function $h^{G'} : M^{G'} \to \mathbb{R}$ on the result of surgery on a Y-link $G' \subset G$ is induced in a similar way. Let $\xi^{G'}$ denote the gradient for $h^{G'}$. The manifold $M^{G'}$ can be considered as the one obtained from $M$ by replacing a cylindrical neighborhood of $\partial V_i$ in $V_i$ with the mapping cylinder of $\varphi_i$ and similarly $M^{G'}$ is obtained by iterating such replacements. Let $f^{G'}$ denote the composition of $\pi\tau$-homology equivalences $M^{G'} \to M \xrightarrow{f} N$ obtained by projecting the mapping cylinders suitably onto $\partial V_i$.

Let $p_1^{(i)}, p_2^{(i)}, p_3^{(i)}$ be the critical points of the Morse function $\mu_i : V_i \to (-\infty, 0]$ of index 1. Let $V_i^{G_i}$ be the part of $M^{G_i}$ corresponding to $V_i$. We also denote by $p_1^{(i)}, p_2^{(i)}, p_3^{(i)}$ the critical points of the induced Morse function $\mu_i^{G_1} : V_i^{G_1} \to (-\infty, 0]$ of index 1. Let $A_k^{(i)}(k = 1, 2, 3)$ be the intersection of $\partial V_i^{(i)}(\xi)$ with $V_i$ and let $B_k^{(i)}(k = 1, 2, 3)$ be the intersection of $\partial V_i^{(i)}(\xi^{G_i})$ with $V_i^{G_i}$. The restrictions $A_k^{(i)} \cap \partial V_i$ and $B_k^{(i)} \cap \partial V_i(k = 1, 2, 3)$ both forms a disjoint triple of simple closed curves in $\partial V_i$. Note that $A_k^{(i)} \cap \partial V_i$ and $B_k^{(i)} \cap \partial V_i$ may intersect and they are bordant in $\partial V_i$. Let $C_k^{(i)}$ be a bordism between $A_k^{(i)} \cap \partial V_i$ and $B_k^{(i)} \cap \partial V_i$ in $\partial V_i$. Namely, there is a diffeomorphism $\phi$ such that the maximal point of $\nu$ is disjoint from every $C_k^{(i)}$.

Lemma–Definition 3.3. The gradients of $\mu_i, \mu_i^{G_i}$, and the bordism $C_k^{(i)}$ can be chosen so that they satisfy the following.

(1) There exists an embedded ribbon graph $T_i \subset \partial V_i$ and $C_k^{(i)}$ is a smaller ribbon graph immersed in $T_i$ for $k = 1, 2, 3$.
(2) Let $T_i^{(0)} \subset T_i^{(1)} = T_i$ be a handle filtration associated with the ribbon structure of $T_i$. Then the image of $C_k^{(i)}$ in each 1-handle of $T_i$ consists of parallel copies of (not necessarily disjoint) bands.
(3) The 0- and 1-handles of $T_i$ can be taken as arbitrarily small tubular neighborhoods of their cores. Namely, for every $\varepsilon > 0$, we may arrange that the diameters of 0-handles are less than $\varepsilon$ and that the thickness of 1-handles are less than $\varepsilon/2$.

We say that such a family $\{(h^{G'}, \xi^{G'})\}_{G' \subset G}$ is coherent.

Proof. Put $\gamma_k^{(i)} = A_k^{(i)} \cap \partial V_i - B_k^{(i)} \cap \partial V_i (k = 1, 2, 3)$. Then $\gamma_k^{(i)}$ can be represented by a map $S^1 \coprod S^1 \to \partial V_i$ and is nullhomologous in $\partial V_i$. Take a minimal Morse function $\nu : \partial V_i \to \mathbb{R}$ such that the maximal point of $\nu$ is disjoint from every $\gamma_k^{(i)}$. Now we define a filtration $T_i^{(0)} \subset T_i^{(1)} = T_i$ by the Morse–Smale filtration for the gradient of $\nu$ up to dimension 1. We modify $\gamma_k^{(i)}$ by applying the negative gradient flow $\Phi_{-\nu} : \mathbb{R} \times \partial V_i \to \partial V_i$ for $\nu$. For a sufficiently large $T$, the cycle $\Phi_{-\nu}(T, \cdot) \circ \gamma_k^{(i)}$ concentrates on a small neighborhood of the 1-skeleton of the cellular decomposition of $\partial V_i$ with respect to the gradient of $\nu$. See Figure 6. Moreover, since $\gamma_k^{(i)}$ is nullhomologous, the image of $\Phi_{-\nu}(T, \cdot) \circ \gamma_k^{(i)}$ in 1-handles of $T_i$ consist of parallel copies of pairs of two arcs with opposite orientations. The intersections between
curves and self-intersections are all squeezed into 0-handles. Then it will be easy to extend each cycle $\gamma_k$ to a nullbordism in $T_i$ so that the extension is an immersed ribbon graph as desired. All the modifications above can be realized by isotopies of $\partial V_i$ and the isotopies can be obtained by modifying the gradient of $\mu_i$. \( \square \)

We remark that the bordisms $C_k^{(i)}$ for different $k$ may intersect each other.

Let $D_k^{(i)}$ be the 2-cycle in the closed 3-manifold $W_i = V_i \cup_\partial (-V_i^G)$ defined by

$$D_k^{(i)} = A_k^{(i)} - B_k^{(i)} + C_k^{(i)}.$$

We denote the class of $D_k^{(i)}$ in $H_2(W_i)$ by $\alpha_k^{(i)}$.

**Lemma 3.4.** $\langle \alpha_1^{(i)}, \alpha_2^{(i)}, \alpha_3^{(i)} \rangle = \pm 1$.

**Proof.** We assume that the mapping cylinder $C(\varphi_i)$ for the Torelli surgery is included in a small neighborhood of $\partial V_i$ in $V_i^G$. Let $\kappa < 0$ be a level such that

- all the critical points of $\mu_i$ and of $\mu_i^G$ lie in $\mu_i^{-1}(-\infty, \kappa)$ and in $(\mu_i^G)^{-1}(-\infty, \kappa)$ respectively and
- $C(\varphi_i) \subset (\mu_i^G)^{-1}(\kappa, 0]$.

Let $\overline{A_k^{(i)}} = A_k^{(i)} \cap \mu_i^{-1}[\kappa, 0]$ and $\overline{B_k^{(i)}} = B_k^{(i)} \cap (\mu_i^G)^{-1}[\kappa, 0]$. Consider the closed 3-manifold

$$M(\varphi_i) = \mu_i^{-1}[\kappa, 0] \cup_\partial - (\mu_i^G)^{-1}[\kappa, 0],$$

where $\mu_i^{-1}(0) = (\mu_i^G)^{-1}(0) = \partial V_i$, and $\mu_i^{-1}(\kappa)$ and $(\mu_i^G)^{-1}(\kappa)$ are glued together by the natural identification of $\mu_i^{-1}(-\infty, \kappa)$ and $(\mu_i^G)^{-1}(-\infty, \kappa)$. Then $M(\varphi_i)$ is the mapping torus of $\varphi_i$. The 2-chains

$$\overline{D_k^{(i)}} = \overline{A_k^{(i)}} - \overline{B_k^{(i)}} + C_k^{(i)} \quad (k = 1, 2, 3)$$

represent 2-cycles in $M(\varphi_i)$. Since $A_k^{(i)} \cap \mu_i^{-1}(-\infty, \kappa)$ and $B_k^{(i)} \cap (\mu_i^G)^{-1}(-\infty, \kappa)$ are mutually disjoint and so are $B_k^{(i)} \cap \mu_i^{-1}(-\infty, \kappa)$ and $A_k^{(i)} \cap (\mu_i^G)^{-1}(-\infty, \kappa)$, we have

$$\langle D_1^{(i)}, D_2^{(i)}, D_3^{(i)} \rangle = \langle \overline{D_1^{(i)}}, \overline{D_2^{(i)}}, \overline{D_3^{(i)}} \rangle.$$

According to [Jo, Second Definition, p.170], the value of the first Johnson homomorphism for $\varphi_i$ is

$$\langle \overline{D_1^{(i)}}, \overline{D_2^{(i)}}, \overline{D_3^{(i)}}, [\gamma_1^{(i)}] \wedge [\gamma_2^{(i)}] \wedge [\gamma_3^{(i)}] \rangle \in \wedge^3 H_1(\partial V_i),$$

where $c_k^{(i)}$ is a 1-cycle of $\partial V_i$ that are homologous to $p_k^{(i)}$. This must agree with $\pm [\gamma_1^{(i)}] \wedge [\gamma_2^{(i)}] \wedge [\gamma_3^{(i)}]$ for the Y-surgery. Hence we have $\langle D_1^{(i)}, D_2^{(i)}, D_3^{(i)} \rangle = \pm 1. \quad \square$
Here we make the following assumption, which will be used later.

**Assumption 3.5.** The indices of $\alpha_1^{(i)}, \alpha_2^{(i)}, \alpha_3^{(i)}$ are chosen so that it is compatible with the cyclic order of $\alpha_1^{(i)}, \alpha_2^{(i)}, \alpha_3^{(i)}$ determined from the vertex-orientation of the $Y$-graph $C_i$. Moreover, we orient the $2$-cycles so that $\langle \alpha_1^{(i)}, \alpha_2^{(i)}, \alpha_3^{(i)} \rangle = 1$.

3.3. Alternating sum for coherent family.

**Lemma 3.6.** If $\{(h_{G'}^G, \xi_{G'}^G)\}_{G' \subset G}$ is coherent, then the map $f_*^G : C_*^G(h_{G'}^G, \xi_{G'}^G; \hat{\Lambda}) \to C_*(f, h; \hat{\Lambda})$ induced by the $\mathbb{Z}\pi$-homology equivalence $f^G : \tilde{M}^G \to M$ is a chain isomorphism.

**Proof.** Since the sets of critical points of $h$ and $h_{G'}^G$ are the same, the underlying $\hat{\Lambda}$-modules can be taken to be the same one. Hence it suffices to check that the boundary operator is invariant under the surgery on $G_i$. The flow-lines that may change the count of flow-lines are those from critical points outside $V_i$ to critical points inside $V_i$. Hence the change of the count under surgery is given by the intersection of the curves $\mathcal{D}_p(\xi) \cap \partial V_i$ and $\mathcal{D}_p(\xi_i) \cap \partial V_i$. Generically, $\mathcal{D}_p(\xi) \cap \partial V_i$ intersects transversally with only edges of the ribbon graph $T_i$ of Lemma 3.3. Since it always intersects both sides of each band of $C_{G_i}^{(i)}$ simultaneously, the intersection number cancels in pair. \hfill $\square$

Under the isomorphism of Lemma 3.6, we may identify all the Morse complexes for $(h_{G'}^G, \xi_{G'}^G)$ ($G' \subset G$). Accordingly, the combinatorial propagators may be taken to be a common one.

**Lemma 3.7.** If $\{(h_{G'}^G, \xi_{G'}^G)\}_{G' \subset G}$ is coherent for each $j \in \{1, 2, \ldots, 3n\}$ and if the parameter $\varepsilon$ in Lemma 3.3 is sufficiently small, then we have the following.

1. The alternating sum of $\tilde{Z}_{2n}$ for $[M, G]$ agrees with that of $Z_{2n}$ (we denote this value by $Z_{2n}([M, G])$).

2. The flow-graphs that may contribute to $Z_{2n}([M, G])$ are such that for each $i$ there are exactly one black vertex in $V_i$ or $V_i^{G_i}$. (We say that such a flow-graph occupies $G_i$.)

**Proof.** Let $X$ be a compact $4$-manifold with $\partial X = M$ considered in the definition of the correction term for $\tilde{Z}_{2n}$, and let $\gamma_X$ be a sequence of sections of $T^\nu X$ that extends $\xi$ on $T^\nu X|_M = TM$. There exists a compact $4$-manifold $X_i$ and a sequence of sections $\gamma_X$, of $T^\nu X_i$ such that $\partial X_i = W_i$, sign $X_i = 0$ and that $\gamma_X$ agrees with $\gamma_X$ on $V_i \subset \partial X_i$. By the additivity of the signature, the signature is unchanged under attaching $X_i$ to $X$ along $V_i$. Moreover, by connected summing several $K3\#(-K3)$ ($K3$ is the Kummer $K3$ surface, $\chi(K3) = 24$) and $T^4$ ($\chi(T^4) = 0$) to $X_i$, we may assume that $\chi(X \cup V_i, X_i) = \chi(X) = 0$ without changing the signature. By the assumption of $X_i$, the term $a_{\Gamma_k}^{n(\Gamma_k)}$ sign $X$ in $\tilde{I}_{\Gamma_k} = \#_\delta \tilde{\mathcal{M}}_{\Gamma_k}(\chi_{\Gamma_k}) - \#_\delta \tilde{\mathcal{M}}_{\Gamma_k}^{\text{local}}(\gamma_{\Gamma_k}) + a_{\Gamma_k}^{n(\Gamma_k)}$ sign $X$ does not change under the surgery. Hence the terms of the form $a_{\Gamma_k}^{n(\Gamma_k)}$ sign $X$ can be factored out as a constant $\mu_{2n}$ sign $X$ for some $n'$. Namely, the alternating sum of the part of $\sum_k (\tilde{I}_{\Gamma_1} \otimes \cdots \otimes \tilde{I}_{\Gamma_k})$ of $\Gamma$ consisting of terms having sign $X$ is a polynomial in sign $X$ whose coefficients are linear combinations of labelled $\pi$-decorated $\tilde{C}$-graphs. Then the problem is reduced to the case of less $n$ without terms of the form $a_{\Gamma_k}^{n(\Gamma_k)}$ sign $X$. 


Corollary 3.8. If \( m = 2n \), then only the flow-graphs that is the disjoint union of 2n \( Y \)-shaped flow-graphs may contribute nontrivially to \( Z_{2n}([M,G]) \).
4. Evaluation of $\tilde{Z}_{2n}$ for coherent family of gradients

4.1. Algebraic Seifert surface of a leaf and equivariant linking number.

Here, we shall give a preliminary for the proof of Theorem 3.1 (2). Let $S_2(\tilde{M})$ be the complex of piecewise smooth singular chains of $\tilde{M}$ with coefficients in $\mathbb{Q}$. In the following, we identify $C_*(f; h; \tilde{\Lambda})$ with the image of a natural embedding

$$\phi^M : C_*(f; h; \tilde{\Lambda}) \to S_2(\tilde{M}) \otimes_{\Lambda} \tilde{\Lambda}$$

defined by the CW-structure of $\tilde{M}$ for the Morse–Smale gradient. Then one can see that $\phi^M$ is a chain equivalence.

From now on, the domain of $h$ is assumed to be $N$. Let $p$ be a critical point of $\mu_i : V_i \to (-\infty, 0]$ of index 1 and let $c = \phi^N(p)$, which corresponds to a leaf of a $Y$-graph $G_i$.

**Lemma 4.1.** For a combinatorial propagator $g$ of the acyclic complex $C_*(id; h; \tilde{\Lambda})$, we have $p = \partial g(p)$ and $c = \partial \phi^N(g(p))$.

**Proof.** By definition of combinatorial propagator, we have $p = (\partial g + g \partial)(p) = \partial g(p)$. □

We shall define the intersection form with holonomy between base pointed cells. For two Morse–Smale gradients $\xi^{(1)}, \xi^{(2)}$ on $M$, we consider the associated CW-structures. We choose a base point of each $d$-dimensional cell as the critical point of index $d$ on the cell. Let $C_1, C_2$ be two cells of the CW-structures for $\xi^{(1)}, \xi^{(2)}$ respectively such that $\dim C_1 + \dim C_2 = 3$ and that they intersect transversally at finitely many points. Let $v_1, v_2$ denote the base points of $C_1, C_2$ respectively. For each point $x$ of $C_1 \cap C_2$, choose a path $\gamma_1$ in $C_1$ that goes from $v_1$ to $x$ and a path $\gamma_2$ in $C_2$ that goes from $x$ to $v_2$, and we define the weight $W(x)$ of the intersection point $x$ as $W(x) = \pm \text{Hol}(\gamma_1 * \gamma_2)$. Here the sign is the sign of the intersection. Then we define

$$\langle C_1, C_2 \rangle_\pi = \sum_{x \in C_1 \cap C_2} W(x) \in \Lambda.$$ 

This does not depend on the choices of $\gamma_1, \gamma_2$ whereas it depends on the choices of $\Sigma_1, \Sigma_2, \Sigma_3$. One can see that $\langle C_2, C_1 \rangle_\pi = \langle C_1, C_2 \rangle_\pi$.

Next, we shall define the triple intersection form with holonomies. For three Morse–Smale gradients $\pm \xi^{(1)}, \pm \xi^{(2)}, \pm \xi^{(3)}$ on $M$, we consider the associated CW-structures. Let $C_1, C_2, C_3$ be three cells of the CW-structures for $\pm \xi^{(1)}, \pm \xi^{(2)}, \pm \xi^{(3)}$ respectively such that the sum of their codimensions is 3 and that they intersect “transversally”, that is, the orthogonal complements of $TC_i$ ($i = 1, 2, 3$) span the direct sum in $TM$. Let $v_1, v_2, v_3$ be the base points of $C_1, C_2, C_3$ respectively. For each point $x$ of $C_1 \cap C_2 \cap C_3$, let $\gamma_k$ be a flow-line of $\xi^{(k)}$ that connects $v_k$ and $x$,
and we define the weight $W(x)$ of $x$ as $W(x) = \pm \text{Hol} (\gamma_1) \otimes \text{Hol} (\gamma_2) \otimes \text{Hol} (\gamma_3)$. Here the sign is the sign of the intersection. Then we define
\[
(C_1, C_2, C_3)_\pi = \sum_{x \in C_1 \cap C_2 \cap C_3} W(x) \in \Lambda^3.
\]

By extending all the above by $\hat{\Lambda}$-bilinearity, the intersection forms with holonomies among two or three $\Lambda$-chains are defined. These forms are defined only for transversal chains. If the chains are not transversal, then we consider the intersection form of them as the one defined after perturbing the gradients slightly so that the transversality is satisfied, not affecting the arguments in other part.

Let $\ell'$ be a critical point of $\mu_{\ell} : V_{\ell} \to (-\infty, 0]$ of index 1 and let $c' = \phi^N (p')$. Take $g$ as above and put $g(c) = \phi^N (g(p))$. Since $c, c'$ represent leaves of $Y$-graphs in $G$, these are nullhomotopic in $\hat{N}$ by the assumption of $Y$-graphs. For two component link $c \cup c'$, their equivariant linking number is defined by
\[
\ell k_\pi (c, c') = \sum_{z \in \pi} (\ell k(\tilde{c}, z\tilde{c}') z) \in \mathbb{Z}^\pi,
\]
where $\tilde{c}, \tilde{c}'$ are lifts of $c, c'$ in $\hat{N}$, and $\ell k(\tilde{c}, z\tilde{c}')$ is the usual linking number in $\hat{N} = \mathbb{R}^3$. The indeterminacy of changing the lifts $\tilde{c}, \tilde{c}'$ is the multiplication by elements of $\pi$. Thus $\ell k_\pi (c, c')$ is well-defined modulo $\pi$. The following lemma is a straightforward analogue of [Les2, Proposition 3.3], with Lemma 4.1 in mind.

**Lemma 4.2.** Let $c$ and $c'$ be as above and put $\hat{\ell}k_\pi (c, c') = \langle g(c), c' \rangle_\pi$. Then the following identity holds.
\[
\hat{\ell}k_\pi (c, c') = \ell k_\pi (c, c') \quad (\text{mod } \pi)
\]

The value of the left hand side of the equation in lemma 4.2 does not depend on the choice of the lifts $\tilde{c}, \tilde{c}'$ and is determined uniquely with respect to $\Sigma_1, \Sigma_2, \Sigma_3$.

**Lemma 4.3.** The two chains of $V_{\ell}$ and $V_{\hat{G}_A}$ that are induced from $g(c)$ induces relative 2-cycles of $(V_{\ell}, \partial V_{\ell})$ and $(V_{\hat{G}_A}, \partial V_{\ell})$ respectively, and they induce the same 1-cycle on $\partial V_{\ell} = \partial V_{\hat{G}_A}$. Hence, by gluing the two relative cycles along the common boundary, a 2-cycle of $W_{\ell} = V_{\ell} \cup_0 (-V_{\hat{G}_A})$ is obtained.

**Proof.** That $g(c)$ induces the same 1-cycle on $\partial V_{\ell} = \partial V_{\hat{G}_A}$ is because the gradient of $h$ above the level 0 does not change under the surgery. By Lemma 4.1, $g(c)$ has no boundary in both $V_{\ell}$ and $V_{\hat{G}_A}$, and hence induces relative 2-cycles of $(V_{\ell}, \partial V_{\ell})$ and $(V_{\hat{G}_A}, \partial V_{\ell})$ respectively.

**Lemma 4.4.** Let $p_{1}^{(\ell)}$ be the critical points of $\mu_{\ell} : V_{\ell} \to (-\infty, 0]$ of index 1 and let $p_{1}^{(\ell)}$, $p_{2}^{(\ell)}$, $p_{3}^{(\ell)}$ be relative 2-cycles that represent the Poincaré–Lefschetz duals of $[p_{1}^{(\ell)}], [p_{2}^{(\ell)}], [p_{3}^{(\ell)}]$. Then the relative homology class in $(V_{\ell}, \partial V_{\ell})$ induced from $g(c)$ is given by the following formula.
\[
\hat{\ell}k_\pi (c, c_{1}^{(\ell)} [p_{1}^{(\ell)*}] + \hat{\ell}k_\pi (c, c_{2}^{(\ell)} [p_{2}^{(\ell)*}] + \hat{\ell}k_\pi (c, c_{3}^{(\ell)} [p_{3}^{(\ell)*}]
\]

**Proof.** This follows immediately from Lemma 4.2.

In $S_{\ell} (\hat{M}) \otimes_\Lambda \hat{\Lambda}$, the relative 2-cycles $p_{1}^{(\ell)*}, p_{2}^{(\ell)*}, p_{3}^{(\ell)*}$ are explicitly given by $s_{p_{1}^{(\ell)}} (\xi) \cap V_{\ell}, s_{p_{2}^{(\ell)}} (\xi) \cap V_{\ell}, s_{p_{3}^{(\ell)}} (\xi) \cap V_{\ell}$ respectively.
4.2. $\tilde{Z}_{2n}$ for alternating sums of degree $2n$. As a $Y$-link $G = \{G_1, G_2, \ldots, G_{2n}\}$, we take the $Y$-link obtained from an embedding $\Gamma \to N$ with ribbon structure that represents a monomial $\pi$-decorated graph $\Gamma(\alpha)$ of degree $2n$. Then $\psi_{2n}([\Gamma(\alpha)])$ is given by the equivalence class of $[N, G]$ in $\mathcal{F}_{2n}(N)/\mathcal{F}_{2n+1}(N)$. We shall compute the value of $\tilde{Z}_{2n}$ for this alternating sum for a coherent family $\{(h^1_\xi, \xi^1_\xi)\} G \subset G$. By Lemma 3.7, it suffices to compute the value of $Z_{2n}$ for the alternating sum.

Let $H$ be an edge-oriented, vertex-oriented labelled trivalent graph of degree $2n$. Let $H^\circ$ be the undecorated $\tilde{C}$-graph of degree $2n$ obtained from $H$ by replacing every edge with separated edges. Then $H^\circ$ is a disjoint union of $Y$-shaped components. Let $H^\circ_i$ ($i = 1, 2, \ldots, 2n$) be the $Y$-component of $H^\circ$ that includes the $i$-th black vertex.

We consider each separate edge $e$ as decomposed into two half-edges $e_+, e_-$ (with input white vertex, $e_-$: with output white vertex) and consider each half-edge as degree 1 object. The exterior product $(e_+^{(1)} \wedge e_-^{(1)}) \wedge (e_+^{(2)} \wedge e_-^{(2)}) \wedge \cdots \wedge (e_+^{(3n)} \wedge e_-^{(3n)})$ for the 3n separated edges of $H^\circ$ can be rewritten as $T_1 \wedge T_2 \wedge \cdots \wedge T_{2n}$, where $T_i$ is a wedge of the three half-edges incident to the $i$-th black vertex, and then $T_i$ determines a vertex-orientation of the $i$-th black vertex.

If the half-edges $e, e', e''$ around the $i$-th black vertex of $H^\circ$ gives $T_i$ in this order, and if the critical points that are attached to the white vertices of $e, e', e''$ are $p, p', p''$ respectively, then we write $H^\circ_i = H^\circ_i(p) (p = (p, p', p''))$. Also, if $H^\circ = H^\circ_1(p_1) \cup \cdots \cup H^\circ_{2n}(p_{2n})$, then we write $H^\circ = H^\circ(p_1, \ldots, p_{2n})$. The orientation of $H^\circ$ is given by the vertex-orientations.

The contribution in $Z_{2n}(f, \xi)$ of the flow-graphs from graphs of the form $H^\circ$ is

$$\sum_{p_1, \ldots, p_{2n}} \text{Tr}_g \left( M H^\circ_1(p_1)(\xi) \cdots M H^\circ_{2n}(p_{2n})(\xi) \cdot H^\circ(p_1, \ldots, p_{2n}) \right),$$

where $p_1, \ldots, p_{2n}$ are such that every separated edge is degree 1. We denote this by $I(\xi, H)$. Each term $M H^\circ_i(p_i)(\xi)$ is the triple product with holonomies for some three 2-chains formed along the vertex-orientation. For example, if among the three white vertices in $H^\circ_i(p_1, p'_1, p''_1)$ the one with $p_1$ attached is input and the other two with $p'_1, p''_1$ attached are outputs, then $M H^\circ_i(p_1)(\xi) = (\mathcal{F}_{p_1}, \mathcal{F}_{p'_1}, \mathcal{F}_{p''_1}) \pi \in H_3^{\circ}$.

The term $M H^\circ_i(p_1)(\xi) \cdots M H^\circ_{2n}(p_{2n})(\xi) \cdot H^\circ(p_1, \ldots, p_{2n})$ is a linear combination of graphs with 3n inputs and 3n outputs. Let $y_1, y_2, \ldots, y_{3n}$ be the critical points that are attached on the input white vertices of $H^\circ(p_1, \ldots, p_{2n})$, and let $x_1, x_2, \ldots, x_{3n}$ be those on the corresponding outputs, so that $y_i$ and $x_i$ are attached on a single separated edge. Put

$$X(x_1, \ldots, x_{3n}; y_1, \ldots, y_{3n}) = M H^\circ_1(p_1)(\xi) \cdots M H^\circ_{2n}(p_{2n})(\xi) \cdot H^\circ(p_1, \ldots, p_{2n}),$$

$$R(x_1, \ldots, x_{3n}; y_1, \ldots, y_{3n}) = M H^\circ_1(p_1)(\xi) \cdots M H^\circ_{2n}(p_{2n})(\xi).$$

Then $I(\xi, H)$ is rewritten as

$$\sum_{x_1, \ldots, x_{3n}} \text{Tr}_g \left( X(x_1, \ldots, x_{3n}; y_1, \ldots, y_{3n}) \right)$$

$$= \left[ \sum_{x_1, \ldots, x_{3n}} R(x_1, \ldots, x_{3n}; g^{(1)}(x_1), \ldots, g^{(3n)}(x_{3n})) \cdot \text{Close}(H^\circ) \right].$$
$R(x_1, \ldots, x_{3n}; y_1, \ldots, y_{3n})$ is extended to sequences of chains by $\overrightarrow{\Lambda}$-linearity and \text{Close}(H^o)$ is the trivalent graph that is obtained from the $\overrightarrow{C}$-graph $H^o$ by identifying the two white vertices in each separated edge. The terms in the sum in the right hand side of (4.1) are linear combinations of \text{Close}(H^o)$ together with holonomy data. The holonomy data is given by a product of $2n$ terms like $(\overrightarrow{\mathcal{F}}_{g^{(1)}(\epsilon_1)}, \overrightarrow{\mathcal{G}}_{x_1}, \overrightarrow{\mathcal{G}}_{x_3})_\pi$, where the product is taken by multiplying the holonomies of the two chains for each separated edge so that the coefficient belongs to $\Lambda^{\otimes 3n}$.

By Corollary 3.8, for the coherent families $\{(h_i^{G'}, \xi_i^{G'})\}_{G' \subset G}$ $(j = 1, 2, \ldots, 3n)$ of Morse–Smale pairs, we have

$$\sum \sum (-1)^{|G'|} I(\xi_i^{G'}, H).$$

Here, the first sum is taken for edge-oriented, vertex-oriented labelled trivalent graphs $H$ of degree $2n$. Let $S_H(\sigma(1), \ldots, \sigma(2n)), \sigma \in S_{2n}$, denote the part in the alternating sum \sum_{G' \subset G}(-1)^{|G'|} I(\xi_i^{G'}, H)$ in (4.2) consisting of terms of flow-graphs such that for each $i$ the $i$-th black vertex lies in $V_{\sigma(i)}$ or $V_{\sigma'(i)}$. Then we have

$$Z_{2n}([N, G]) = \frac{1}{2^{2n}(2n)!((3n)!)} \sum_{G' \subset G} S_H(\sigma(1), \ldots, \sigma(2n)).$$

**Lemma 4.5.** The following identity holds.

$$S_H(1, 2, \ldots, 2n) = \left[ \sum_{x_1, \ldots, x_{3n}} 1 \prod_i (Q_{x_i}^{(1)}(0) - Q_{x_i}^{(1)}(1)) \cdot \text{Close}(H^o) \right]$$

Here, $x = (x_1, \ldots, x_{3n}; g^{(1)}(x_1), \ldots, g^{(3n)}(x_{3n}))$, and for $\epsilon_i = 0$ or 1,

$$Q_{x_i}^{(1)}(\epsilon_i) = \begin{cases} \# H_{\pi_1}^\circ(p_i)(\xi_{G_i}) & (\epsilon_i = 1) \\ \# H_{\pi_1}^\circ(p_i)(\xi_{G_i}) & (\epsilon_i = 0) \end{cases}$$

and $p_1, \ldots, p_{2n}$ are sequences of linear combinations of critical points that correspond to $x$.

**Proof.** By (4.1), $S_H(1, 2, \ldots, 2n)$ is

$$\sum_{G' \subset G} (-1)^{|G'|} \left[ \sum_{x_1, \ldots, x_{3n}} Q_{x_1}^{(1)}(\epsilon_1)Q_{x_2}^{(2)}(\epsilon_2) \cdots Q_{x_{3n}}^{(2n)}(\epsilon_{2n}) \cdot \text{Close}(H^o) \right].$$

The right hand side of the formula of the lemma is obtained by rewriting this expression. \hfill \square

For $S_H(\sigma(1), \ldots, \sigma(2n))$, similar formula will be obtained by replacing $Q_{x_i}^{(1)}(1)$ in the formula of Lemma 4.5 with $\# H_{\pi_1}^\circ(p_i)(\xi_{G_{\sigma(i)}})$.

The indices of the critical points $x_1, \ldots, x_{3n}$ in the formula of Lemma 4.5 may be restricted as follows.

**Lemma 4.6.** After perturbing the gradients $\xi_i^{G'}$ without affecting the previous assumptions, we may restrict the Y-shaped flow-graphs in Corollary 3.8 to those such that the critical points attached on input white vertices are all of index 2 and that the critical points attached on output white vertices are all of index 1.
Proof. Let $I$ be a flow-graph in $N^{G'}$ for some $G' \subset G$ with $2n$ black vertices consisting only of $Y$-shaped components and suppose that $I$ occupies $G$. This implies that every output white vertices of a $Y$-shaped component are mapped to critical points in some $V_\ell$ or $V_\ell^{G'}$. We may assume the following.

(1) $I$ has no output white vertex on which a critical point of index 2 is attached.

(2) $I$ has no input white vertex on which a critical point of index 3 is attached.

For (2), if $I$ had such a white vertex, then the input vertex were paired with an output white vertex on which a critical point of index 2 is attached, contradicting (1).

We define the index of a $Y$-shaped component $J$ in $I$ as $(a_1, \ldots, a_k|a_{k+1}, \ldots, a_3)$ if the indices of the critical points attached on the input white vertices of $J$ are $a_1, \ldots, a_k$ and if the indices of the critical points attached on the output white vertices of $J$ are $a_{k+1}, \ldots, a_3$. By the condition that the dimension of the moduli space is 0, the possibilities for the indices of $Y$-components in $I$ are as follows.

$(2, 2, 2), (1, 2, 3), (1, 1, 1), (0, 1, 2), (2, 2|1), (2, 1|0), (3, 1|1), (2|1, 1), (1|0, 1), (3|1, 2)$

Among these, $(2, 2, 2), (1, 1, 1), (2, 2|1), (2|1, 1)$ satisfy the condition of the lemma. Further, by the result of the previous paragraph, we do not need to consider the cases $(1, 2, 3), (0, 1, 2), (3, 1|1), (3|1, 2)$. Hence it suffices to check that the remaining cases $(2, 1|0), (1|0, 1)$ does not contribute to the alternating sum $Z_{2n}([N, G])$.

Let $K$ be a $Y$-shaped component in $I$ of index $(2|1, 0)$. Suppose that $K$ has a black vertex in $V_i$ or $V_i^{G'}$, and that the half-edges with critical points of indices 2, 1, 0 are labelled by $k, \ell, m$, respectively. By (4.1), we may consider that on the input white vertex of $K$ of index 2, the chain $g(p)$ for some critical point $p$ of $h_\ell^{G'}$ of index 1 with negative critical value is attached, extending $I$ to chains. Let $q$ be the critical point of $h_\ell^{G'}$ of index 1 that is attached on an input white vertex of $K$. According to Lemma 4.3, $g(p)$ induces a relative 2-cycle in $(V_i, \partial V_i)$, which we now denote by $\sigma$.

If $q$ lies in $V_i$, then $q$ is a 1-cycle in the Morse complex. By Lemma 3.2, the holonomies of the half-edges $\ell, m$ are both 1. Let $r$ be the critical point of index 0 attached on $m$. By Lemma 4.2, $K$ gives

$$\langle \sigma, \overline{\phi_r \phi_q^N} \rangle_\pi = \pm \hat{k}_\pi(\phi^N(q), \phi^N(p)) \otimes 1 \otimes 1.$$
Since this value does not change under the surgery on \( G_i \), the flow-graph \( I \) does not contribute to the alternating sum \( Z_{2n}([N,G]) \) in this case.

If \( q \) lies outside \( V_i \), then a leg of \( \mathcal{D}_q \) converges to a critical point \( s \) of index 0 in \( V_i \) or \( V^G_i \) by assumption. We remark that since \( \mathcal{D}_q \) induces a relative 1-cycle of \( (V_i, \partial V_i \cup \{s\}) \), the value of \( \langle \sigma, \mathcal{D}_r, \mathcal{D}_q \rangle_{\pi} \) will not change by adding to \( \sigma \) the boundary of a 3-chain in \( V_i \) that does not meet \( \partial V_i \cup \{s\} \). Now we assume that the gradients for Morse functions \( \mu_i, \mu_i^G \) are as follows. We assume that the support of the Torelli surgery for the surgery on \( G_i \) is included in a ribbon graph \( R \) in \( \partial V_i \). Then we may assume that for a real number \( \kappa \) with small absolute value, both the manifolds \( V_i, V_i^G \) and the gradients for \( \mu_i, \mu_i^G \) agree on the complement of \( R \times [\kappa, 0] \), respectively. Here, the direct product structure \( R \times [\kappa, 0] \) is the one generated by the gradient of \( \mu_i \), and we consider that \( V_i^G \) is obtained from \( V_i \) by surgery within \( R \times [\kappa, 0] \). Furthermore, we may assume that this property is satisfied for every pair \( \xi, \xi^G \). By perturbing the gradient of \( h_e \) in \( N \setminus V_i \), we may assume that \( \mathcal{D}_q \cup \{s\} \) does not meet \( R \times [\kappa, 0] \) in both \( V_i \) and \( V_i^G \). Note that this perturbation is independent of the gradients in \( V_i \)'s and does not affect all the previous vanishing results. By Lemma 4.3 again, \( g(p) \) induces a relative 2-cycle of \( (V^G_i, \partial V_i) \), which we now denote by \( \sigma^G \), whose boundary agrees with that of \( \sigma \). We may assume that the relative cycles \( \sigma, \sigma^G \) are both disjoint from \( \{s\} \). By modifying \( \sigma^G \) by adding the boundary of a 3-chain in \( V_i^G \), that does not meet \( \partial V_i \cup \{s\} \), we may assume that \( \sigma \) and \( \sigma^G \) agree on the complement of \( R \times [\kappa, 0] \).

See Figure 8. Under all the above assumptions, \( \langle \sigma, \mathcal{D}_r, \mathcal{D}_q \rangle_{\pi} \) is invariant under the surgery on \( G_i \), and hence the flow-graph \( I \) does not contribute to the alternating sum \( Z_{2n}([N,G]) \) in this case, too.

Next, let \( L \) be a Y-shaped component in \( I \) of index \((1|0,1)\). Suppose that \( L \) has a black vertex in \( V_i \) or \( V^G_i \) and that the input and the output half-edges of \( L \) with critical points of index 1 are labelled by \( k, \ell \) respectively, and the other half-edge is labelled by \( m \). Let \( q, r, s \) be the critical points of index 1,1,0 attached on the half-edge \( k, \ell, m \) respectively. Then \( L \) gives \( \langle \mathcal{D}_q, \mathcal{D}_s, \mathcal{D}_r \rangle_{\pi} \). By the same argument as above, the difference of \( \mathcal{D}_r \) before and after the surgery on \( G_i \) can be squeezed into \( R \times [\kappa, 0] \) for a ribbon graph \( R \). Since \( \mathcal{D}_q \) can be taken so that it does not meet \( R \times [\kappa, 0] \), the intersection \( \langle \mathcal{D}_q, \mathcal{D}_s, \mathcal{D}_r \rangle_{\pi} \) does not change under the surgery, and the flow-graph \( I \) does not contribute to the alternating sum \( Z_{2n}([N,G]) \) in this case.

Finally, notice that the gradients may be perturbed so that every Y-shaped component in any \( I \) with indices \((2,1|0)\) and \((1|0,1)\) vanish simultaneously.

**Lemma 4.7.** Recall that \( W_i = V_i \cup q (-V^G_i) \). Let \( \beta_1^{(i)}, \beta_2^{(i)}, \beta_3^{(i)} \) be the basis of \( H_1(W_i) \) given by the three 1-handles of \( V_i \). The following identity holds.

\[
\sum_{x_1, \ldots, x_{3n}} \prod_{i=1}^{2n} (Q_x^{(i)}(0) - Q_x^{(i)}(1)) \prod_{e=(j,k) \in \text{Edges}(H^r)} \sum_{1 \leq a, b \leq 3} \hat{q}_{kx}(\beta_a^{(j)}, \beta_b^{(k)}) \prod_{\ell=1}^{2n} \alpha_1^{(\ell)}, \alpha_2^{(\ell)}, \alpha_3^{(\ell)}
\]

(4.3)
Proof. We consider the difference \( \# \mathcal{I}_H^i(p_j)(\xi) - \# \mathcal{I}_H^i(p_j)(\xi_{G'}) \) of triple products such that the index of \( H_i^p(p_j) \) defined in the proof of Lemma 4.6 satisfies the condition of Lemma 4.6. By the same reason as in the proof of Lemma 3.7, this difference may not vanish only when there are flow-graphs from \( H_i^p(p_j) \) with the \( i \)-th vertex included in \( V_i \) or \( V_i^{G'} \). There exist 2-chains \( T_m^{(i)}(\xi) \) \((m = 1, 2, 3)\) of the form \( \mathcal{I}_g(x_k)(\xi) \) or \( \mathcal{I}_x(\xi) \cap V_i \) such that the difference \( \# \mathcal{I}_H^i(p_j)(\xi) - \# \mathcal{I}_H^i(p_j)(\xi_{G'}) \) can be written as follows.

\[
\left\langle T_1^{(i)}(\xi_{G}), T_2^{(i)}(\xi_{G}), T_3^{(i)}(\xi_{G}^e), T_4^{(i)}(\xi_{G}^e) \right\rangle \quad (4.4)
\]

This is the count of triple intersections in \( W_i \) with weights. If \( T_m^{(i)} \) is a \( \mathcal{I}_g \) of \( g^{(i)}(x_k) \), then the boundaries of \( T_m^{(i)}(\xi) \) and \( T_m^{(i)}(\xi_{G'}) \) agree on \( \partial V_i \), and then \( T_m^{(i)}(\xi) - T_m^{(i)}(\xi_{G'}) \) is a 2-cycle in \( W_i \) (Lemma 4.3). By Lemma 4.4,

\[
[T_m^{(i)}(\xi) - T_m^{(i)}(\xi_{G'})] = \hat{\mathcal{I}}_x(x_k, c_1^{(i)}) \alpha_1^{(i)} + \hat{\mathcal{I}}_x(x_k, c_2^{(i)}) \alpha_2^{(i)} + \hat{\mathcal{I}}_x(x_k, c_3^{(i)}) \alpha_3^{(i)}
\]

in \( H_2(W_i) \). If \( T_m^{(i)} \) is an ascending manifold, then by closing \( T_m^{(i)}(\xi) - T_m^{(i)}(\xi_{G'}) \) by a ribbon graph as Lemma 3.3, we obtain a 2-cycle in \( W_i \) that is homologous to \( \alpha_1^{(i)} \).

Further, if the ribbon graphs that close the three chains are chosen sufficiently thin, then the intersections between the ribbon graphs in \( \partial V_i \) are those induced from transversal double points between edges. Since we may arrange that there are no triple intersections among three 1-dimensional objects on \( \partial V_i \), we may assume that there are no triple intersection of the 2-chains involving the ribbon graphs after perturbing the 2-chains slightly. This shows that the additions of the ribbon graphs to \( T_m^{(i)}(\xi) - T_m^{(i)}(\xi_{G'}) \) does not affect the value of the triple product (4.4).

Therefore, the triple product (4.4) is a linear combination of terms of the forms \( \langle \alpha_{k_1}^{(i)}, \alpha_{k_2}^{(i)}, \alpha_{k_3}^{(i)} \rangle \) with coefficients given by products of equivariant linking numbers between leaves of \( \mathcal{I}_g \)-graphs.

Let \( e = (j, k) \) be a separated edge of \( H^c \). Consider the following element of \( H_4(W_j \times W_k; \Lambda) \) with untwisted \( A \)-coefficients

\[
L_e = \sum_{1 \leq a, b \leq 3} \hat{\mathcal{I}}_x(x_k, c_1^{(i)}) \alpha_a^{(i)} \alpha_b^{(k)}.
\]

This is represented by the moduli space of flow-graphs from a separated edge whose pairs of black vertices lie in \( W_j \) and \( W_k \), respectively. Let \( \hat{L} \) be the element of \( H_{6n-2}(W_1 \times W_2 \times \cdots \times W_{2n}; \Lambda) \) obtained from \( W_1 \otimes W_2 \otimes \cdots \otimes W_{2n} \) by replacing the \( j \)-th and the \( k \)-th factors with \( L_e \). By the argument above, the following identity holds.

\[
\sum_{x_1, \ldots, x_{2n}} \prod_{i=1}^{2n} (Q_x^{(i)}(0) - Q_x^{(i)}(1)) = \prod_{e \in \text{Edges}(H^c)} \hat{L}_e
\]

The product in the right hand side is the usual intersection form

\[
H_{6n-2}(\prod_{i=1}^{2n} W_i; \Lambda) \otimes \Lambda^{3n} \to H_0(\prod_{i=1}^{2n} W_i; \Lambda \otimes \Lambda^{3n}) = \Lambda \otimes \Lambda^{3n}.
\]

This product can be rewritten as the right hand side of the formula of the lemma.

\( \square \)
Although the sign in the right hand side of (4.3) depends on the graph orientation of \( H^0 \), it is cancelled out by the graph orientation after multiplying (4.3) to a graph. For the details, we refer the reader to [KT, Les1].

The following lemma is immediate.

**Lemma 4.8.** \( \prod_{e=(j,k) \in \text{Edges}(H^0)} \sum_{a,b} \hat{k}_e(\beta_a^{(j)}, \beta_b^{(k)}) \neq 0 \) iff there exists an automorphism \( H \cong \pm \Gamma \) that preserves the labels of vertices.

For an abstract trivalent graph \( \Gamma \), let \( \text{Aut}_v \Gamma \) be the group of automorphisms of \( \Gamma \) that fix all vertices. Let \( \text{Aut}_e \Gamma \) be the group of permutations of vertices of \( \Gamma \) that give automorphism of \( \Gamma \). We have \( |\text{Aut}_v \Gamma| = |\text{Aut}_e \Gamma||\text{Aut}_e \Gamma| \).

**Lemma 4.9.** The following identity folds.

\[
\frac{1}{2^{6n}(2n)!}(3n)! \sum_{H} \sum_{\sigma \in S_{2n}} S_H(\sigma(1), \ldots, \sigma(2n)) = \frac{1}{2^{3n}|\Gamma(\alpha)|}. 
\]

**Proof.** By Assumption 3.5, the value of the right hand side of (4.3) agrees with \( \prod_{e=(j,k) \in \text{Edges}(H^0)} \sum_{a,b} \hat{k}_e(\beta_a^{(j)}, \beta_b^{(k)}) \), whose value is the sum of monomials that correspond to the decorations of \( \Gamma \) obtained by composing edge automorphisms of \( \Gamma \) and a map \( \Gamma \to N \) whose holonomy gives the monomial \( \pi \)-decorated graph \( \Gamma(\alpha) \). Each term becomes the same element in \( S_{2n}(\pi) \) after multiplying it to \( H^0 \). Thus

\[
S_H(1, 2, \ldots, 2n) = \begin{cases} 
|\text{Aut}_v \Gamma||\Gamma(\alpha)| & \text{if there exists an automorphism } H \cong \pm \Gamma \\
0 & \text{otherwise}
\end{cases}
\]

The value of \( S_H(\sigma(1), \ldots, \sigma(2n)) \) is the same as this. Among the permutations \( \sigma \in S_{2n} \) of the set of vertices, there are \( |\text{Aut}_e \Gamma| \) elements satisfying the condition of Lemma 4.8. For a trivalent graph, there are \( \frac{3^3n(2n)!3n!}{|\text{Aut}_e \Gamma|} \) different ways of giving edge-orientations and labellings. Therefore,

\[
\frac{1}{2^{6n}(2n)!}(3n)! \sum_{H} \sum_{\sigma \in S_{2n}} S_H(\sigma(1), \ldots, \sigma(2n)) = \frac{1}{2^{3n}(2n)!3n!}|\text{Aut}_v \Gamma||\Gamma(\alpha)| = \frac{1}{2^{3n}|\Gamma(\alpha)|}.
\]

This completes the proof. \( \square \)

4.3. **Completing the proof of Theorem 2.4.** To complete the proof of Theorem 2.4, we shall prove that the alternating sums of \( \hat{Z}_{2n}(f, (\varepsilon_1\xi_1, \ldots, \varepsilon_{3n}\xi_{3n})) \) and of \( \hat{Z}_{2n}(f, (\xi_1, \ldots, \xi_{3n})) \) agree for all \( \varepsilon_i = \pm 1 \) for the perturbed family of gradients \( \{(\xi^1_G, \ldots, \xi^3_G)\}_{G \subseteq \mathcal{G}} \) considered in §3.2–§4.2.

Flow-graphs for non-vanishing terms in \( \hat{Z}_{2n}(f, (\varepsilon_1\xi_1, \ldots, \varepsilon_{3n}\xi_{3n})) \) are exactly the same as those for \( \hat{Z}_{2n}(f, (\xi_1, \ldots, \xi_{3n})) \). Here, although the orientations of the flow-graphs may change under \( \xi_i \to \varepsilon_i \xi_i \), the orientations of graphs change accordingly and the changes do not affect the product \( \#_H(\xi) : \Gamma \). Thus we need only to show that in such a flow-graph a separated edge of \( \varepsilon_j \xi_j, \varepsilon_j = 1 \), gives the same coefficient as that of \( \varepsilon_j = 1 \). The analogue of Lemma 4.1 for \( \varepsilon_j = -1 \) is \( p = \partial^p g^\alpha(p) \),
where $\partial^*$, $g^*$ are given by the adjoint matrices for $\partial$, $g$, respectively. The analogue of Lemma 4.2 is

$$\ell k_\pi(c, c') = \langle g^*(c), c' \rangle_\pi = \langle c, g(c') \rangle_\pi \pmod{\pi}.$$ 

There are no changes in Lemmas 4.3, 4.4 for $\varepsilon_j = -1$ except that $g(c)$ becomes $g^*(c)$, and there are no other changes in the argument of §4.2 except that $\mathcal{D}$ and $\mathcal{A}$ are exchanged. Hence by Lemma 4.9 we have $\tilde{Z}_{2n}(f, (\varepsilon_1 \xi_1, \ldots, \varepsilon_{3n} \xi_{3n})) = \tilde{Z}_{2n}(f, (\xi_1, \ldots, \xi_{3n})) = \frac{1}{2\pi} \left[ \Gamma(\alpha) \right]$. 

5. The first invariant $\tilde{Z}_2$ for connected sum of $N$ with an integral homology sphere

Let $N = \Sigma^3$ and let $f : M \to N$ be a $\mathbb{Z}_\pi$-homology equivalence that represents an element of $\mathcal{K}(N)$. If we put

$$\lambda_\pi(M) = \tilde{Z}_2(M) - \tilde{Z}_2(N),$$

then by Theorem 2.4, $\lambda_\pi$ is an invariant of $\mathbb{Z}_\pi$-homology equivalences over $\Sigma^3$. If $M = N \# S$ for an integral homology sphere $S$, then $M$ can be represented by an algebraically split framed link in a small ball in $N$, and it determines a $\mathbb{Z}_\pi$-homology equivalence $f : M \to N$ uniquely up to homotopy.

**Theorem 5.1.** For $N = \Sigma^3$ and an integral homology sphere $S$, the following identity holds.

$$\lambda_\pi(N \# S) = \frac{\lambda(S)}{2} [\Theta],$$

where $\lambda$ is the Casson invariant, and $\Theta$ is the 1-decorated theta-graph with vertex-orientation given by blackboard ribbon structure and counter-clockwise order at each trivalent vertex.

**Proof.** Consider the exact sequence

$$0 \to \mathcal{F}_\chi^Y(N) / \mathcal{F}_\chi^Y(N) \to \mathcal{F}_\chi^Y(N) / \mathcal{F}_\chi^Y(N) \to \mathcal{P}_\chi^Y(N) / \mathcal{P}_\chi^Y(N) \to 0.$$ 

Here, since $\mathcal{F}_\chi^Y(N) = \mathcal{F}_\chi^Y(N)$, we have $\mathcal{P}_\chi^Y(N) / \mathcal{P}_\chi^Y(N) = \mathcal{P}_\chi^Y(N) / \mathcal{P}_\chi^Y(N)$, which is one-dimensional and is generated by the class of $\id : N \to N$. The space $\mathcal{F}_\chi^Y(N) / \mathcal{F}_\chi^Y(N) = \text{Ker} \eta$ is generated by elements of the form $[M] - [N]$ ($[M] \in \mathcal{K}(N)$). In particular, if $M = N \# S$, then by Theorem 3.1(2), $\lambda_\pi(M)$ is a linear combination of 1-decorated graphs of degree 2. If $S$ is represented by a framed link $L$ in $\Sigma^3$, then this value agrees with the difference $\tilde{I}_1((\Sigma^3)^L) - \tilde{I}_1(\Sigma^3)$ of the (unframed) Kontsevich–Kuperberg–Thurston invariant ([Ko, KT]), and by a result of [KT], this value agrees with $\frac{\lambda(S)}{2} [\Theta]$ (see also [Les1]).

By Theorem 5.1, $\lambda_\pi$ can be considered as an extension of the Casson invariant up to a constant. This can be computed by modifying $N^k - N$ for a $\pi$-algebraically split framed link $L$ to a linear combination of the images of $\psi_2$ of 2-loop graphs in $\mathcal{F}_\chi^Y(N) / \mathcal{F}_\chi^Y(N)$, by the method given in [GL, §4.3].
6. Further generalizations

6.1. The case $N = S^3$, $π = 0$. The classification of $\mathbb{Z}π$-homology equivalences $f : M → N$ to $N = S^3$ is equivalent to that of $\mathbb{Z}$-homology 3-spheres ($\mathbb{Z}$HS’s). A perturbative invariant for $\mathbb{Z}$HS’s similar to $\tilde{Z}_{2n}$ has been constructed in [Wa1]. The proof of Theorem 3.1 can be applied to $\mathbb{Z}$HS’s by just forgetting the holonomy, and gives a Morse theoretic proof of the isomorphism $\mathcal{A}_{2n}(0) ≅ \mathcal{F}_{2n}^{Y}(S^3)/\mathcal{F}_{2n+1}^{Y}(S^3)$ ([Le, KT]), which is Theorem 1.1 for $N = S^3$.

6.2. The case $N = S^2 × S^1$, $π = \mathbb{Z}$. The construction of the perturbative invariant $\tilde{Z}_{2n}$ in §2 also woks for $\mathbb{Z}π$-homology equivalences $f : M → N$ to $N = S^2 × S^1$. In this case, $\pi_2 N = \mathbb{Z} ≠ 0$ and as in [GL], one should consider finite type invariants for the equivalence relation $\sim$ among $\mathbb{Z}π$-homology equivalences, which is coarser than the diffeomorphism equivalence. Roughly, the relation $\sim$ corresponds to an operation that changes the image in $N$ from a 2-handle in a 4-bordism between $f$ and $id : N → N$, by an element of $π_2 N$. Then Theorems 1.1, 3.1 and 5.1 still hold, with $Λ = Q[u^±1]$ and $\hat{Λ} = Q(t)$. The invariant obtained is a special case of that of [Wa2].

6.3. The case $N = L(p,q)$, $π = \mathbb{Z}_p$. If $N$ is a lens space $L(p,q)$ ($p,q > 0, p ≠ 1$), then $\pi = \mathbb{Z}_p$, $\pi_2 N = 0$ and the perturbative invariant $\tilde{Z}_{2n}$ can be defined as an invariant of $\mathbb{Z}π$-homology equivalences $f : M → N$ in a similar way. In this case, the definition of holonomy must be modified as follows, since $Ω_2(N) = 0$. Choose a Morse function $h : N → \mathbb{R}$ having only one critical point in each index, and choose a Morse–Smale gradient $ξ$ for $h$. As in [Fr], this gives an (oriented) CW structure with one cell $σ_i$ in each dimension $i = 0, 1, 2, 3$ such that $∂σ_2$ covers $σ_1$ $p$ times. Then, for a generic bounded path $γ : J → N$, we define its holonomy $\text{Hol}(γ)$ by $ζ^m$, where $m$ is the intersection number of $γ$ with $σ_2$, and $ζ = e^{2πξ}$. One can see that $\text{Hol}(γ)$ is invariant under a homotopy of $γ$ fixing the endpoints. With this holonomy, the perturbative invariant $\tilde{Z}_{2n}$ is defined in the same way as before. Indeed, the twisted Morse complex $C$ is given by

$\cdots → 0 → Λ^z → Λ^u → Λ^z → Λ^v → 0 → \cdots$

where $qq ≡ 1$ (mod $p$) and $Λ = Q[ζ]$. Then $C$ is acyclic and a combinatorial propagator for $C$ is given by $g(x) = 0$, $g(y) = (1 - ζ^q)^{-1}x$, $g(z) = 0$, $g(w) = (1 - ζ)^{-1}z$. Then Theorems 1.1, 3.1 and 5.1 still hold, with $Λ = Λ = Q[ζ]$.

For example, homotopy equivalence $L(p,q') → L(p,q)$ is obviously a $\mathbb{Z}π$-homology equivalence. It is known that there are non-homeomorphic pair $L(p,q')$, $L(p,q)$ with the same homotopy type, e.g., $L(25, 4)$ and $L(25, 9)$ are homotopy equivalent, non-homeomorphic, and moreover, they cannot be distinguished by the LMO invariant ([BL]). See [BL, Mo] for relevant results.

Problem 6.1. Is there a pair $q' ≠ q$ of positive integers such that $λπ$ is nontrivial for a homotopy equivalence $L(p,q') → L(p,q)$?

6.4. The cases $N = K(G, 1)$. Let $G$ be a free abelian group of rank $p$ and let $M$ be a closed, connected, oriented 3-manifold equipped with a smooth map $f :
$M \to K(G,1) = TP$. Such a pair $(M, f)$ is called a 3-manifold over $K(G,1)$.\footnote{In [Tu], such a pair $(M, f)$ is called a $G$-manifold.} We say that two 3-manifolds over $K(G,1)$, $(M, f)$ and $(M', f')$, are diffeomorphically equivalent if there exists a diffeomorphism $g : M \to M'$ such that $f$ is homotopic to $f' \circ g$. For a 3-manifold $M$ over $K(G,1)$, choose a Morse function $h : M \to \mathbb{R}$ and a Morse–Smale gradient $\xi$, and consider the twisted Morse complex $C = C_*(f; h; \hat{\Lambda})$, $\hat{\Lambda} = Q(\Lambda)$, $\Lambda = Q[t_1^{\pm 1}, \ldots, t_p^{\pm 1}]$. We make the following assumption for the pair $(M, f)$.

**Assumption 6.2.** $H_i(\text{End}_\hat{\Lambda}(C)) = 0$ for $i = 0, 1$.

Under Assumption 6.2, the perturbative invariant $\tilde{Z}_{2n}$ can be defined in a similar way as above, as an invariant of $(M, f)$ under diffeomorphism equivalence. Namely, it follows from Assumption 6.2 that (1): the identity $1 \in \text{End}_\hat{\Lambda}(C)_0$ is nullhomologous,\footnote{This corresponds to the condition that the diagonal $\Delta_M$ in $M \times M$ is nullhomologous (in a twisted complex).} which implies that there exists a combinatorial propagator for $C$. Moreover, (2): for any two combinatorial propagators $g, g'$ for $C$, the cycle $g' - g \in \text{End}_\hat{\Lambda}(C)_1$ is nullhomologous, which implies that there exists $h \in \text{End}_\hat{\Lambda}(C)_2$ such that $g' - g = \partial h - h \partial$. The facts (1) and (2) were the only constraints on the homological properties of $C$ used in the proof of well-definedness of $\tilde{Z}_{2n}$ (see [Wa1, Wa2] for detail).

For general abelian group $G$, one may also define perturbative invariant $\tilde{Z}_{2n}$ for 3-manifolds $(M, f)$ over $K(G,1)$ if holonomy can be defined suitably, and if Assumption 6.2 holds. At present, we do not have finite type invariant that $\tilde{Z}_{2n}$ for this general case fits into. Possibly related results in this direction can be found in [HW].

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**References**


