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TWO CHARACTERIZATIONS OF HOMOGENEOUS HOPF HYPERSURFACES IN NONFLAT COMPLEX SPACE FORMS

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ABSTRACT. In this expository paper we survey two characterizations of homogeneous Hopf hypersurfaces M^{2n-1} in nonflat complex space forms $\widetilde{M}_n(c)$. These characterizations are established by observing some geodesics on M and by investigating the holomorphic distribution T^0M on M, respectively.

1. INTRODUCTION

Standard examples play an important role in geometry. We denote by $\widetilde{M}_n(c)$ a complex $n\,(\geqq 2)$-dimensional complete and simply connected nonflat complex space form of constant holomorphic sectional curvature $c(\neq 0)$, namely a complex projective space $\mathbb{C}P^n(c)(c > 0)$ or a complex hyperbolic space $\mathbb{C}H^n(c)(c < 0)$. In the theory of real hypersurfaces in $\widetilde{M}_n(c)$ it is interesting to investigate geometric properties of *homogeneous* examples. Here, a real hypersurface M^{2n-1} in $\widetilde{M}_n(c)$ is said to be homogeneous if M is an orbit of some subgroup of the isometry group $\mathrm{I}(\widetilde{M}_n(c))$ of the ambient space. Takagi ([19]), Berndt and Tamaru ([8]) classified homogeneous real hypersurfaces in $\mathbb{C}P^n(c)$ and $\mathbb{C}H^n(c)$, respectively. Due to their works we can see that in the case of $c > 0$ every homogeneous real hypersurface is a Hopf hypersurface but in the case of $c < 0$ there exist many non-Hopf hypersurfaces as well as many Hopf hypersurfaces (for the definition of Hopf hypersurfaces see Section 2).

In this context we pay particular attention to homogeneous Hopf hypersurfaces in a nonflat complex space form $\widetilde{M}_n(c)$. Motivated by a fact that the ambient space $\widetilde{M}_n(c)$ has no totally umbilic real hypersurfaces M, we count the number of geodesics on M that are mapped to circles in $\widetilde{M}_n(c)$. We next recall a fact that in $\widetilde{M}_n(c)$ there does not exist a Hopf hypersurface M such that the holomorphic distribution T^0M on M is integrable (for the definition of T^0M see Section 2). We

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characterize all homogeneous Hopf hypersurfaces in a nonflat complex space form from these viewpoints (see Theorems 1 and 2).

2. Basic terminologies and facts

Let M^{2n-1} be a real hypersurface immersed into a nonflat complex space form $\widetilde{M}_n(c)$ through an isometric immersion with a unit normal local vector field $\mathcal{N}$. The Riemannian connections $\widetilde{\nabla}$ of $\widetilde{M}_n(c)$ and ∇ of M are related by the following formulas of Gauss and Weingarten:

$$\widetilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)\mathcal{N}, \tag{2.1}$$

$$\widetilde{\nabla}_X \mathcal{N} = -AX \tag{2.2}$$

for arbitrary vector fields X and Y on M, where g is the Riemannian metric of M induced from the standard metric of the ambient space $\widetilde{M}_n(c)$ and A is the shape operator of M in $\widetilde{M}_n(c)$. An eigenvector of the shape operator A is called a *principal curvature vector* of M in $\widetilde{M}_n(c)$ and an eigenvalue of A is called a *principal curvature* of M in $\widetilde{M}_n(c)$. We set $V_\lambda = \{v \in TM \mid Av = \lambda v\}$ which is called the principal distribution associated to the principal curvature λ.

It is well-known that M has an almost contact metric structure induced from the Kähler structure (J, g) of the ambient space $\widetilde{M}_n(c)$. That is, we have a quadruple (ϕ, ξ, η, g) defined by

$$g(\phi X, Y) = g(JX, Y), \quad \xi = -J\mathcal{N} \quad \text{and} \quad \eta(X) = g(\xi, X) = g(JX, \mathcal{N}).$$

Then they satisfy

$$\phi^2 X = -X + \eta(X)\xi, \ \eta(\xi) = 1 \quad \text{and} \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for all vectors $X, Y \in TM$. It is known that these equations imply that $\phi\xi = 0$ and $\eta(\phi X) = 0$. In the following, we call ϕ, ξ and η *the structure tensor*, *the characteristic vector* and *the contact form* on M, respectively.

It follows from (2.1), (2.2), $\widetilde{\nabla} J = 0$ and $JX = \phi X + \eta(X)\mathcal{N}$ that

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \tag{2.3}$$

$$\nabla_X \xi = \phi AX. \tag{2.4}$$

Denoting the curvature tensor of M by R, we have the equation of Gauss given by

$$\begin{aligned} g((R(X,Y)Z, W) = (c/4)\{&g(Y,Z)g(X,W) \\ &- g(X,Z)g(Y,W) + g(\phi Y, Z)g(\phi X, W) - g(\phi X, Z)g(\phi Y, W) \\ &- 2g(\phi X, Y)g(\phi Z, W)\} \\ &+ g(AY, Z)g(AX, W) - g(AX, Z)g(AY, W). \end{aligned} \tag{2.5}$$

The following is called the equation of Codazzi.

$$(\nabla_X A)Y - (\nabla_Y A)X = (c/4)(\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi), \tag{2.6}$$

Let K be the sectional curvature of M. That is, K is defined by $K(X,Y) = g(R(X,Y)Y,X)$, where X and Y are orthonormal vectors on M. Then it follows from (2.5) that

$$K(X,Y) = (c/4)\big(1 + 3g(\phi X,Y)^2\big) + g(AX,X)g(AY,Y) - g(AX,Y)^2. \tag{2.7}$$

We usually call M a *Hopf hypersurface* if the characteristic vector ξ of M is a principal curvature vector at each point of M. The following lemma clarifies fundamental properties of principal curvatures of a Hopf hypersurface M in $\widetilde{M}_n(c)$.

Lemma 1 ([12, 15])**.** *Let M be a Hopf hypsurface of a nonflat complex space form $\widetilde{M}_n(c), n \geqq 2$. Then the following hold.*

1. *If a nonzero vector $v \in TM$ orthogonal to ξ satisfies $Av = \lambda v$, then $(2\lambda - \delta)A\phi v = (\delta\lambda + (c/2))\phi v$, where δ is the principal curvature associated with ξ. In particular, when $c > 0$, we have $A\phi v = \big((\delta\lambda + (c/2))/(2\lambda - \delta)\big)\phi v$.*
2. *The principal curvature δ associated with ξ is locally constant.*

The discussion in the proof of Lemma 1 gives the following:

Lemma 2 ([18])**.** *There exist no real hypersurfaces with $\phi A + A\phi = 0$ in a nonflat complex space form $\widetilde{M}_n(c), n \geqq 2$.*

By virtue of Lemma 2 we obtain the following fundamental property of all Hopf hypersurfaces in a nonflat complex space form.

Proposition 1 ([10])**.** *For every Hopf hypersurface M in a nonflat complex space form $\widetilde{M}_n(c), n \geqq 2$, the holomorphic distribution $T^0M = \{X \in TM | \eta(X) = 0\}$ on M is not integrable.*

We see easily that a real hypersurface M is a Hopf hypersurface if and only if every integral curve of ξ is a geodesic on M (see (2.4)). The following gives another characterization of all Hopf hypersurfaces in a nonflat complex space form.

Proposition 2 ([3])**.** *For a real hypersurface M in a nonflat complex space form $\widetilde{M}_n(c), n \geqq 2$ the following two conditions are mutually equivalent.*

1. *M is a Hopf hypersurface in $\widetilde{M}_n(c)$.*
2. *At each point $x \in M$, if we take a totally geodesic complex curve $\widetilde{M}_1(c)$ in $\widetilde{M}_n(c)$ through x whose tangent space $T_x\widetilde{M}_1(c)$ is the complex one dimensional linear subspace of $T_x\widetilde{M}_n(c)$ spanned by ξ_x, then the normal section $N_x = M \cap \widetilde{M}_1(c)$ given by $\widetilde{M}_1(c)$ is the integral curve through the point x of the characteristic vector field ξ of M.*

3. Classification of homogeneous real hypersurfaces in $\mathbb{C}P^n(c)$

Takagi ([19]) classified homogeneous real hypersurfaces in $\mathbb{C}P^n(c)$, namely they are orbits under analytic subgroups of the projective unitary group $PU(n+1)$. We explain his results briefly.

Theorem A ([19, 20])**.** *All homogeneous real hypersurfaces of* $\mathbb{C}P^n(c), n \geqq 2$ *are Hopf hypersurfaces with constant principal curvatures in this ambient space. They are said to be of types* $(\mathrm{A}_1), (\mathrm{A}_2), (B), (C), (D)$ *and* (E)*. The numbers of distinct constant principal curvatures of these Hopf hypersurfaces are* $2, 3, 3, 5, 5, 5$*, respectively. Every homogeneous real hypersurface in* $\mathbb{C}P^n(c)$ *is constructed by using an effective Hermitian orthogonal symmetric Lie algebra of ran two.*

Sketch of proof. Without loss of generality we set $c = 4$. We denote by $(\mathfrak{u}, \theta)$ an effective orthogonal symmetric Lie algebra of compact type of rank two, where $\mathfrak{u}$ is a compact semisimple Lie algebra and θ is an involutive automorphism of $\mathfrak{u}$ (see [11]). Let $\mathfrak{u} = \mathfrak{t} + \mathfrak{p}$ be the decomposition of $\mathfrak{u}$ into the eigenspaces of θ for the eigenvalues $+1$ and -1, respectively. Then $\mathfrak{t}$ and $\mathfrak{p}$ satisfy $[\mathfrak{t}, \mathfrak{t}] \subset \mathfrak{t}, [\mathfrak{t}, \mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{t}$. For the Killing form B of $\mathfrak{u}$ we define a positive definite inner product $\langle\ ,\ \rangle$ on $\mathfrak{p}$ by $\langle X, Y\rangle = -B(X, Y)$ for $X, Y \in \mathfrak{p}$. Let K be the analytic subgroup of the group of inner automorphisms of $\mathfrak{u}$ with Lie algebra $\mathrm{ad}(\mathfrak{t})$. Then $K(\mathfrak{p}) = \mathfrak{p}$ and K is an orthogonal tranformation group on $\mathfrak{p}$ with respect to $\langle\ ,\ \rangle$. We define a representation ρ of K on $\mathfrak{p}$ by $\rho(k) = k|\mathfrak{p}$ for $k \in K$. The differentiation ρ_* of ρ is an automorphism of $\mathfrak{t}$ into the Lie algebra of the orthogonal group of $\mathfrak{p}$ and satisfies $(\rho_* X)Y = [X, Y]$ for all $X \in \mathfrak{t}$ and all $Y \in \mathfrak{p}$.

We denote by S the unit hypersphere in $\mathfrak{p}$ centered at the origin and A a regular element of $\mathfrak{p}$ in S. Then the orbit $N = \rho(K)A$ of A under $\rho(K)$ is a hypersurface in the ambient sphere S. It is known that there exists an element Z_0 in the center of $\mathfrak{t}$ such that ([11])

$$(\rho_* Z_0)^2 = -1, \quad \langle(\rho_* Z_0)X, (\rho_* Z_0)Y\rangle = \langle X, Y\rangle \text{ for } X, Y \in \mathfrak{p}.$$

Thus we may regard $\mathfrak{p}$ as a complex vector space $\mathbb{C}^{n+1}$ with complex structure $J = \rho_* Z_0$ and Hermitian inner product $\langle\ ,\ \rangle$, where $\dim \mathfrak{p} = 2(n+1)$. We denote by π the Hopf fibration of $\mathbb{C}^{n+1} - \{0\}$ onto $\mathbb{C}P^n(4)$ and V a vector field on $\mathfrak{p}$ which is defined by $V_X = JX$ for $X \in \mathfrak{p}$. Since the one-parameter subgroup $\rho(\exp \mathbb{R}Z_0)$ of $\rho(K)$ induces V and leaves N invariant, the image $M = \pi(N)$ becomes a homogeneous real hypersurface of $\mathbb{C}P^n(4)$. Indeed, the group $G = \rho(K)/\rho(C_0)$ is a compact analytic subgroup of $PU(n+1) = U(n+1)/\rho(C_0)$ which acts on M transtively as isometries of M, where C_0 is the subgroup of K generated by Z_0. We call here this manifold M a standard real hypersurface of $\mathbb{C}P^n(4)$. Conversely, we can say that every homogeneous real hypersurface in $\mathbb{C}P^n(4)$ is locally congruent to one of standard real hypersurfaces (see Lemmas 2.2 and 2.3 in [19]). Therefore by virtue of a complete classification theorem of effective Hermitian orthogonal symmetric Lie algebras we can obtain a classification theorem of homogeneous real hypersurfaces in $\mathbb{C}P^n(4)$ (see [19]). Needless to say, all principal curvatures of homogeneous real hypersurfaces are constant. In consideration of Theorem 2 in [20] and the work of Araki ([5]) we can compute principal curvatures of all homogeneous real hypersurfaces in $\mathbb{C}P^n(4)$. □

Remark 1. The discussion in the proof of Theorem A shows that a real hypersurface M is homogeneous in $\mathbb{C}P^n(c)$ if and only if $\pi^{-1}(M)$ is homogeneous in a $(2n+1)$-dimensional sphere $S^{2n+1}(c/4)$ of constant sectional curvature $c/4$ through the Hopf fibration $\pi : S^{2n+1}(c/4) \to \mathbb{C}P^n(c)$, that is $\pi^{-1}(M)$ is an orbit of a subgroup of $SO(2n+2)$ which is the full isometry group of $S^{2n+1}(c/4)$.

Theorem A is written in an algebraic style. Using the works of Cecil and Ryan ([9]) and Kimura ([13]), we can rewrite geometrically Theorem A. The following list is the so-called *Takagi's list.*

In $\mathbb{C}P^n(c)$ $(n \geqq 2)$, a homogeneous real hypersurface is locally congruent to one of the following Hopf hypersurfaces all of whose principal curvatures are constant:

(A_1) A geodesic sphere of radius r, where $0 < r < \pi/\sqrt{c}$;
(A_2) A tube of radius r around a totally geodesic $\mathbb{C}P^\ell(c)$ $(1 \leqq \ell \leqq n-2)$, where $0 < r < \pi/\sqrt{c}$;
(B) A tube of radius r around a complex hyperquadric $\mathbb{C}Q^{n-1}$, where $0 < r < \pi/(2\sqrt{c}\,)$;
(C) A tube of radius r around the Segre embedding of $\mathbb{C}P^1(c) \times \mathbb{C}P^{(n-1)/2}(c)$, where $0 < r < \pi/(2\sqrt{c}\,)$ and n $(\geqq 5)$ is odd;
(D) A tube of radius r around the Plüker embedding of a complex Grassmannian $\mathbb{C}G_{2,5}$, where $0 < r < \pi/(2\sqrt{c}\,)$ and $n = 9$;
(E) A tube of radius r around a Hermitian symmetric space SO(10)/U(5), where $0 < r < \pi/(2\sqrt{c}\,)$ and $n = 15$.

These real hypersurfaces are said to be of types (A_1), (A_2), (B), (C), (D) and (E). Unifying real hypersurfaces of types (A_1) and (A_2), we call them hypersurfaces of type (A). The numbers of distinct principal curvatures of these real hypersurfaces are $2, 3, 3, 5, 5, 5$, respectively. The principal curvatures of these real hypersurfaces in $\mathbb{C}P^n(c)$ are given as follows:

	(A_1)	(A_2)	(B)	(C, D, E)
λ_1	$\frac{\sqrt{c}}{2}\cot\bigl(\frac{\sqrt{c}}{2}r\bigr)$	$\frac{\sqrt{c}}{2}\cot\bigl(\frac{\sqrt{c}}{2}r\bigr)$	$\frac{\sqrt{c}}{2}\cot\bigl(\frac{\sqrt{c}}{2}r-\frac{\pi}{4}\bigr)$	$\frac{\sqrt{c}}{2}\cot\bigl(\frac{\sqrt{c}}{2}r-\frac{\pi}{4}\bigr)$
λ_2	—	$-\frac{\sqrt{c}}{2}\tan\bigl(\frac{\sqrt{c}}{2}r\bigr)$	$\frac{\sqrt{c}}{2}\cot\bigl(\frac{\sqrt{c}}{2}r+\frac{\pi}{4}\bigr)$	$\frac{\sqrt{c}}{2}\cot\bigl(\frac{\sqrt{c}}{2}r+\frac{\pi}{4}\bigr)$
λ_3	—	—	—	$\frac{\sqrt{c}}{2}\cot\bigl(\frac{\sqrt{c}}{2}r\bigr)$
λ_4	—	—	—	$-\frac{\sqrt{c}}{2}\tan\bigl(\frac{\sqrt{c}}{2}r\bigr)$
δ	$\sqrt{c}\cot(\sqrt{c}\,r)$	$\sqrt{c}\cot(\sqrt{c}\,r)$	$\sqrt{c}\cot(\sqrt{c}\,r)$	$\sqrt{c}\cot(\sqrt{c}\,r)$

One should notice that in $\mathbb{C}P^n(c)$ a tube of radius r $(0 < r < \pi/\sqrt{c}\,)$ around a totally geodesic $\mathbb{C}P^\ell(c)$ $(0 \leqq \ell \leqq n-1)$ is congruent to a tube of radius $\bigl((\pi/\sqrt{c}\,)-r\bigr)$ around a totally geodesic $\mathbb{C}P^{n-\ell-1}(c)$. So, in particular by setting $\ell = 0$ we know that a geodesic sphere $G(r)$ of radius r $(0 < r < \pi/\sqrt{c}\,)$ in $\mathbb{C}P^n(c)$ is congruent to a tube of radius $\bigl((\pi/\sqrt{c}\,) - r\bigr)$ around a totally geodesic hypersurface $\mathbb{C}P^{n-1}(c)$. Then we can see that all homogeneous real hypersurfaces of $\mathbb{C}P^n(c)$ are realized as tubes of constant radius around compact Hermitian symmetric spaces of rank 1 or

2. The multiplicities of these principal curvatures are given as follows (cf. [20]):

	(A_1)	(A_2)	(B)	(C)	(D)	(E)
$m(\lambda_1)$	$2n-2$	$2n-2\ell-2$	$n-1$	2	4	6
$m(\lambda_2)$	—	2ℓ	$n-1$	2	4	6
$m(\lambda_3)$	—	—	—	$n-3$	4	8
$m(\lambda_4)$	—	—	—	$n-3$	4	8
$m(\delta)$	1	1	1	1	1	1

4. Classification of homogeneous real hypersurfaces in $\mathbb{C}H^n(c)$

We next study homogeneous real hypersurfaces in an n-dimensional complex hyperbolic space $\mathbb{C}H^n(c)$ of constant holomorphic sectional curvature $c < 0$. For this purpose, we recall the works of Montiel ([17]), Berndt ([6]) and Berndt, Tamaru ([8]), which are significant in the theory of real hypersurfaces in $\mathbb{C}H^n(c)$.

Montiel ([17]) classified real hypersurfaces in $\mathbb{C}H^n(c)$ $(n \geqq 3)$ with at most two distinct principal curvatures at its each point.

Theorem B ([17])**.** *Let M be a connected real hypersurface in $\mathbb{C}H^n(c)$ $(n \geqq 3)$ with at most two distinct principal curvatures at its each point. Then M is locally congruent to one of the following*:

(A_0) *A horosphere in $\mathbb{C}H^n(c)$;*
$(A_{1,0})$ *A geodesic sphere of radius r $(0 < r < \infty)$;*
$(A_{1,1})$ *A tube of radius r around a totally geodesic $\mathbb{C}H^{n-1}(c)$, where $0 < r < \infty$;*
(B) *A tube of radius $r = (1/\sqrt{|c|}\,)\log_e(2+\sqrt{3}\,)$ around a totally real totally geodesic $\mathbb{R}H^n(c/4)$.*

On the other hand, Berndt ([6]) classified Hopf hypersurfaces with constant principal curvatures in $\mathbb{C}H^n(c)$.

Theorem C ([6])**.** *Let M be a connected Hopf hypersurface all of whose principal curvatures are constant in $\mathbb{C}H^n(c)$ $(n \geqq 2)$. Then M is locally congruent to one of the following*:

(A_0) *A horosphere in $\mathbb{C}H^n(c)$;*
$(A_{1,0})$ *A geodesic sphere of radius r $(0 < r < \infty)$;*
$(A_{1,1})$ *A tube of radius r around a totally geodesic $\mathbb{C}H^{n-1}(c)$, where $0 < r < \infty$;*
(A_2) *A tube of radius r around a totally geodesic $\mathbb{C}H^{\ell}(c)$ $(1 \leqq \ell \leqq n-2)$, where $0 < r < \infty$;*
(B) *A tube of radius r around a totally real totally geodesic $\mathbb{R}H^n(c/4)$, where $0 < r < \infty$.*

We remark that the list in Theorem C contains that of Theorem B completely. In this paper, we call the list in Theorem C *Montiel-Berndt's list*. In [18], this list is called Montiel's list. The real hypersurfaces in Theorems B and C are said to be of types (A_0), (A_1), (A_1), (A_2) and (B). Here, type (A_1) means either type ($A_{1,0}$) or type ($A_{1,1}$). Unifying real hypersurfaces of types (A_0), (A_1) and (A_2), we call them hypersurfaces of type (A). A real hypersurface of type (B) with radius $r = (1/\sqrt{|c|}\,)\log_e(2+\sqrt{3}\,)$ has two distinct constant principal curvatures $\lambda_1 = \delta = \sqrt{3|c|}/2$ and $\lambda_2 = \sqrt{|c|}/(2\sqrt{3}\,)$. Except for this real hypersurface, the numbers of distinct principal curvatures of Hopf hypersurfaces with constant principal curvatures are $2, 2, 2, 3, 3$, respectively. The principal curvatures of these real hypersurfaces in $\mathbb{C}H^n(c)$ are given as follows (see [6]):

	(A_0)	($A_{1,0}$)	($A_{1,1}$)	(A_2)	(B)
λ_1	$\frac{\sqrt{\lvert c\rvert}}{2}$	$\frac{\sqrt{\lvert c\rvert}}{2}\coth(\frac{\sqrt{\lvert c\rvert}}{2}r)$	$\frac{\sqrt{\lvert c\rvert}}{2}\tanh(\frac{\sqrt{\lvert c\rvert}}{2}r)$	$\frac{\sqrt{\lvert c\rvert}}{2}\coth(\frac{\sqrt{\lvert c\rvert}}{2}r)$	$\frac{\sqrt{\lvert c\rvert}}{2}\coth(\frac{\sqrt{\lvert c\rvert}}{2}r)$
λ_2	—	—	—	$\frac{\sqrt{\lvert c\rvert}}{2}\tanh(\frac{\sqrt{\lvert c\rvert}}{2}r)$	$\frac{\sqrt{\lvert c\rvert}}{2}\tanh(\frac{\sqrt{\lvert c\rvert}}{2}r)$
δ	$\sqrt{\lvert c\rvert}$	$\sqrt{\lvert c\rvert}\coth(\sqrt{\lvert c\rvert}\,r)$	$\sqrt{\lvert c\rvert}\coth(\sqrt{\lvert c\rvert}\,r)$	$\sqrt{\lvert c\rvert}\coth(\sqrt{\lvert c\rvert}\,r)$	$\sqrt{\lvert c\rvert}\tanh(\sqrt{\lvert c\rvert}\,r)$

The multiplicities of these principal curvatures are given as follows (see [6]):

	(A_0)	($A_{1,0}$)	($A_{1,1}$)	(A_2)	(B)
$m(\lambda_1)$	$2n-2$	$2n-2$	$2n-2$	$2n-2\ell-2$	$n-1$
$m(\lambda_2)$	—	—	—	2ℓ	$n-1$
$m(\delta)$	1	1	1	1	1

Berndt and Tamaru ([8]) classified all homogeneous real hypersurfaces in $\mathbb{C}H^n(c)$. The following theorem shows that there exist many homogeneous real hypersurfaces which are *not* Hopf hypersurfaces in $\mathbb{C}H^n(c)$.

Theorem D ([8]). *Let M be a homogeneous real hypersurface in $\mathbb{C}H^n(c), n \geqq 2$. Then M is locally congruent to one of the following*:

(A_0) *A horosphere in $\mathbb{C}H^n(c)$;*
($A_{1,0}$) *A geodesic sphere of radius r, where $0 < r < \infty$;*
($A_{1,1}$) *A tube of radius r around a totally geodesic $\mathbb{C}H^{n-1}(c)$, where $0 < r < \infty$;*
(A_2) *A tube of radius r around a totally geodesic $\mathbb{C}H^{\ell}(c)$ $(1 \leqq \ell \leqq n-2)$, where $0 < r < \infty$;*
(B) *A tube of radius r around a totally real totally geodesic $\mathbb{R}H^n(c/4)$, where $0 < r < \infty$;*
(S) *The minimal ruled real hypersurface S determined by a horocycle in a totally geodesic $\mathbb{R}H^2(c/4)$ in $\mathbb{C}H^n(c)$, or an equidistant hypersurface from S at distance r, where $0 < r < \infty$;*
(W_1) *A tube of radius r around the minimal ruled submanifold W^{2n-k} with $k \in \{2, \dots, n-1\}$, where $0 < r < \infty$;*
(W_2) *A tube of radius r around the minimal ruled submanifold W^{2n-k}_{φ} for some $\varphi \in (0, \pi/\sqrt{|c|}\,)$ and $k \in \{2, \dots, n-1\}$, where k is even and where $0 < r < \infty$;*

In this paper, we call the list in Theorem D *Berndt-Tamaru's list.* Note that all examples of (S), (W_1), (W_2) in Berndt-Tamaru's list are non-Hopf hypersurfaces. The following three propositions give information on the principal curvatures of homogeneous non-Hopf hypersurfaces in $\mathbb{C}H^n(c)$.

Proposition A ([7]). (1) *When M is congruent to the homogeneous minimal ruled real hypersurface S in $\mathbb{C}H^n(c)$, M has three distinct constant principal curvatures $\lambda_1 = \sqrt{|c|}/2$ with multiplicity 1, $\lambda_2 = -\sqrt{|c|}/2$ with multiplicity 1 and $\lambda_3 = 0$ with multiplicity $2n-3$.*

(2) *When M is congruent to an equidistant hypersurface from S at distance $r\,(0 < r < \infty)$ in $\mathbb{C}H^n(c)$, M has the following three distinct constant principal curvatures λ_1 with multiplicity 1, λ_2 with multiplicity 1 and λ_3 with multiplicity $2n-3$:*

$$\begin{cases} \lambda_1 = \dfrac{3\sqrt{|c|}}{4}\tanh\dfrac{\sqrt{|c|}\,r}{2} + \dfrac{\sqrt{|c|}}{2}\sqrt{1-\dfrac{3}{4}\tanh^2\dfrac{\sqrt{|c|}\,r}{2}}\,, \\ \lambda_2 = \dfrac{3\sqrt{|c|}}{4}\tanh\dfrac{\sqrt{|c|}\,r}{2} - \dfrac{\sqrt{|c|}}{2}\sqrt{1-\dfrac{3}{4}\tanh^2\dfrac{\sqrt{|c|}\,r}{2}}\,, \\ \lambda_3 = \dfrac{\sqrt{|c|}}{2}\tanh\dfrac{\sqrt{|c|}\,r}{2}. \end{cases}$$

Proposition B ([7]). (1) *When M is congruent to a homogeneous real hypersurface of type (W_1) with radius $r \neq (1/\sqrt{|c|}\,)\log_e(2+\sqrt{3}\,)$, M has the following four distinct constant principal curvatures λ_1 with multiplicity 1, λ_2 with multiplicity 1, λ_3 with multiplicity $2n-k-2$ and λ_4 with multiplicity $k-1$:*

$$\begin{cases} \lambda_1 = \dfrac{3\sqrt{|c|}}{4}\tanh\dfrac{\sqrt{|c|}\,r}{2} - \dfrac{\sqrt{|c|}}{2}\sqrt{1-\dfrac{3}{4}\tanh^2\dfrac{\sqrt{|c|}\,r}{2}}\,, \\ \lambda_2 = \dfrac{3\sqrt{|c|}}{4}\tanh\dfrac{\sqrt{|c|}\,r}{2} + \dfrac{\sqrt{|c|}}{2}\sqrt{1-\dfrac{3}{4}\tanh^2\dfrac{\sqrt{|c|}\,r}{2}}\,, \\ \lambda_3 = \dfrac{\sqrt{|c|}}{2}\tanh\dfrac{\sqrt{|c|}\,r}{2}, \\ \lambda_4 == \dfrac{\sqrt{|c|}}{2}\coth\dfrac{\sqrt{|c|}\,r}{2}. \end{cases}$$

(2) *When M is congruent to a homogeneous real hypersurface of type (W_1) with radius $r = (1/\sqrt{|c|}\,)\log_e(2+\sqrt{3}\,)$, M has three distinct constant principal curvatures $\lambda_1 = 0$ with multiplicity 1, $\lambda_2 = \lambda_4 = \sqrt{3|c|}/2$ with multiplicity k and $\lambda_3 = \sqrt{|c|}/(2\sqrt{3}\,)$ with multiplicity $2n-k-2$.*

Proposition C ([7]). (1) *When M is congruent to a homogeneous real hypersurface of type (W_2) with $k \neq 2$, M has the following five distinct constant* *principal curvatures* λ_1 *with multiplicity* 1, λ_2 *with multiplicity* 1, λ_3 *with multiplicity* 1, λ_4 *with multiplicity* $2n-k-2$ *and* λ_5 *with multiplicity* $k-2$:

$$\lambda_i = -\frac{\sqrt{|c|}}{6}\Big(\coth\frac{\tilde{r}}{2}\Big(u^i_{\tilde{r},\varphi} + \frac{1}{u^i_{\tilde{r},\phi}}\Big) - \mathrm{cseh}\frac{\tilde{r}}{2}\mathrm{sech}\frac{\tilde{r}}{2} - 4\tanh\frac{\tilde{r}}{2}\Big)$$

for $i = 1, 2, 3$,

$$\lambda_4 = \frac{\sqrt{|c|}}{2}\tanh\frac{\sqrt{|c|}\,r}{2},\ \lambda_5 = \frac{\sqrt{|c|}}{2}\coth\frac{\sqrt{|c|}\,r}{2}.$$

Here, $\tilde{r} = \sqrt{|c|}\,r$, k *is even with* $2 \leqq k \leqq n-1$ *and the number* $u^i_{\tilde{r},\varphi}$ *is the* i*-th cubic root of* $(\beta_{\tilde{r},\varphi} + \sqrt{\beta^2_{\tilde{r},\varphi} - 4}\,)/2$, *where* $\beta_{\tilde{r},\varphi} = 27\sin^2(\varphi)\tanh^2(\tilde{r}/2)\cdot \mathrm{sech}^4(\tilde{r}/2) - 2$ *and* $0 < \varphi < \pi/2$.

(2) *When* M *is congruent to a homogeneous real hypersurface of type* (W_2) *with* $k = 2$, M *has the same four distinct constant principal curvatures* $\lambda_1, \lambda_2, \lambda_3$ *and* λ_4 *as those in the above case.*

5. Characterizations of all homogeneous Hopf hypersurfaces in nonflat complex space forms

We first characterize all homogeneous Hopf hypersurfaces in a nonflat complex space form by using the notion of circles in Riemannian geometry.

A smooth real curve $\gamma = \gamma(s)$ $(s \in I)$ in a Riemannian manifold M with a Riemmanian metric g is called a *circle* of curvature k if the ordinary differential equations $\nabla_{\dot\gamma}\dot\gamma = kY_s$ and $\nabla_{\dot\gamma}Y_s = -k\dot\gamma$ hold for each $s \in I$, where $\nabla_{\dot\gamma}$ is the covariant differentiation along γ with respect to the Riemannian connection ∇ of M and k is a nonnegative constant. A circle of null curvature is nothing but a geodesic. The definition of circles can be rewritten as follows: A smooth real curve $\gamma = \gamma(s)$ $(s \in I)$ in a Riemannian manifold M is called a circle if it satisfies the ordinary differential equation

$$\nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma + g(\nabla_{\dot\gamma}\dot\gamma, \nabla_{\dot\gamma}\dot\gamma)\dot\gamma = 0. \tag{5.1}$$

The following is fundamental.

Proposition 3. *For a connected hypersurface* M^n *isometrically immersed into a Riemannian manifold* $\widetilde{M}^{n+1}$, *the following three conditions are mutually equivalent*:

(1) *Every geodesic* γ *on* M^n *is mapped to a circle in* $\widetilde{M}^{n+1}$;
(2) *Every geodesic* γ *on* M^n *is mapped to a circle of the same curvature which is independent of the choice of* γ *in* $\widetilde{M}^{n+1}$;
(3) M^n *is totally umbilic in* $\widetilde{M}^{n+1}$ *and* M^n *has constant mean curvature, namely* Trace A *is constant on* M^n, *where* A *is the shape operator of* M^n *in* $\widetilde{M}^{n+1}$.

Proof. We suppose Condition (1). Then, from (5.1) every geodesic γ of M^n, considered as a curve in the ambient space $\widetilde{M}^{n+1}$, satisfies the following ordinary differential equation:

$$\widetilde\nabla_{\dot\gamma}\widetilde\nabla_{\dot\gamma}\dot\gamma + g(\widetilde\nabla_{\dot\gamma}\dot\gamma, \widetilde\nabla_{\dot\gamma}\dot\gamma)\dot\gamma = 0. \tag{5.2}$$

On the other hand, in consideration of (2.1) and (2.2) for the hypersurface M^n in $\widetilde{M}^{n+1}$, we can rewrite (5.2) as follows:

$$-g(A\dot{\gamma},\dot{\gamma})A\dot{\gamma} + g(A\dot{\gamma},\dot{\gamma})^2\dot{\gamma} + g((\nabla_{\dot{\gamma}}A)\dot{\gamma},\dot{\gamma})\mathcal{N} = 0. \tag{5.3}$$

Hence, taking the tangential component and the normal component of (5.3) for the hypersurace M^n in $\widetilde{M}^{n+1}$, we obtain

$$g(A\dot{\gamma},\dot{\gamma})A\dot{\gamma} = g(A\dot{\gamma},\dot{\gamma})^2\dot{\gamma} \quad \text{and} \quad g((\nabla_{\dot{\gamma}}A)\dot{\gamma},\dot{\gamma}) = 0 \tag{5.4}$$

for each geodesic γ on M^n. Equation (5.4) means that

$$g(AX,X)AX = g(AX,X)^2X \quad \text{and} \quad g((\nabla_X A)X,X) = 0 \tag{5.5}$$

for all $X \in TM$ with $\|X\| = 1$. Note that the former equation in (5.5) means

$$g(AX,X)g(AX,Y) = 0 \tag{5.6}$$

for each pair of orthonormal vectors X and Y on M, which is equivalent to saying that

$$g(A_pX,X)^2 \text{ is constant at each point } p \in M \tag{5.7}$$

for every unit vector $X \in T_pM$.

Indeed, let $f : S^{n-1}(1)(\subset \mathbb{R}^n) \to \mathbb{R}$ be the differential function on a subset $S^{n-1}(1) \cong \{u \in T_pM \| \|u\| = 1\}$ defined by $f(u) = g(A_pu,u)^2$, where A_p is the shape operator of M in $\widetilde{M}^{n+1}$ at the point $p \in M$. If v is a vector tangent to $S^{n-1}(1)$ at u (hence $u \perp v$), we find $v(f) = 4g(A_pu,u)g(A_pu,v) = 0$ by (5.6). Thus f is a constant function on $S^{n-1}(1)$.

Then we can set $\lambda^2(p) = g(AX,X)^2$ for each unit vector $X \in T_pM$ with $\lambda(p) \geqq 0$ at every point $p \in M$. When M^n is not totally geodesic in $\widetilde{M}^{n+1}$, there exists a point $x \in M$ with $\lambda(x) > 0$. Then the continuity of the function λ shows that there exists some open neighborhood U_x of the point x such that $\lambda > 0$ on U_x. We here choose a local field of orthonormal frames $e_1, \ldots, e_n$ on U_x in such a way that $Ae_i = \lambda_i e_i$ $(1 \leqq i \leqq n)$. Hence, from (5.7) we see that $\lambda_1^2 = \ldots = \lambda_n^2 = \lambda^2$. In this case, we suppose that there exist an orthonormal pair of vectors e_i and e_j such that $Ae_i = \lambda e_i$ and $Ae_j = -\lambda e_j$. Then we find that

$$g(A(e_i+e_j)/\sqrt{2}, (e_i+e_j)/\sqrt{2}\,) = 0,$$

which is a contradiction. So, we know that $Ae_i = \lambda e_i$ $(1 \leqq i \leqq n)$, which shows that every point $y \in U_x$ is an umbilic point. Thus we can see that M^n is totally umbilic in $\widetilde{M}^{n+1}$. Furthermore, the latter equation in (5.5) yields that the function λ is constant on M. Therefore we get Conditions (2) and (3) in our proposition.

By virtue of the above argument in the proof of our proposition we can see that each of Conditions (2) and (3) implies Condition (1). $\square$

As an immediate consequence of Proposition 3 we get

Lemma 3. *Let M^n be a hypersurface isometrically immersed into a Riemannian manifold $\widetilde{M}^{n+1}$. If a geodesic $\gamma = \gamma(s)$ $(s \in I)$ on M is mapped to a circle of* *positive curvature k, then the shape operator A of M^n in $\widetilde{M}^{n+1}$ satisfies either $A\dot\gamma(s) = k\dot\gamma(s)$ for all $s \in I$ or $A\dot\gamma(s) = -k\dot\gamma(s)$ for all $s \in I$.*

In view of a fact that $\widetilde{M}_n(c)$ has no totally umbilic real hypersurfaces and Proposition 3 we see that there exist no real hypersurfaces all of whose geodesics are mapped to circles in a nonflat complex space form. By weakening this condition we obtain the following characterization of all homogeneous Hopf hypersurfaces in $\widetilde{M}_n(c)$.

Theorem 1 ([1, 10])**.** *Let M^{2n-1} be a connected real hypersurface in a nonflat complex space form $\widetilde{M}_n(c), n \geqq 2$. Then M is locally congruent to a homogeneous Hopf hypersurface if and only if there exist orthonormal vectors $v_1, \dots, v_{2n-2}$ orthogonal to the characteristic vector ξ at each point p of M such that all geodesics $\gamma_i = \gamma_i(s)$ $(1 \leqq i \leqq 2n-2)$ on M with $\gamma_i(0) = p$ and $\dot\gamma_i(0) = v_i$ are mapped to circles of positive curvature in $\widetilde{M}_n(c)$.*

Proof. We first prove the "only if" part.

Case of $c > 0$. Let M be of type (A_1). Then M has two distinct constant principal curvatures $\delta = \sqrt{c}\,\cot(\sqrt{c}\,r)$ with multiplicity 1 and $\lambda = (\sqrt{c}/2)\cot(\sqrt{c}\,r/2)$ with multiplicity $2n-2$. We take and fix an arbitrary point $p \in M$ and a unit vector $v \in T_pM$ orthogonal to ξ_p. Let $\gamma = \gamma(s)$ be a geodesic on M through $\gamma(0) = p$ with initial vector $\dot\gamma(0) = v$. It follows from an equality $\phi A = A\phi$, (2.4), the skew-symmetry of ϕ and the symmetry of A that

$$\begin{aligned}\nabla_{\dot\gamma} g(\dot\gamma, \xi) &= g(\dot\gamma, \nabla_{\dot\gamma}\xi) = g(\dot\gamma, \phi A\dot\gamma)\\ &= g(\dot\gamma, A\phi\dot\gamma) = -g(\phi A\dot\gamma, \dot\gamma) = 0,\end{aligned}$$

so that $g(\dot\gamma(s), \xi_{\gamma(s)})$ is constant along the curve $\gamma = \gamma(s)$. This, together with the initial condition $g(\dot\gamma(0), \xi_{\gamma(0)}) = 0$, shows that $\dot\gamma(s)$ is orthogonal to the characteristic vector $\xi_{\gamma(s)}$ for each s. Hence

$$A\dot\gamma(s) = (\sqrt{c}/2)\cot(\sqrt{c}\,r/2)\dot\gamma(s) \quad \text{for each } s. \tag{5.8}$$

It follows from (2.1), (2.2) and (5.8) that the curve γ is mapped to a circle of positive curvature $(\sqrt{c}/2)\cot(\sqrt{c}\,r/2)$ in $\mathbb{C}P^n(c)$. Thus, choosing the vectors $v_1, \dots, v_{2n-2}$ at p as orthonormal vectors orthogonal to ξ_p, we get the desired result.

Let M be of type (A_2). Then M has three distinct constant principal curvatures $\delta = \sqrt{c}\,\cot(\sqrt{c}\,r)$ with multiplicity 1, $\lambda_1 = (\sqrt{c}/2)\cot(\sqrt{c}\,r/2)$ with multiplicity $2n-2\ell-2$ and $\lambda_2 = -(\sqrt{c}/2)\tan(\sqrt{c}\,r/2)$ with multiplicity 2ℓ. We take and fix an arbitrary point $p \in M$ and a unit vector $v \in T_pM$ orthogonal to ξ_p with $v \in V_{\lambda_i} := \{X \in T_pM | AX = \lambda_i X\}$, where $i = 1$ or 2. Let $\gamma = \gamma(s)$ be a geodesic on M through $\gamma(0) = p$ with initial vector $\dot\gamma(0) = v$. In consideration of a differential equation $(\nabla_X A)Y = -(c/4)(g(\phi X, Y)\xi + \eta(Y)\phi X)$ for $X, Y \in TM$ and the properties of ϕ, A we see

$$\begin{aligned}\nabla_{\dot\gamma}\|A\dot\gamma(s) - \lambda_i\dot\gamma(s)\|^2 &= \nabla_{\dot\gamma}(g(A\dot\gamma, A\dot\gamma)) - 2\lambda_i\nabla_{\dot\gamma}(g(A\dot\gamma, \dot\gamma))\\ &= 2g((\nabla_{\dot\gamma}A)\dot\gamma, A\dot\gamma) - 2\lambda_i g((\nabla_{\dot\gamma}A)\dot\gamma, \dot\gamma)\\ &= -(c/2)\eta(\dot\gamma)g(\phi\dot\gamma, A\dot\gamma) = 0,\end{aligned}$$

which, combined with $A\dot\gamma(0) - \lambda_i\dot\gamma(0) = Av - \lambda_i v = 0$, yields

$$A\dot\gamma(s) = \lambda_i\dot\gamma(s) \quad \text{for each } s, \tag{5.9}$$

where $\dot\gamma(0) \in V_{\lambda_i}$, $i = 1$ or 2. Thus, from (2.1), (2.2) and (5.9) we know that the curve γ is mapped to a circle of positive curvature $|\lambda_i|$ in $\mathbb{C}P^n(c)$. Hence, choosing the vectors $v_1, \dots, v_{2n-2}$ at p in such a way that $v_1, \dots, v_{2n-2\ell-2}$ and $v_{2n-2\ell-1}, \dots, v_{2n-2}$ are orthonormal bases of V_{λ_1} and V_{λ_2}, respectively, we get the desired result.

Let M be of type (B). Then M has three distinct constant principal curvatures $\delta = \sqrt{c}\,\cot(\sqrt{c}\,r)$ with multiplicity 1, $\lambda_1 = (\sqrt{c}/2)\cot(\sqrt{c}\,r/2 - \pi/4)$ with multiplicity $n-1$ and $\lambda_2 = (\sqrt{c}/2)\cot(\sqrt{c}\,r/2 + \pi/4)$ with multiplicity $n-1$. Note that $\phi V_{\lambda_1} = V_{\lambda_2}$ (see Lemma 1). We shall verify that the principal distribution V_{λ_1} (resp. V_{λ_2}) on M is integrable, and moreover that any leaf L_{λ_1} of V_{λ_1} (resp. L_{λ_2} of V_{λ_2}) is a totally geodesic submanifold of M. It suffices to check that $\nabla_X Y \in V_{\lambda_1}$ for all $X, Y \in V_{\lambda_1}$. First we have

$$\begin{aligned} A\nabla_X Y &= \nabla_X(AY) - (\nabla_X A)Y \\ &= \lambda_1 \nabla_X Y - (\nabla_X A)Y. \end{aligned}$$

For any $Z \in TM$, since $\nabla_X A$ is symmetric, from Codazzi equation (2.6) we see

$$\begin{aligned} g((\nabla_X A)Y, Z) &= g((\nabla_X A)Z, Y) \\ &= g((\nabla_Z A)X \\ &\qquad + (c/4)\big(\eta(X)\phi Z - \eta(Z)\phi X - 2g(\phi X, Y)\xi\big), Y) \\ &= g((\nabla_Z A)X, Y) \\ &= g(\nabla_Z(AX) - A\nabla_Z X, Y) \\ &= g((Z\lambda_1)X + (\lambda_1 I - A)\nabla_Z X, Y) = 0. \end{aligned}$$

Hence $\nabla_X Y \in V_{\lambda_1}$ for any $X, Y \in V_{\lambda_1}$, so that every leaf V_{λ_1} of the principal distribution V_{λ_1} is totally geodesic in M. Therefore the manifold L_{λ_1} is totally umbilic in $\mathbb{C}P^n(c)$ (see (2.1)). L_{λ_1} is locally congruent to a totally umbilic hypersurface of constant sectional curvature d with $\sqrt{d - (c/4)} = |\lambda_1|$ in $\mathbb{R}P^n(c/4)$ of constant sectional curvature $c/4$, which is totally real totally geodesic in $\mathbb{C}P^n(c)$. This implies that every geodesic of L_{λ_1} (which is totally geodesic in our real hypersurface M) is mapped to a circle of positive curvature $|\lambda_1|$ in the ambient space $\mathbb{C}P^n(c)$. Hence we can check the case of type (B).

Next, let M be one of types (C), (D) and (E). Then M has five distinct constant principal curvatures $\delta = \sqrt{c}\,\cot(\sqrt{c}\,r)$, $\lambda_1 = (\sqrt{c}/2)\cot(\sqrt{c}\,r/2 - \pi/4)$, $\lambda_2 = (\sqrt{c}/2)\cot(\sqrt{c}\,r/2 + \pi/4)$, $\lambda_3 = (\sqrt{c}/2)\cot(\sqrt{c}\,r/2)$, $\lambda_4 = -(\sqrt{c}/2)\tan(\sqrt{c}\,r/2)$. Note that $\phi V_{\lambda_1} = V_{\lambda_2}$, $\phi V_{\lambda_2} = V_{\lambda_1}$, $\phi V_{\lambda_3} = V_{\lambda_3}$ and $\phi V_{\lambda_4} = V_{\lambda_4}$ (see Lemma 1). By virtue of the discussion in the case of type (B) it suffices to study the principal distributions λ_3 and λ_4. We remark that both of V_{λ_3} and V_{λ_4} are not integrable. Our aim here is to show that $V_{\lambda_3} \oplus \{\xi\}_{\mathbb{R}}$ (resp. $V_{\lambda_4} \oplus \{\xi\}_{\mathbb{R}}$) is integrable and any leaf of the distribution $V_{\lambda_3} \oplus \{\xi\}_{\mathbb{R}}$ (resp. $V_{\lambda_4} \oplus \{\xi\}_{\mathbb{R}}$) is a totally geodesic submanifold of M. Let $\mathfrak{T} = V_{\lambda_3} \oplus \{\xi\}_{\mathbb{R}}$. Then we can verify the following:

$$\nabla_X \xi \in \mathfrak{T},\ \nabla_\xi X \in \mathfrak{T} \quad \text{and} \quad \nabla_X Y \in \mathfrak{T} \text{ for any } X, Y \in V_{\lambda_3}.$$

In fact, from (2.4) $\nabla_X \xi = \phi AX = \lambda_3 \phi X \in \mathfrak{T}$. Next, from (2.4) we get

$$\begin{aligned}(\nabla_\xi A)X - (\nabla_X A)\xi &= \nabla_\xi (AX) - A\nabla_\xi X - \nabla_X (A\xi) + A\nabla_X \xi \\ &= (\lambda_3 I - A)\nabla_\xi X - \delta\phi AX + A\phi AX \\ &= (\lambda_3 - A)\nabla_\xi X + \lambda_3(\lambda_3 - \delta)\phi X.\end{aligned}$$

On the other hand, it follows from (2.6) that

$$(\nabla_\xi A)X - (\nabla_X A)\xi = (c/4)\phi X \in V_{\lambda_3}.$$

Thus for any $Z \in V_\mu$ ($\mu = \lambda_1, \lambda_2, \lambda_4, \delta$), we have

$$g((\lambda_3 - A)\nabla_\xi X, Z) = 0,$$

so that $\nabla_\xi X \in V_{\lambda_3} \subset \mathfrak{T}$. Finally, for any $X, Y \in V_{\lambda_3}$ and for any $Z \in V_\mu$ ($\mu = \lambda_1, \lambda_2, \lambda_4$) we see

$$\begin{aligned}g((\nabla_X A)Y, Z) &= g(\nabla_X (AY) - A\nabla_X Y, Z) \\ &= g((\lambda_3 I - A)\nabla_X Y, Z) \\ &= (\lambda_3 - \mu) g(\nabla_X Y, Z).\end{aligned}$$

On the other hand, it follows from the symmetry of ∇A and (2.6) that

$$\begin{aligned}g((\nabla_X A)Y, Z) &= g((\nabla_X A)Z, Y) = g((\nabla_Z A)X, Y) \\ &= g(\nabla_Z (AX) - A\nabla_Z X, Y) \\ &= g((\lambda_3 I - A)\nabla_Z X, Y) = 0.\end{aligned}$$

Hence, $\nabla_X Y \in \mathfrak{T}$. Thus we can see that any leaf L of the distribution $\mathfrak{T}$ is a totally geodesic submanifold of M. We consider the distribution $\widetilde{\mathfrak{T}} := V_{\lambda_3} \oplus \{\xi\}_{\mathbb{R}} \oplus \{\mathcal{N}\}_{\mathbb{R}}$ along M in the ambient space $\mathbb{C}P^n(c)$, where $\mathcal{N}(= J\xi)$ is a local unit normal vector on M. Here, the vectors $X(\in V_{\lambda_3}), \xi$ and $\mathcal{N}$ are extended by the parallel displacement along the geodesic $\gamma = \gamma(s)$ in $\mathbb{C}P^n(c)$ with $\gamma(0) = p$ and $\dot\gamma(0) = \mathcal{N}_p$ at each point $p \in M$. Then by a similar computation we have $\widetilde{\nabla}_X Y \in \widetilde{\mathfrak{T}}$ for all $X, Y \in \widetilde{\mathfrak{T}}$, which implies that the distribution $\widetilde{\mathfrak{T}}$ is integrable and each of its leaves is a totally geodesic submanifold $\mathbb{C}P^{m+1}(c)$ of $\mathbb{C}P^n(c)$, where $2m = \dim V_{\lambda_3}$ (for the dimension of V_{λ_3}, see the table of the principal curvatures in Takagi's list). Therefore we can see that any leaf L of the distribution $\mathfrak{T}$ is locally congruent to a geodesic sphere $G(r)$ of radius r $(0 < r < \pi/\sqrt{c}\,)$ in $\mathbb{C}P^{m+1}(c)$. So, the discussion in the case of type (A_1) shows that every geodesic on a totally geodesic submanifold L of M whose initial vector is orthogonal to ξ is mapped to a circle of positive curvature $|\lambda_3|$ in $\mathbb{C}P^n(c)$. The same discussion as above yields that any leaf K of the distribution $V_{\lambda_4} \oplus \{\xi\}_{\mathbb{R}}$ is a totally geodesic submanifold M. The manifold K is locally congruent a geodesic sphere $G(\pi/\sqrt{c}\ - r)$ of radius $\pi/\sqrt{c}\ - r$ of a totally geodesic submanifold $\mathbb{C}P^{k+1}(c)$ of $\mathbb{C}P^n(c)$, where $2k = \dim V_{\lambda_4}$. Every geodesic on a totally geodesic submanifold K of M whose initial vector is orthogonal to ξ is mapped to a circle of positive curvature $|\lambda_4|$ in $\mathbb{C}P^n(c)$. Therefore we can check the cases of types (C), (D) and (E).

The discussion in the case of $c < 0$ follows directly from that in the case of $c > 0$ (see the argument in the case where M is of either type (A_1), type (A_2) or type (B)).

We next prove the "if" part. It suffices to consider the case of $c < 0$. The discussion in the case of $c > 0$ is a bit simpler than that in the case of $c < 0$. Assume that at each point $p \in M$ there exist $2n$–2 orthonormal vectors $v_1, \dots, v_{2n-2}$ orthogonal to ξ_p such that the geodesics $\gamma_i = \gamma_i(s)$ $(1 \leqq i \leqq 2n-2)$ on M with $\gamma_i(0) = p$ and $\dot\gamma_i(0) = v_i$ are mapped to circles of positive curvature (, say) k_i in $\mathbb{C}H^n(c)$. Then it follows from Lemma 3 that

$$Av_i = k_i v_i \quad \text{or} \quad Av_i = -k_i v_i \quad \text{for } i = 1, \dots, 2n-2, \tag{5.10}$$

which implies that M is a Hopf hypersurface in $\mathbb{C}H^n(c)$ because $g(A\xi, v_i) = g(\xi, Av_i) = 0$ for $i \in \{1, \dots, 2n-2\}$.

Now, consider the open dense subset $\mathcal{U}$ of M defined by

$$\begin{aligned} \mathcal{U} = \{p \in M | &\text{the multiplicity of each principal curvature} \\ &\text{of } M \text{ in } \mathbb{C}H^n(c) \text{ is constant on some neighborhood } \mathcal{V}_p \text{ of } p\}. \end{aligned} \tag{5.11}$$

Recall that principal curvatures are differentiable on $\mathcal{U}$ and, moreover the principal curvature vectors can be chosen to be smooth on a sufficiently small neighborhood of each point $p \in \mathcal{U}$. In the following, we shall study on such a fixed neighborhood $\mathcal{V}_p$. At the fixed point p, the linear subspace $T_p^0 M = \{X \in T_pM | X \perp \xi_p\}$ of the tangent space T_pM is decomposed as:

$$\begin{aligned} T_p^0 M = &\{v \in T_p^0 M | Av = -k_{i_1} v\} \oplus \{v \in T_p^0 M | Av = k_{i_1} v\} \\ &\oplus \cdots \oplus \{v \in T_p^0 M | Av = -k_{i_g} v\} \oplus \{v \in T_p^0 M | Av = k_{i_g} v\}, \end{aligned}$$

where $0 < k_{i_1} < \cdots < k_{i_g}$ and g denotes the number of positive distinct $k_j, j = 1, \dots, 2n-2$. Note that every k_{i_j} is differentiable on $\mathcal{V}_p$.

Now we prove the constancy of $k_{i_1}, \dots, k_{i_g}$. It suffices to prove the check the case of $Av_{i_j} = k_{i_j} v_{i_j}$. By hypothesis we have $v_{i_j} k_{i_j} = 0$.

In the following, we consider the case of $k_{i_j}(p) \neq \delta$. So we may suppose that $k_{i_j} \neq \delta$ at each point of a sufficiently small neighborhood of the point p. Since ∇A is symmetric, we find

$$g((\nabla_{v_{i_j}} A) v_\ell, v_{i_j}) = g(v_\ell, (\nabla_{v_{i_j}} A) v_{i_j}), \quad 1 \leqq \ell \neq i_j \leqq 2n-2. \tag{5.12}$$

In order to compute Equation (5.12) easily, we extend $v_\ell, v_{i_j} \in T_pM$ to vector fields V_ℓ, V_{i_j} on some sufficiently small neighborhood $\mathcal{W}_p (\subset \mathcal{V}_p)$ as follows.

We define V_ℓ to be a vector field on $\mathcal{W}_p$ satisfying $(V_\ell)_p = v_\ell$ and $V_\ell \perp \xi$. In order to define V_{i_j}, we first set a smooth vector field W_{i_j} on $\mathcal{W}_p$ by using the parallel displacement of the vector v_{i_j} along each geodesic on M through the point p. We remark that although W_{i_j} is not principal on $\mathcal{W}_{i_j}$ in general, but we have $AW_{i_j} = k_{i_j} W_{i_j}$ along the geodesic $\gamma = \gamma(s)$ on M with $\gamma(0) = p$ and $\dot\gamma(0) = v_{i_j}$. We define a vector field U_{i_j} on $\mathcal{W}_p$ by $U_{i_j} = \Big(\Pi_{\alpha \neq k_{i_j}} (A - \alpha I)\Big) W_{i_j}$, where α runs over the set of all distinct principal curvatures of M except for the principal curvature k_{i_j}. We note that the vector field U_{i_j} is perpendicular to ξ because $k_{i_j} \neq \delta$. Furthermore, we have

$$\begin{aligned} AU_{i_j} &= A\Big(\Pi_{\alpha\neq k_{i_j}}(A-\alpha I)\Big)W_{i_j} \\ &= \Big(\Pi_{\alpha\neq k_{i_j}}(A-\alpha I)\Big)A(V_{i_j} \text{ component of } W_{i_j}) \\ &= k_{i_j}U_{i_j} \neq 0 \end{aligned}$$

on $\mathcal{W}_p$. We note that $U_{i_j} \neq 0$ on the neighborhood $\mathcal{W}_p$ because $(U_{i_j})_p \neq 0$. We next define the vector field V_{i_j} by normalizing U_{i_j} in some sense. That is, when $\Pi_{\alpha\neq k_{i_j}}(k_{i_j}-\alpha)(p) > 0$ (resp. $\Pi_{\alpha\neq k_{i_j}}(k_{i_j}-\alpha)(p) < 0$), we define $V_{i_j} = U_{i_j}/\|U_{i_j}\|$ (resp. $V_{i_j} = -U_{i_j}/\|U_{i_j}\|$). Our construction gurarantees that $AV_{i_j} = k_{i_j}, (V_{i_j})p = v_{i_j}$ and the integral curve of V_{i_j} through the point p is a geodesic of M. In particular, we have $(\nabla_{V_{i_j}}V_{i_j})_p = 0$. Since Codazzi equation (2.6) implies

$$g((\nabla_X A)Y, Z) = g((\nabla_Y A)X, Z) \quad \text{for } \forall X, Y, Z \perp \xi,$$

at the point p we find

$$\begin{aligned} (\text{The left hand side of (5.12)}) &= g((\nabla_{v_\ell}A)v_{i_j}, v_{i_j}) \\ &= g((\nabla_{V_\ell}A)V_{i_j}, V_{i_j})_p \\ &= g((V_\ell k_{i_j})V_{i_j} + (k_{i_j}I - A)\nabla_{V_\ell}V_{i_j}, V_{i_j})_p \\ &= v_\ell k_{i_j}. \end{aligned}$$

Similarly, at the point p we get

$$\begin{aligned} (\text{The right hand side of (5.12)}) &= g(V_\ell, (\nabla_{V_{i_j}}A)V_{i_j})_p \\ &= g(V_\ell, \nabla_{V_{i_j}}(k_{i_j}V_{i_j}) - A\nabla_{V_{i_j}}V_{i_j})_p \\ &= g(v_\ell, (v_{i_j}k_{i_j})v_{i_j}) = 0. \end{aligned}$$

These imply that $Xk_{i_j} = 0$ for any $X(\in T_pM)$ which is perpendicular to ξ.

We next prove $\xi k_{i_j} = 0$. We divide our discussion into two cases.

Case of (I_a). Suppose that $2k_{i_j} - \delta \neq 0$ at some point $p \in \mathcal{U}$. In this case, from Lemma 1 we see that

$$A\phi V_{i_j} = \frac{\delta k_{i_j} + (c/2)}{2k_{i_j} - \delta}\phi V_{i_j}$$

on some neighborhood $\mathcal{W}_p(\subset \mathcal{V}_p)$ of the point p. This, together with (2.4), yields

$$\begin{aligned} (\nabla_\xi A)V_{i_j} - (\nabla_{V_{i_j}}A)\xi &= \nabla_\xi(AV_{i_j}) - A\nabla_\xi V_{i_j} - \nabla_{V_{i_j}}(\delta\xi) + A\nabla_{V_{i_j}}\xi \\ &= \nabla_\xi(k_{i_j}V_{i_j}) - A\nabla_\xi V_{i_j} - \delta\phi AV_{i_j} + A\phi AV_{i_j} \\ &= (\xi k_{i_j})V_{i_j} + (k_{i_j}I - A)\nabla_\xi V_{i_j} \\ &\quad - k_{i_j}\Big(\delta - \frac{\delta k_{i_j} + (c/2)}{2k_{i_j} - \delta}\Big)\phi V_{i_j}. \end{aligned}$$

On the other hand, from (2.6) we have

$$(\nabla_\xi A)V_{i_j} - (\nabla_{V_{i_j}} A)\xi = (c/4)\phi V_{i_j}.$$

By combining these two equations we find $\xi k_{i_j} = 0$. Consequently, k_{i_j} is constant on $\mathcal{W}_p$.

Case of (I_b). Suppose that $2k_{i_j} - \delta = 0$ at some point $p \in \mathcal{U}$. Then, by the continuity of the principal curvature k_{i_j} and the local constancy of k_{i_j} in Case of (I_a) there exists some sufficiently small neighborhood $\mathcal{W}_p$ of the point p such that $2k_{i_j} - \delta \equiv 0$ on $\mathcal{W}_p$. Hence, $k_{i_j} = \delta/2$ is constant on $\mathcal{W}_p$ (see Lemma 1).

Therefore, in the case of $k_{i_j} \neq \delta$ we can see that the function k_{i_j} is constant locally. When $k_{i_j}(p) = \delta$, by the above discussion, the continuity of k_{i_j} and the constancy of δ we also see that the function k_{i_j} is constant locally.

Thus, we know that every principal curvature of M is constant locally on the open dense subset $\mathcal{U}$ of M. This, combined with the assumption that M is connected and the fact that all principal curvature functions are continuous on M, implies that every principal curvature is constant on M. Hence we can see that our real hypersurface M is a Hopf hypersurface all of whose principal curvatures are constant in $\mathbb{C}H^n(c)$, so that we obtain the desired conclusion. □

Remark 2. (1) In Theorem 1, we do not need to assume that $\{v_1, \ldots, v_{2n-2}\}$ is a local field of orthonormal frames on M. However, for each homogeneous Hopf hypersurface M we can take a local smooth field of orthonormal frames in M satisfying the hypothesis of Theorem 1.
(2) Every circle in $\mathbb{C}P^n(c)$ (resp. $\mathbb{C}H^n(c)$) given in Theorem 1 is a simple curve lying on some totally real totally geodesic $\mathbb{R}P^2(c/4)$ (resp. $\mathbb{R}H^2(c/4)$). However, the feature of circles in $\mathbb{C}H^n(c)$ is more complicated than that of circles in $\mathbb{C}P^n(c)$. In the case of $\mathbb{C}P^n(c)$ every circle given by Theorem 1 is closed. On the other hand, in the case of $\mathbb{C}H^n(c)$ a circle given by Theorem 1 is closed if and only if its curvature is greater than $\sqrt{|c|}/2$ (for details, see [4, 2]).

As an immediate consequence of the proof of Theorem 1 we have the following:

Lemma 4. *For every homogeneous Hopf hypersurface M in a nonflat complex space form $\widetilde{M}_n(c), n \geqq 2$, take a unit principal curvature λ which is orthogonal to ξ_p at an arbitrary point $p \in M$. Then the geodesic $\gamma = \gamma(s)$ with $\gamma(0) = p$ and $\dot\gamma(0) = v$ on M is mapped to a circle of positive curvature $|\lambda|$ in the ambient space $\widetilde{M}_n(c)$.*

Next, we shall give another characterization of all homogeneous Hopf hypersurfaces in a nonflat complex space form. We here recall that for every Hopf hypersurface M in a nonflat complex space form $\widetilde{M}_n(c)$, the holomorphic distribution $T^0M = \{X \in TM | \eta(X) = 0\}$ on M is not integrable (see Proposition 1). Note that there exists a real hypersurface M of type (B) with two distinct constant principal curvatures $\lambda_1 = \delta = \sqrt{3|c|}/2$ and $\lambda_2 = \sqrt{|c|}/(2\sqrt{3}\,)$ in $\mathbb{C}H^n(c)$. So we need the following notion. For a real hypersurface M in $\widetilde{M}_n(c), n \geqq 2$, we call $V^0_\lambda = \{X \in T^0M | AX = \lambda X\}$ the *restricted principal distribution* associated to the principal curvature λ of M in $\widetilde{M}_n(c)$.

The following is another characterization of all homogeneous Hopf hypersurfaces in $\widetilde{M}_n(c)$.

Theorem 2 ([10, 14, 16])**.** *Let M^{2n-1} be a connected real hypersurface in a nonflat complex space form $\widetilde{M}_n(c), n \geqq 2$. Then M is locally congruent to a homogeneous Hopf hypersurface if and only if M satisfies the following two conditions.*

(1) *The holomorphic distribution $T^0M = \{X \in TM | X \perp \xi\}$ of M is decomposed as the direct sum of restricted principal distributions $V^0_{\lambda_i} = \{X \in T^0M | AX = \lambda_i X\}$.*

(2) *Every restricted principal distribution $V^0_{\lambda_i}$ in Condition* (1) *satisfies one of the following two conditions*:

  2_a) *The distribution $V^0_{\lambda_i} \oplus \{\xi\}_{\mathbb{R}}$ is integrable*;

  2_b) *$V^0_{\lambda_i}$ is integrable and every leaf of $V^0_{\lambda_i}$ is a totally geodesic submanifold of the real hypersurface M.*

Proof. It suffices to study the "if" part (see the proof in Theorem 1). It suffices to prove the "if" part when $c < 0$, since the discussion in the case of $c < 0$ is much more complicated than that in the case of $c > 0$.

First of all note that a real hypersurface M satisfying Condition (1) is a Hopf hypersurface in $\mathbb{C}H^n(c)$. In fact, for each $X = \sum_i X^i v_i \in T^0M$, where v_i is a unit vector in the restricted principal distribution $V^0_{\lambda_i}$ of Condition (1), we find $g(A\xi, X) = g(\xi, AX) = \sum_i g(\xi, X^i \lambda_i v_i) = 0$.

We here study the case where $V^0_\lambda \oplus \{\xi\}_{\mathbb{R}}$ is integrable for some principal curvature λ of M in $\mathbb{C}H^n(c)$. Then, for any X in V^0_λ we have $\nabla_X\xi - \nabla_\xi X \in V^0_\lambda \oplus \{\xi\}_{\mathbb{R}}$. Since ξ is a unit principal curvature vector, we also have $g(\nabla_\xi X, \xi) = -g(X, \nabla_\xi \xi) = 0$ and $g(\nabla_X \xi, \xi) = 0$ for each X in V^0_λ. These show that $\nabla_X\xi - \nabla_\xi X$ is perpendicular to ξ for all $X \in V^0_\lambda$. Therefore

$$A(\nabla_X\xi - \nabla_\xi X) = \lambda(\nabla_X\xi - \nabla_\xi X) \quad \text{for } \forall X \in V^0_\lambda. \tag{5.13}$$

Now, we consider the case that $2\lambda - \delta \neq 0$ on some neighborhood of an arbitrary fixed point p. It follows from (2.4), Lemma 1 and (5.13) that

$$(A - \lambda I)\nabla_\xi X = \lambda\Big(\frac{\delta\lambda + (c/2)}{2\lambda - \delta} - \lambda\Big)\phi X. \tag{5.14}$$

In consideration of (2.4), Lemma 1 and (5.14) we obtain

$$\begin{aligned}(\nabla_X A)\xi - (\nabla_\xi A)X &= \nabla_X(\delta\xi) - A\nabla_X\xi - \nabla_\xi(AX) + A\nabla_\xi X \\ &= \lambda(\delta - \lambda)\phi X - (\xi\lambda)X.\end{aligned}$$

On the other hand, from (2.6) we see

$$(\nabla_X A)\xi - (\nabla_\xi A)X = -(c/4)\phi X.$$

These two equations imply that the principal curvature λ is a solution to the quadratic equation:

$$\lambda^2 - \delta\lambda - (c/4) = 0, \tag{5.15}$$

so that λ is constant locally.

Next, consider the case that $2\lambda-\alpha=0$ at some point p. In this case, from the discussion in the case $2\lambda-\delta\neq 0$ and the continuity of $2\lambda-\delta$ we find that $2\lambda-\delta=0$ holds on some neighborhood of p. This, together with Lemma 1, implies that either $\lambda=\sqrt{|c|}/2,\ \delta=\sqrt{|c|}$ or $\lambda=-\sqrt{|c|}/2,\ \delta=-\sqrt{|c|}$ holds on some neighborhood of the point p. Note that those λ and δ satisfy (5.15). Hence, for a principal curvature λ of M in $\mathbb{C}H^n(c)$, we see that if $V^0_\lambda\oplus\{\xi\}_{\mathbb{R}}$ is integrable, then λ is a solution to (5.15), so that λ is constant.

So, it is sufficient to study Condition 2_b) in our Theorem. Our discussion is divided into two Cases of (I): $\dim T^0M=2$ and (II): $\dim T^0M\geqq 4$.

Case (I): $\dim T^0M=2$.
Recall that T^0M is not integrable (see Proposition 1). Then, in this case $T^0M=V^0_{\lambda_1}\oplus V^0_{\lambda_2}$, where $\lambda_1\neq\lambda_2$ and $\dim V^0_{\lambda_1}=\dim V^0_{\lambda_2}=1$. Case (I) is divided into the following three cases:

(I_a) All integral curves of $V^0_{\lambda_1}$ and $V^0_{\lambda_2}$ are geodesics in M;
(I_b) $V_{\lambda_1}\oplus\{\xi\}_{\mathbb{R}}$ is integrable and all integral curves of V_{λ_2} are geodesics in M;
(I_c) Both of $V^0_{\lambda_1}\oplus\{\xi\}_{\mathbb{R}}$ and $V^0_{\lambda_2}\oplus\{\xi\}_{\mathbb{R}}$ are integrable.

(I_a) By hypothesis we see that all integral curves of $V^0_{\lambda_1}$ and $V^0_{\lambda_2}$ are geodesics in M. We take a local field of orthonormal frames $\{v,\phi v,\xi\}$ on M in such a way that $v\in V^0_{\lambda_1}$ and $\phi v\in V^0_{\lambda_2}$. By hypothesis we have $\nabla_v v=\nabla_{\phi v}(\phi v)=0$. This, together with (2.4), shows

$$\nabla_v\xi=\lambda_1\phi v,\ \nabla_{\phi v}\xi=-\lambda_2 v,\ \nabla_v(\phi v)=-\lambda_1\xi,\ \nabla_{\phi v}v=\lambda_2\xi. \tag{5.16}$$

Codazzi equation (2.6) implies

$$(\nabla_v A)\phi v-(\nabla_{\phi v}A)v=-(c/2)\xi.$$

On the other hand, Equation (5.16) yields

$$\begin{aligned}(\nabla_v A)\phi v-(\nabla_{\phi v}A)v&=\nabla_v(A\phi v)-A\nabla_v(\phi v)-\nabla_{\phi v}(Av)+A\nabla_{\phi v}v\\&=(v\lambda_2)\phi v+(\lambda_2 I-A)\nabla_v(\phi v)-(\phi v\lambda_1)v\\&\qquad-(\lambda_1 I-A)\nabla_{\phi v}v\\&=-(\phi v\lambda_1)v+(v\lambda_2)\phi v+\{\delta(\lambda_1+\lambda_2)-2\lambda_1\lambda_2\}\xi.\end{aligned}$$

It follows from these two equations that

$$-c/2=\delta(\lambda_1+\lambda_2)-2\lambda_1\lambda_2. \tag{5.17}$$

$$\phi v\lambda_1=0. \tag{5.18}$$

$$v\lambda_2=0. \tag{5.19}$$

We here show that $\xi\lambda_1=\xi\lambda_2=0$. It follows from (2.6) that

$$(\nabla_v A)\xi-(\nabla_\xi A)v=-(c/4)\phi v.$$

On the other hand, we also have

$$\begin{aligned}(\nabla_v A)\xi-(\nabla_\xi A)v&=\nabla_v(A\xi)-A(\nabla_v\xi)-\nabla_\xi(Av)+A\nabla_\xi v\\&=\delta\lambda_1\phi v-\lambda_1\lambda_2\phi v-(\xi\lambda_1)v-(\lambda_1 I-A)\nabla_\xi v.\end{aligned}$$

By these two equations we get $\xi\lambda_1 = 0$. Similarly, we see $\xi\lambda_2 = 0$. Next, since δ is constant locally, from (5.17) and (5.18) we know

$$(\delta - 2\lambda_1)(\phi v\lambda_2) = 0. \tag{5.20}$$

Similarly, from (5.17) and (5.19) we see

$$(\delta - 2\lambda_2)(v\lambda_1) = 0. \tag{5.21}$$

So, we divide Case (I_a) into the following three cases.

Case $(\mathrm{I}_a)_1$. Suppose that $\delta \equiv 2\lambda_2$ locally and $\delta \neq 2\lambda_1$ at some point $p \in M$. In this case, Equation (5.17) shows $(\lambda_2)^2 = |c|/4$. Without loss of generality we may assume $\lambda_2 = \sqrt{|c|}/2$, so that $\delta = \sqrt{|c|}$. For simplicity, putting $\lambda_1 = \lambda$, we find

$$\nabla_v v = \nabla_{\phi v}(\phi v) = \nabla_\xi \xi = 0, \nabla_v(\phi v) = -\lambda\xi,$$
$$\nabla_{\phi v} v = (\sqrt{|c|}/2)\xi, \nabla_v \xi = \lambda\phi v, \nabla_{\phi v}\xi = -(\sqrt{|c|}/2)v.$$

It follows from the continuity of λ that $\lambda \neq \sqrt{|c|}/2$ on some neighborhood of the point p. Putting $\nabla_\xi v = \mu\phi v$, we see from Codazzi equation (2.6) that

$$(\nabla_v A)\xi - (\nabla_\xi A)v = -(c/4)\phi v.$$

On the other hand, we obtain

$$\begin{aligned}(\nabla_v A)\xi - (\nabla_\xi A)v &= \sqrt{|c|}\,(\nabla_v\xi) - A\nabla_v\xi - \nabla_\xi(\lambda v) + A\nabla_\xi v\\ &= -(\xi\lambda)v + \big((\sqrt{|c|}/2)(\lambda+\mu) - \mu\lambda\big)\phi v.\end{aligned}$$

These two equations yield

$$(\sqrt{|c|}/2)(\lambda+\mu) - \mu\lambda = -c/4,$$

so that $\mu = \sqrt{|c|}/2$, since $\lambda \neq \sqrt{|c|}/2$. Then, by the same computation as in (I_b) we have

$$\begin{aligned}g(R(v,\phi v)\phi v, v) &= g(-\nabla_{\phi v}(\nabla_v(\phi v)) - \nabla_{\nabla_v(\phi v)-\nabla_{\phi v}v}(\phi v), v)\\ &= -\sqrt{|c|}\,\lambda + (c/4).\end{aligned}$$

On the other hand, from (2.5) we get

$$g(R(v,\phi v)\phi v, v) = c + (\sqrt{|c|}/2)\lambda.$$

Thus $\lambda = \sqrt{|c|}/2$, which is a contradiction. Hence, this case does not occur.

Case $(\mathrm{I}_a)_2$. Suppose that $\delta \equiv 2\lambda_1$ locally and $\delta \neq 2\lambda_2$ at some point $p \in M$. This case cannot occur by the same discussion as in Case $(\mathrm{I}_a)_1$.

Case $(\mathrm{I}_a)_3$. Suppose that $\delta \neq 2\lambda_1$ and $\delta \neq 2\lambda_2$ at some point $p \in M$. In this case, Equations (5.18), (5.19), (5.20) and (5.21) yield

$$v\lambda_1 = v\lambda_2 = \phi v\lambda_1 = \phi v\lambda_2 = 0,$$

on some neighborhood of the point p. This, together with $\xi\lambda_1 = \xi\lambda_2 = 0$, shows that both of λ_1 and λ_2 are constant locally. Moreover, $\phi V^0_{\lambda_1} = V^0_{\lambda_2}$. Therefore we find that M is of type (B) in $\mathbb{C}H^2(c)$.

Case (I_b). We suppose that $V_{\lambda_1} \oplus \{\xi\}_{\mathbb{R}}$ is integrable and all integral curves of V_{λ_2} are geodesics in M. Since $\dim V^0_{\lambda_1} = 1$, we may suppose that $\lambda_1 = \sqrt{|c|}/2$

and $\delta = \sqrt{|c|}$ (see (5.15)). (In fact, if $\lambda_1 \neq \sqrt{|c|}/2$, then Equation (5.15) shows $\phi V^0_{\lambda_1} = V^0_{\lambda_1}$.) So we may take a local field of orthonormal frames $\{v, \phi v, \xi\}$ on M in such a way that

$$Av = (\sqrt{|c|}/2)v,\ A\phi v = \lambda v,\ A\xi = \sqrt{|c|}\,\xi,\ \nabla_{\phi v}(\phi v) = 0, \lambda \neq \sqrt{|c|}/2.$$

We get

$$\nabla_{\phi v} v = \lambda \xi. \tag{5.22}$$

Indeed, since $0 = \phi v(g(v, \phi v)) = g(\nabla_{\phi v} v, \phi v)$ and $g(\nabla_{\phi v} v, v) = 0$, we have

$$\begin{aligned}\nabla_{\phi v} v &= g(\nabla_{\phi v} v, \xi)\xi = -g(v, \nabla_{\phi v}\xi)\xi \\ &= -g(v, \phi A \phi v)\xi = -g(v, \lambda \phi^2 v)\xi = \lambda \xi.\end{aligned}$$

Also, using (5.22) and $A\xi = \sqrt{|c|}\,\xi$, we see

$$(\nabla_v A)\phi v - (\nabla_{\phi v} A)v = (v\lambda)\phi v + (\lambda I - A)\nabla_v(\phi v) + (\sqrt{|c|}/2)\lambda\xi.$$

On the other hand, from (2.6) we get

$$(\nabla_v A)\phi v - (\nabla_{\phi v} A)v = -(c/2)\xi.$$

Hence, from these two equations we have

$$\left(\lambda - \frac{\sqrt{|c|}}{2}\right) g(\nabla_v(\phi v), v) = 0,$$

so that

$$g(\nabla_v(\phi v), v) = 0.$$

This, together with $g(\nabla_v(\phi v), \phi v) = 0$, shows

$$\nabla_v(\phi v) = -(\sqrt{|c|}/2)\xi. \tag{5.23}$$

In fact,

$$\begin{aligned}\nabla_v(\phi v) &= g(\nabla_v(\phi v), \xi)\xi = -g(\phi v, \nabla_v \xi)\xi \\ &= -g(\phi v, \phi A v)\xi = -\frac{\sqrt{|c|}}{2} g(\phi v, \phi v)\xi = -\frac{\sqrt{|c|}}{2}\xi.\end{aligned}$$

Next, we have

$$(\nabla_{\phi v} A)\xi - (\nabla_\xi A)\phi v = -(\sqrt{|c|}/2)\lambda v - (\xi\lambda)\phi v + (A - \lambda I)\nabla_\xi(\phi v).$$

On the other hand, it follows from (2.6) that

$$(\nabla_v A)\phi v - (\nabla_{\phi v} A)v = (c/4)v.$$

Hence, we get

$$-(\sqrt{|c|}/2)\lambda + \big((\sqrt{|c|}/2) - \lambda\big) g(\nabla_\xi(\phi v), v) = c/4,$$

so that

$$g(\nabla_\xi(\phi v), v) = -\frac{\sqrt{|c|}}{2}$$

which, combined with

$$g(\nabla_\xi(\phi v), \phi v) = g(\nabla_\xi(\phi v), \xi) = 0,$$

yields

$$\nabla_\xi(\phi v) = -\frac{\sqrt{|c|}}{2} v. \tag{5.24}$$

Thus, in consideration of (5.22), (5.23), (5.24) and $\nabla_{\phi v}(\phi v) = 0$ by a direct computation we can see that

$$g(R(v, \phi v)\phi v, v) = -\sqrt{|c|}\,\lambda + (c/4).$$

On the other hand, it follows from (2.5) that

$$g(R(v, \phi v)\phi v, v) = c + (\sqrt{|c|}/2)\lambda.$$

Thus, from these two equations we find $\lambda = \sqrt{|c|}/2$, which is a contradiction.

Case (I_c). Finally, we suppose the case that both of $V^0_{\lambda_1} \oplus \{\xi\}_{\mathbb{R}}$ and $V^0_{\lambda_2} \oplus \{\xi\}_{\mathbb{R}}$ are integrable. Since $\dim V_{\lambda_1} = \dim V_{\lambda_2} = 1$, $\lambda_1 = \lambda_2 = \sqrt{|c|}/2$, which is a contradiction. Therefore we can check Case (I): $\dim T^0M = 2$.

Case (II): $\dim T^0M \geqq 4$. We divide Case (II) into two cases.

Case (II_a) : $\dim V^0_{\lambda_i} \geqq 2$. It follows from Condition 2_b) in our theorem that

$$(\nabla_X A)Y - (\nabla_Y A)X = (X\lambda_i)Y - (Y\lambda_i)X \quad \text{for } \forall X, Y \in V^0_{\lambda_i}.$$

On the other hand, from Codazzi equation (2.6) we find

$$(\nabla_X A)Y - (\nabla_Y A)X = -(c/2)g(\phi X, Y)\xi \quad \text{for } \forall X, Y \in V^0_{\lambda_i}.$$

Choosing X, Y as arbitrary two independent vectors in $V^0_{\lambda_i}$ in these two equations, we see that $X\lambda_i = Y\lambda_i = g(\phi X, Y) = (\nabla_X A)Y = 0$, so that

$$(\nabla_X A)Y = g(\phi X, Y) = 0 \quad \text{for } \forall X, Y \in V^0_{\lambda_i}. \tag{5.25}$$

Therefore, for each unit vector $X \in V^0_{\lambda_i}$ and each $Z \in TM$, from (2.6), (5.25) and the symmetry of ∇A we obtain

$$\begin{aligned} 0 &= g((\nabla_X A)X, Z) = g((\nabla_X A)Z, X) \\ &= g((\nabla_Z A)X, X) = g(\nabla_Z(AX) - A\nabla_Z X, X) \\ &= g((Z\lambda_i)X + (\lambda_i I - A)\nabla_Z X, X) = Z\lambda_i, \end{aligned} \tag{5.26}$$

so that λ_i is constant.

Case (II_b) : $\dim V^0_{\lambda_i} = 1$. In order to verify the constancy of λ_i, we only need to consider the case that $2\lambda_i - \delta \neq 0$ on some neighborhood of an arbitrary fixed point. Let v be a unit vector in $V^0_{\lambda_i}$ so that $Av = \lambda_i v$. Then Lemma 1 implies $A\phi v = \big((\delta\lambda_i + (c/2))/(2\lambda_i - \delta)\big)\phi v$. Hence, $\phi v \in V^0_{\lambda_j}$ for some j with $\lambda_j = \big((\delta\lambda_i + (c/2))/(2\lambda_i - \delta)\big)(\neq \lambda_i)$. So, when $\dim V^0_{\lambda_j} \geqq 2$, we find that λ_j is constant (see the discussion in the case that $V_{\lambda_j} \oplus \{\xi\}_{\mathbb{R}}$ is integrable, and the discussion in Case of (II_a)), so that if $2\lambda_j - \delta \neq 0$, then λ_i is also constant.

Next, we consider the case of $2\lambda_j - \delta = 0$. Hence we may suppose that $2\lambda_j = \delta = \sqrt{|c|}$. For simplicity, we set $\lambda = \lambda_i$. So we see that

$$Av = \lambda v,\ A\phi v = (\sqrt{|c|}/2)\phi v,\ A\xi = \sqrt{|c|}\,\xi,\ \nabla_v v = 0,\ \lambda \neq \sqrt{|c|}/2.$$

We shall verify some equalities in order to show that the case of $2\lambda_j - \delta = 0$ does not occur. It follows from Codazzi equation (2.6) that

$$(\nabla_\xi A)\phi v - (\nabla_{\phi v} A)\xi = (-c/4)v.$$

On the other hand, by a direct computation we have

$$(\nabla_\xi A)\phi v - (\nabla_{\phi v} A)\xi = \Big(\frac{\sqrt{|c|}}{2}I - A\Big)\nabla_\xi(\phi v) + \frac{\sqrt{|c|}}{2}(\sqrt{|c|} - \lambda)v.$$

Taking the inner products of these two equations and the unit vector v, since $\lambda \neq \sqrt{|c|}/2$ we have

$$g((\nabla_\xi(\phi v), v) = -\sqrt{|c|}/2. \tag{5.27}$$

Again, by using Codazzi equation (2.6) we find that

$$(\nabla_v A)\phi v - (\nabla_{\phi v} A)v = -(c/2)\xi.$$

On the other hand, by a direct computation we get

$$(\nabla_v A)\phi v - (\nabla_{\phi v} A)v = (\sqrt{|c|}/2)\lambda\xi - (\phi v\lambda)v + (A - \lambda I)\nabla_{\phi v} v.$$

These two equations, combined with the fact that $\lambda \neq \sqrt{|c|}/2$ and $g(\nabla_{\phi v} v, v) = 0$, yield that $\nabla_{\phi v} v = g(\nabla_{\phi v} v, \xi)\xi$, so that

$$\nabla_{\phi v} v = (\sqrt{|c|}/2)\xi. \tag{5.28}$$

We here recall the following

$$\nabla_v(\phi v) = -\lambda\xi. \tag{5.29}$$

Using these equalities (2.4), (5.27), (5.28) and (5.29), by a direct computation we obtain

$$\begin{aligned} R(v, \phi v)\phi v &= \nabla_v \nabla_{\phi v}(\phi v) - \nabla_{\phi v}\nabla_v(\phi v) - \nabla_{[v,\phi v]}(\phi v) \\ &= (\phi v\lambda)\xi - \frac{\sqrt{|c|}}{2}\lambda v + \left(\lambda + \frac{\sqrt{|c|}}{2}\right)\nabla_\xi(\phi v), \end{aligned}$$

so that

$$g(R(v, \phi v)\phi v, v) = -\sqrt{|c|}\,\lambda + (c/4).$$

On the other hand, from (2.5) we see

$$g(R(v, \phi v)\phi v, v) = c + (\sqrt{|c|}/2)\lambda.$$

Therefore, from these two equations we know $\lambda = \sqrt{|c|}/2$, which is a contradiction.

Thus Case (II_b) reduces to the case of $\dim V_{\lambda_i} = \dim V_{\lambda_j} = 1$. We set $\mathfrak{T} = \{\xi, v, \phi v\}_{\mathbb{R}}$ with $Av = \lambda v$ and $A\phi v = \big(\delta\lambda + (c/2)\big)/(2\lambda - \delta)\big)\phi v$. For simplicity we denote $\big(\delta\lambda + (c/2)\big)/(2\lambda - \delta)$ by μ. Note that $\lambda \neq \mu$. Now, we prove that $\mathfrak{T}$ is integrable and each leaf of $\mathfrak{T}$ is a totally geodesic submanifold of the real hypersurface M. First, notice that $\nabla_v v = \nabla_{\phi v}(\phi v) = 0$. The reason why $\nabla_{\phi v}(\phi v) = 0$ is that if $V_\mu^0 \oplus \{\xi\}_{\mathbb{R}}$ is integrable, then $2\mu = \delta = \sqrt{|c|}$, which is a contradiction (see the argument in (I_b)).

Also, it is easy to verify that $\nabla_\xi \xi,\ \nabla_v \xi,\ \nabla_v(\phi v) \in \mathfrak{T}$. Now we prove $\nabla_\xi v \in \mathfrak{T}$. For this, we have

$$\begin{aligned}(\nabla_\xi A)v - (\nabla_v A)\xi &= \nabla_\xi(Av) - A\nabla_\xi v - \nabla_v(A\xi) + A\nabla_v \xi \\ &= (\xi\lambda)v + (\lambda I - A)\nabla_\xi v - \delta\lambda\phi v + \lambda\mu\phi v.\end{aligned}$$

On the other hand, Codazzi equation (2.6) gives

$$(\nabla_\xi A)v - (\nabla_v A)\xi = (c/4)\phi v.$$

Thus we find

$$(\lambda I - A)\nabla_\xi v = -\{\lambda(\mu - \delta) - (c/4)\}\phi v,$$

which, together with $g(\nabla_\xi v, v) = g(\nabla_\xi v, \xi) = 0$, implies $\nabla_\xi v \in \{\phi v\}_{\mathbb{R}} \subset \mathfrak{T}$. Similarly, we have $\nabla_\xi(\phi v) \in \mathfrak{T}$. Next, we prove $\nabla_{\phi v} v \in \mathfrak{T}$. It follows from $Av = \lambda v$ and $A\phi v = \mu\phi v$ that

$$\begin{aligned}(\nabla_v A)\phi v - (\nabla_{\phi v} A)v = (v\mu)\phi v + (\mu I - A)\nabla_v(\phi v) - (\phi v\lambda)v \\ - (\lambda I - A)\nabla_{\phi v} v.\end{aligned}$$

On the other hand, from (2.3) and $\nabla_v v = 0$ we also have

$$(\mu I - A)\nabla_v(\phi v) = -\lambda(\mu - \delta)\xi.$$

Moreover, Codazzi equation (2.6) yields

$$(\nabla_v A)\phi v - (\nabla_{\phi v} A)v = -(c/2)\xi.$$

It follows from these three equations that

$$-(c/2)\xi = (v\mu)\phi v - \lambda(\mu - \delta)\xi - (\phi v\lambda)v - (\lambda I - A)\nabla_{\phi v} v,$$

which implies $\nabla_{\phi v} v \in \{\xi, \phi v\}_{\mathbb{R}} \subset \mathfrak{T}$. Consequently, $\mathfrak{T} = \{v, \phi v, \xi\}_{\mathbb{R}}$ is integrable and each leaf (say,) L of $\mathfrak{T}$ is a totally geodesic submanifold of the real hypersurface M. Moreover, the distribution $\widetilde{\mathfrak{T}} := \{v, \phi v, \xi, \mathcal{N}\}_{\mathbb{R}}$ is also integrable and each leaf (, say) $\widetilde{L}$ of $\widetilde{\mathfrak{T}}$ is a totally geodesic submanifold $\mathbb{C}H^2(c)$ in the ambient space $\mathbb{C}H^n(c)$ (see the proof of Theorem 1). Then by the discussion in Case of (I) we know that λ must be constant locally on L, so that $(\nabla_v A)v = 0$. This, combined with the same computation as in (5.26), yields that λ is constant locally on our real hypersurface M.

Therefore our real hypersurface M is a Hopf hypersurface with constant principal curvatures in $\mathbb{C}H^n(c)$. □

Remark 3. (1) There does not exist $V_{\lambda_i}^0$ satisfying both of Conditions 2_a) and 2_b) in Theorem 2.
(2) We here give a comment on Condition (1) in Theorem 2. Needless to say, if M satisfies Condition (1) in Theorem 2, then M is a Hopf hypersurface. But, in general the converse does not hold. However, every Hopf hypersurface M satisfies locally Condition (1) on the open dense subset $\mathcal{U}$ given by (5.11).

As an immediate consequence of Theorem 2 we have the following which is a characterization of homogeneous real hypersurfaces of types (C), (D) and (E) in a complex projective space.

Theorem 3 ([16])**.** *Let M^{2n-1} be a connected real hypersurface in $\mathbb{C}P^n(c)$, $n \geqq 2$. Then M is locally congruent to one of homogeneous real hypersurfaces of types* (C), (D) *and* (E) *if and only if M satisfies the following three conditions.*

(1) *The holomorphic distribution $T^0M = \{X \in TM | X \perp \xi\}$ of M is decomposed as the direct sum of restricted principal distributions $V^0_{\lambda_i} = \{X \in T^0M | AX = \lambda_i X\}$.*
(2) *Every restricted principal distribution $V^0_{\lambda_i}$ in Condition* (1) *satisfies one of the following two conditions*:
   2_a) *The distribution $V^0_{\lambda_i} \oplus \{\xi\}_{\mathbb{R}}$ is integrable*;
   2_b) *$V^0_{\lambda_i}$ is integrable and every leaf of $V^0_{\lambda_i}$ is a totally geodesic submanifold of the real hypersurface M.*
(3) *There exist a restricted principal distribution $V^0_{\lambda_i}$ which satisfies Condition 2_a) and also a restricted principal distribution V_{λ_j} which satisfies Condition 2_b).*

The following result shows that Theorem 2 in the case of $c > 0$ is no longer true if we omit the condition that every leaf of $V^0_{\lambda_i}$ is a totally geodesic submanifold of the real hypersurface in Condition 2_b).

Theorem 4 ([10])**.** *There exists a real hypersurface M^{2n-1} in $\mathbb{C}P^n(c)$ with $n \geqq 3$ which satisfy the following properties.*

(1) *The holomorphic distribution $T^0M = \{X \in TM|\ X \perp \xi\}$ of M is decomposed as the direct sum of restricted principal distributions $V^0_{\lambda_i} = \{X \in T^0M|\ AX = \lambda_i X\}$.*
(2) *Every $V^0_{\lambda_1}$ in Condition* (1) *is integrable.*
(3) *There exists some $V^0_{\lambda_j}$ in Condition* (1) *satisfying that not every leaf of $V^0_{\lambda_j}$ is a totally geodesic submanifold of M.*
(4) *M has a non-constant principal curvature in $\mathbb{C}P^n(c)$.*

Proof. We take the Hopf fibration $\pi : S^{2n+1} \to \mathbb{C}P^n$ of $S^{2n+1} = \{z \in \mathbb{C}^{n+1} | \|z\| = 1\}$ onto $\mathbb{C}P^n$, which is a principal fibre bundle with structure group $S^1 \equiv \{e^{i\theta} | \theta \in \mathbb{R}\}$. For each $z \in S^{2n+1}$, let T'_z denote the horizontal subspace of $T_z S^{2n+1}$ consisting of all vectors at z which are orthogonal to z and $\sqrt{-1}\, z$. Since T' is invariant by the action of S^1, π induces an isomorphsm $\pi_* : T'_z \to T_{[z]}\mathbb{C}P^n$ for any $z \in S^{2n+1}$, where $[z] = \pi(z)$.

The Fubini-Study metric $\tilde{g}$ on $\mathbb{C}P^n$ of constant holomorphic sectional curvature 1 is defined as follows: For $\tilde{X}, \tilde{Y} \in T_{[z]}\mathbb{C}P^n$, let z be any point in $\pi^{-1}([z])$ and let $X, Y \in T'_z$ be the horizontal lifts of $\tilde{X}, \tilde{Y}$ at z. Then $\tilde{g}_{[z]}(\tilde{X}, \tilde{Y}) = 4g_z(X, Y)$. The metric $\tilde{g}$ is well defined because if z' is also a representation of $[z]$, $z' = e^{i\theta}z$ and so $g_{z'}(e^{i\theta}X, e^{i\theta}Y) = g_z(X, Y)$.

Let

$$f_{a,k}(z) = z_0^2 + \cdots + z_k^2 + a(z_{k+1}^2 + \cdots + z_n^2), \tag{5.30}$$

where $z = (z_0, z_1, \cdots, z_n) \in \mathbb{C}^{n+1}$, $k \geqq 1$, $n-k \geqq 2$ and $a(\neq 1)$ is a positive constant. We put

$$V^{n-1}_{a,k} = \{[z] = [z_0, \cdots, z_n] \in \mathbb{C}P^n | f_{a,k}(z) = 0\}$$

which is a complex hypersurface of $\mathbb{C}P^n$. For each $[z] \in V^{n-1}_{a,k}$, the tangent space $T_{[z]}V^{n-1}_{a,k}$ can be identified

$$\begin{aligned} T_z = \{X \in \mathbb{C}^{n+1} | \langle X, z\rangle &= \langle X, \sqrt{-1}\, z\rangle \\ &= \langle X, \partial f_{a,k}/\partial z\rangle = \langle X, \sqrt{-1}\, \partial f_{a,k}/\partial z\rangle = 0\}, \end{aligned} \tag{5.31}$$

where $\langle\ ,\ \rangle$ denotes the Euclidean inner product induced from $\mathbb{C}^{n+1}$. The vector field $\mathcal{N} = \frac{\partial f_{a,k}}{\partial z} / \big(2\|\frac{\partial f_{a,k}}{\partial z}\|\big)$ can be regarded as a unit normal vector field of $V^{n-1}_{a,k}$ in $\mathbb{C}P^n$. For each $X \in T_{[z]}V^{n-1}_{a,k}$, the shape operator $A_{\mathcal{N}}$ at $[z] \in V^{n-1}_{a,k}$ is given by (cf. [21]):

$$A_{\mathcal{N}}X = -\frac{1}{2\|\frac{\partial f_{a,k}}{\partial z}\|}\overline{X}\overline{\Big(\frac{\partial^2 f_{a,k}}{\partial z_i \partial z_j}\Big)} + \zeta \overline{\frac{\partial f_{a,k}}{\partial z}}, \tag{5.32}$$

where $\overline{w}$ is the complex conjugate of w, $\|\ \|$ is the Euclidean norm and

$$\zeta = \frac{1}{2\|\frac{\partial f_{a,k}}{\partial z}\|^3}\overline{X}\overline{\Big(\frac{\partial^2 f_{a,k}}{\partial z_i \partial z_j}\Big)}\Big(\frac{\partial f_{a,k}}{\partial z}\Big)^T. \tag{5.33}$$

We define three subsets U_1, U_2, U_3 of $V^{n-1}_{a,k}$ by

$$\begin{aligned} U_1 &= \{[z_0, \ldots, z_n] \in V^{n-1}_{a,k} | z_{k+1} = \cdots = z_n = 0\}, \\ U_2 &= \{[z_0, \ldots, z_n] \in V^{n-1}_{a,k} | z_0 = \cdots = z_k = 0\}, \\ U_3 &= \left\{[z_0, \ldots, z_n] \in V^{n-1}_{a,k} | \sum_{j=o}^{k} \|z_j\| \neq 0 \text{ and } \sum_{i=k+1}^{n} \|z_i\| \neq 0\right\}. \end{aligned}$$

Clearly, U_3 is an open dense subset of $V^{n-1}_{a,k}$ and U_1 can be identified with a complex hyperquadric $\mathbb{C}Q^{k-1} = \{[z_0, \ldots, z_k] | \sum_{j=0}^{k} z_j^2 = 0\}$ in a totally geodesic $\mathbb{C}P^k$ in $\mathbb{C}P^n$, which is defined by $\mathbb{C}P^k = \{[z] \in \mathbb{C}P^n | z_{k+1} = \cdots = z_n = 0\}$. Similarly, U_2 can be regarded as a complex hyperquadric in $\mathbb{C}P^{n-k-1} \subset \mathbb{C}P^n$.

We shall determine the eigenvalues of the shape operator of $V^{n-1}_{a,k}$ in $\mathbb{C}P^n$.

Case (a). We take a point $p \in U_1$. Since the group $\mathrm{A}(\mathbb{C}P^k)$ of holomorphic isometries of $\mathbb{C}P^k$ acts transitively on $\mathbb{C}Q^{k-1}$ and $\mathrm{A}(\mathbb{C}P^k)$ is a subgroup of $\mathrm{A}(\mathbb{C}P^n)$, by applying a suitable holomorphic isometry on $\mathbb{C}P^n$ if necessary we may assume that the homogeneous coordinates of p take the form $[z]_p = [z_0, z_1, \ldots, 0]$ with

$z_0, z_1 \neq 0$. By a direct computation we obtain

$$T_{[z]_p} V^{n-1}_{a,k} = \{(0, 0, v_2, \dots, v_n) | v_2, \dots, v_n \in \mathbb{C}\}, \tag{5.34}$$

$$\Big(\frac{\partial f_{a,k}}{\partial z}\Big)_p = 2(z_0, z_1, 0, \dots, 0), \tag{5.35}$$

$$\Big(\frac{\partial^2 f_{a,k}}{\partial z_i \partial z_j}\Big)_p = 2 \begin{pmatrix} I_{k+1} & 0 \\ 0 & aI_{n-k} \end{pmatrix}, \tag{5.36}$$

where I_r is the identity matrix of order r. It follows from (5.32), (5.33), (5.34), (5.35) and (5.36) that $\zeta = 0$ and $\|\frac{\partial f_{a,k}}{\partial z}\| = 2$ at p, and also

$$A_{\mathcal{N}} V = -(1/2)\big(0, 0, \overline{v_2}, \dots, \overline{v_k}, a\overline{v_{k+1}}, \dots, a\overline{v_n}\big) \tag{5.37}$$

for $V = (0, 0, v_2, \dots, v_n) \in T_{[z]_p} V^{n-1}_{a,k}$. Hence, from (5.37) we see that

$$\begin{aligned} V_1 &= \{(0, 0, r_2, \dots, r_k, 0, \dots, 0) | r_2, \dots, r_k \in \mathbb{R}\}, \\ V_2 &= \{\sqrt{-1}\,(0, 0, r_2, \dots, r_k, 0, \dots, 0) | r_2, \dots, r_k \in \mathbb{R}\}, \\ V_3 &= \{(0, \dots, 0, r_{k+1}, \dots, r_n) | r_{k+1}, \dots, r_n \in \mathbb{R}\}, \\ V_4 &= \{\sqrt{-1}\,(0, \dots, 0, r_{k+1}, \dots, r_n) | r_{k+1}, \dots, r_n \in \mathbb{R}\}, \end{aligned}$$

are eigenspaces of $A_{\mathcal{N}}$ with eigenvalues $-1/2,\ 1/2,\ -a/2,\ a/2$ and with multiplicities $k-1,\ k-1,\ n-k,\ n-k$, respectively. By using orthonormal bases defined by

$$\{e_2, \dots, e_k\},\ \{\sqrt{-1}\, e_2, \dots, \sqrt{-1}\, e_k\},\ \{e_{k+1}, \dots, e_n\},$$
$$\{\sqrt{-1}\, e_{k+1}, \dots, \sqrt{-1}\, e_n\}$$

of V_1, V_2, V_3, V_4, respectively, we find

$$A_{\mathcal{N}} = \frac{1}{2} \begin{pmatrix} -I_{k-1} & 0 & 0 & 0 \\ 0 & I_{n-k} & 0 & 0 \\ 0 & 0 & -aI_{n-k} & 0 \\ 0 & 0 & 0 & aI_{n-k} \end{pmatrix}. \tag{5.38}$$

Therefore, by $A_{J\mathcal{N}} = JA_{\mathcal{N}}$ and (5.38) we get

$$A_{J\mathcal{N}} = -\frac{1}{2} \begin{pmatrix} 0 & I_{k-1} & 0 & 0 \\ I_{n-k} & 0 & 0 & 0 \\ 0 & 0 & 0 & aI_{n-k} \\ 0 & 0 & aI_{n-k} & 0 \end{pmatrix}. \tag{5.39}$$

It follows from (5.38) and (5.39) that, for any unit normal vector $\eta = (\cos\theta)\mathcal{N} + (\sin\theta)J\mathcal{N}$, we have

$$A_\eta = -\frac{1}{2} \begin{pmatrix} \cos\theta\, I_{k-1} & \sin\theta\, I_{k-1} & 0 & 0 \\ \sin\theta\, I_{k-1} & -\cos\theta\, I_{k-1} & 0 & 0 \\ 0 & 0 & a\cos\theta\, I_{n-k} & a\sin\theta\, I_{n-k} \\ 0 & 0 & a\sin\theta\, I_{n-k} & -a\cos\theta\, I_{n-k} \end{pmatrix}. \tag{5.40}$$

which implies that, for each unit normal vector η A_η has eigenvalues $-1/2, 1/2, -a/2, a/2$ with multiplicities $k-1, k-1, n-k, n-k$, respectively.

Case (b). We take a point $p \in U_2$. By the argument similar to Case (a) we find that for each unit normal vector η of U_2 at p A_η has eigenvalues $-1/(2a), 1/(2a), -1/2, 1/2$ with multiplicities $k+1, k+1, n-k-2, n-k-2$, respectively.

Case (c). We take a point $p \in U_3$. We first show that there is a holomorphic isometry of $\mathbb{C}P^n$ which carries p to a point $\hat{p} \in U_3$ whose homogeneous coordinates take the form $[z]_{\hat{p}} = [z_0, 0, \ldots, z_n]$ with $z_0, z_n \neq 0$. This can be seen as follows:

We take a point $p = [w_0, \ldots, w_n] \in U_3$. Then $\sum_{j=0}^{k} \|w_j\| \neq 0$ and $\sum_{j=k+1}^{n} \|w_j\| \neq 0$. Let $U_3^1 = \{[v_0, \ldots, v_n] \in U_3 | v_{k+1} = w_{k+1}, \ldots, v_n = w_n\}$ which is the intersection of U_3 with the $n-k$ linear subspaces of $\mathbb{C}P^n$ defined by $z_j = w_j$, $j = k+1, \ldots, n$. It follows from $\sum_{j=k+1}^{n} \|w_j\| \neq 0$ that there is a point q in U_3^1 whose homogeneous coordinate takes the form $[z_0, 0, \ldots, 0, w_{k+1}, \ldots, w_n]$ with $z_0 \neq 0$. Since the group $\mathrm{A}(\mathbb{C}P^n)$ acts transitively on $\mathbb{C}P^n$, there is a holomorphic isometry of $\mathbb{C}P^n$ which carries p to q. Similarly, there is a holomorphic isometry of $\mathbb{C}P^n$ which carries q to a point $\hat{p} \in U_3$ whose homogeneous coordinates take the form $[z]_{\hat{p}} = [z_0, 0, \ldots, 0, z_n]$ with $z_0, z_n \neq 0$. Consequently, without loss of generality, we may assume that the homogeneous coordinates of p take the form $[z_0, 0, \ldots, 0, z_n]$ with $z_0, z_n \neq 0$, by applying a suitable holomorphic isometry of $\mathbb{C}P^n$ if necessary.

By a direct computation we obtain

$$T_{[z]_p} V_{a,k}^{n-1} = \{(0, v_1, \ldots, v_{n-1}, 0) | v_1, \ldots, v_{n-1} \in \mathbb{C}\}, \tag{5.41}$$

$$\left(\frac{\partial^2 f_{a,k}}{\partial z_i \partial z_j}\right)_p = 2 \begin{pmatrix} I_{k+1} & 0 \\ 0 & aI_{n-k} \end{pmatrix}. \tag{5.42}$$

It follows from (5.32), (5.41) and (5.42) that $\zeta = 0$ and

$$A_{\mathcal{N}} V = -\frac{1}{\|\frac{\partial f_{a,k}}{\partial z}\|}(0, \overline{v_1}, \ldots, \overline{v_k}, a\overline{v_{k+2}}, \ldots, a\overline{v_{n-1}}, 0) \tag{5.43}$$

for $V = (0, v_1, \ldots, v_{n-1}, 0) \in T_{[z]_p} V_{a,k}^{n-1}$. Equation (5.43) implies that $A_{\mathcal{N}}$ has eigenvalues $-\|\frac{\partial f_{a,k}}{\partial z}\|^{-1}, \|\frac{\partial f_{a,k}}{\partial z}\|^{-1}, -a\|\frac{\partial f_{a,k}}{\partial z}\|^{-1}$ and $a\|\frac{\partial f_{a,k}}{\partial z}\|^{-1}$ with multiplicities $k, k, n-k-1$ and $n-k-1$, respectively. Due to the same argument as in Case (a) we know that for each unit normal vector η, A_η has the same eigenvalues and with the same multiplicities as those of $A_{\mathcal{N}}$.

Since $k \geqq 1$ and $n-k \geqq 2$, U_3 has four distinct principal curvatures and they satisfy the following two properties:

(1) Each eigenvalue of the shape operator of U_3 with respect to any given unit normal vector η is not zero.
(2) The multiplicity of each eigenvalue of A_η with respect to any unit normal vector η is constant.

Let M be the real hypersurface in $\mathbb{C}P^n$ given by the tube of radius $r(>0)$ over the complex hypersurface U_3. Then M has at most five distinct principal curvatures in $\mathbb{C}P^n$. Moreover, the holomorphic distribution T^0M of M satisfies properties (1) and (2) in Theorem 4 (cf. [9]). Furthermore, because M is not in Takagi's list, properties (3) and (4) must hold according to Theorem 2. □

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