

REFINEMENTS AND REVERSES FOR THE RELATIVE OPERATOR ENTROPY $S(A|B)$ WHEN $B \geq A$

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ABSTRACT. In this paper we obtain new refinements and reverse inequalities for the relative operator entropy $S(A|B)$ of two positive invertible operators when $B \geq A$. Applications for the operator entropy $\eta(C)$ in the case of positive contractions C are also given.

1. INTRODUCTION

Kamei and Fujii [6], [7] defined the *relative operator entropy* $S(A|B)$, for positive invertible operators A and B , by

$$(1.1) \quad S(A|B) := A^{\frac{1}{2}} \left(\ln \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right) \right) A^{\frac{1}{2}},$$

which is a relative version of the *operator entropy* considered by Nakamura-Umegaki [15].

For the *entropy function* $\eta(t) = -t \ln t$, the operator entropy has the following expression:

$$\eta(A) = -A \ln A = S(A|1_H) \geq 0$$

for positive contraction A . This shows that the relative operator entropy (1.1) is a relative version of the operator entropy

In [18], A. Uhlmann has shown that the relative operator entropy $S(A|B)$ can be represented as the strong limit

$$(1.2) \quad S(A|B) = s\text{-}\lim_{t \rightarrow 0} \frac{A \sharp_t B - A}{t},$$

where

$$A \sharp_\nu B := A^{1/2} \left(A^{-1/2} B A^{-1/2} \right)^\nu A^{1/2}, \quad \nu \in [0, 1]$$

is the *weighted geometric mean* of positive invertible operators A and B . For $\nu = \frac{1}{2}$ we denote $A \sharp B$.

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This definition of the weighted geometric mean can be extended for any real number ν with $\nu \neq 0$.

Following [11, p. 149-p. 155], we recall some important properties of relative operator entropy for A and B positive invertible operators:

(i) We have the equalities:

$$(1.3) \quad S(A|B) = -A^{1/2} (\ln A^{1/2} B^{-1} A^{1/2}) A^{1/2} = B^{1/2} \eta(B^{-1/2} A B^{-1/2}) B^{1/2};$$

(ii) We have the inequalities

$$(1.4) \quad S(A|B) \leq A (\ln \|B\| - \ln A) \text{ and } S(A|B) \leq B - A;$$

(iii) For any C, D positive invertible operators we have that

$$S(A + B|C + D) \geq S(A|C) + S(B|D);$$

(iv) If $B \leq C$ then

$$S(A|B) \leq S(A|C);$$

(v) If $B_n \downarrow B$ then

$$S(A|B_n) \downarrow S(A|B);$$

(vi) For $\alpha > 0$ we have

$$S(\alpha A|\alpha B) = \alpha S(A|B);$$

(vii) For every operator T we have

$$T^* S(A|B) T \leq S(T^* A T | T^* B T).$$

The relative operator entropy is *jointly concave*, namely, for any positive invertible operators A, B, C, D we have

$$S(tA + (1-t)B | tC + (1-t)D) \geq tS(A|C) + (1-t)S(B|D)$$

for any $t \in [0, 1]$.

For other results on the relative operator entropy see [3], [8], [12], [13], [14] and [16].

For $t > 0$ and the positive invertible operators A, B we define the *Tsallis relative operator entropy* (see also [10]) by

$$T_t(A|B) := \frac{A \sharp_t B - A}{t}.$$

We observe that, for the function

$$f(x) = \frac{1}{t} (1 - x^{-t}) = \frac{x^t - 1}{t} x^{-t}, \quad x > 0,$$

we have

$$\begin{aligned} A^{1/2} f(A^{-1/2} B A^{-1/2}) A^{1/2} &= -A T_t(A^{-1} | B^{-1}) A = T_t(A|B) (A^{-1} \sharp_t B^{-1}) A \\ &= T_t(A|B) (A \sharp_t B)^{-1} A \end{aligned}$$

for any positive invertible operators A, B and $t > 0$.

The following result providing upper and lower bounds for relative operator entropy in terms of $T_t(\cdot|\cdot)$ has been obtained in [6] for $0 < t \leq 1$. However, it holds for any $t > 0$.

Theorem 1. *Let A, B be two positive invertible operators, then for any $t > 0$ we have*

$$(1.5) \quad T_t(A|B) (A \sharp_t B)^{-1} A \leq S(A|B) \leq T_t(A|B).$$

In particular, we have

$$(1.6) \quad A - AB^{-1}A \leq S(A|B) \leq B - A \quad [6]$$

and

$$(1.7) \quad \frac{1}{2}A \left(1_H - (B^{-1}A)^2\right) \leq S(A|B) \leq \frac{1}{2}(BA^{-1}B - A).$$

The case $t = \frac{1}{2}$ is of interest as well, since in this case we get from (1.5) that

$$(1.8) \quad 2(1_H - A(A \sharp B)^{-1})A \leq S(A|B) \leq 2(A \sharp B - A) \leq B - A.$$

This inequality provides a refinement and a reverse for (1.4).

The following upper and lower bounds for the operator entropy also hold for any positive invertible operator C and any $t > 0$:

$$(1.9) \quad \frac{1}{t}C(1_H - C^t) \leq \eta(C) \leq \frac{1}{t}C^{1-t}(1_H - C^t).$$

In particular, we have

$$(1.10) \quad C(1_H - C) \leq \eta(C) \leq 1_H - C,$$

$$(1.11) \quad \frac{1}{2}C(1_H - C^2) \leq \eta(C) \leq \frac{1}{2}(C^{-1} - C)$$

and

$$(1.12) \quad 2C(1_H - C^{1/2}) \leq \eta(C) \leq 2C^{1/2}(1_H - C^{1/2}).$$

Motivated by the above results, in this paper we obtain new refinements and reverse inequalities for the relative operator entropy $S(A|B)$ of two positive invertible operators when $B \geq A$. Applications for the operator entropy $\eta(C)$ in the case of positive contractions C are also given.

2. SOME REFINEMENTS

We start with the following sequence of scalar inequalities:

Lemma 1. *For any $y \geq 1$ we have the inequalities*

$$(2.1) \quad \begin{aligned} 0 &\leq \frac{y-1}{y} \leq \frac{2(y-1)}{y+1} \leq \ln y \leq \frac{y-1}{\sqrt{y}} \\ &\leq \frac{y-1}{y+1} + \frac{y^2-1}{4y} \leq \frac{y^2-1}{2y} \leq y-1. \end{aligned}$$

Proof. We prove only the third, fourth and fifth inequalities, the other ones are obvious due to the fact that $y \geq 1$.

We use the first *Hermite-Hadamard inequality* for *convex functions*, namely [5]

$$(2.2) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t) dt,$$

where $f : [a, b] \rightarrow \mathbb{R}$ is a convex function.

If we take in (2.2) $a = 1$ and $b = y$, then we get the third inequality in (2.1).

It is known that, if $G(a, b) := \sqrt{ab}$ is the *geometric mean* of $a, b > 0$ and

$$L(a, b) := \begin{cases} \frac{b-a}{\ln b - \ln a} & \text{if } b \neq a, \\ a & \text{if } b = a \end{cases}$$

is the *logarithmic mean* of a, b , then

$$(2.3) \quad G(a, b) \leq L(a, b).$$

Now, if we take in (2.2) $f(t) = \frac{1}{t}$, $a = 1$ and $b = y$, then we get the fourth inequality in (2.1).

By arithmetic mean-geometric mean inequality,

$$\frac{1}{y+1} + \frac{y+1}{4y} \geq 2\sqrt{\left(\frac{1}{y+1}\right) \frac{y+1}{4y}} = \frac{1}{\sqrt{y}}$$

for $y > 0$, which proves the fifth inequality in (2.1). ■

The following result provides an improvement of (1.5) in the case that $B \geq A$.

Theorem 2. *Let A, B be two positive invertible operators and $B \geq A$, then for any $t > 0$ we have*

$$(2.4) \quad \begin{aligned} 0 &\leq T_t(A|B) (A \sharp_t B)^{-1} A \\ &\leq 2T_t(A|B) (A \sharp_t B + A)^{-1} A \\ &\leq S(A|B) \leq T_t(A|B) (A \sharp_{t/2} B)^{-1} A \\ &\leq T_t(A|B) (A \sharp_t B + A)^{-1} A + \frac{1}{2} T_{2t}(A|B) (A \sharp_t B)^{-1} A \\ &\leq T_{2t}(A|B) (A \sharp_t B)^{-1} A \leq T_t(A|B). \end{aligned}$$

Proof. Let $x \geq 1$ and $t > 0$, then by taking $y = x^t$ in (2.1) we get

$$(2.5) \quad \begin{aligned} 0 &\leq \frac{x^t - 1}{tx^t} \leq \frac{2(x^t - 1)}{t(x^t + 1)} \leq \ln x \leq \frac{x^t - 1}{tx^{t/2}} \\ &\leq \frac{x^t - 1}{t(x^t + 1)} + \frac{x^{2t} - 1}{4tx^t} \leq \frac{x^{2t} - 1}{2tx^t} \leq \frac{x^t - 1}{t}. \end{aligned}$$

Using the functional calculus for the operator $X \geq 1_H$, then by (2.5) we get

$$(2.6) \quad \begin{aligned} 0 &\leq \frac{X^t - 1}{t} X^{-t} \leq 2 \frac{(X^t - 1)}{t} (X^t + 1)^{-1} \leq \ln X \\ &\leq \frac{X^t - 1}{t} X^{-t/2} \leq \frac{(X^t - 1)}{t} (X^t + 1)^{-1} + \frac{1}{2} \frac{X^{2t} - 1}{2t} X^{-t} \\ &\leq \frac{X^{2t} - 1}{2t} X^{-t} \leq \frac{X^t - 1}{t}. \end{aligned}$$

If $B \geq A$, then by multiplying both sides by $A^{-1/2}$ we get $A^{-1/2}BA^{-1/2} \geq 1_H$ and if we write the inequality for $X = A^{-1/2}BA^{-1/2}$, we get

$$\begin{aligned}
(2.7) \quad 0 &\leq \frac{(A^{-1/2}BA^{-1/2})^t - 1}{t} (A^{-1/2}BA^{-1/2})^{-t} \\
&\leq 2 \frac{\left((A^{-1/2}BA^{-1/2})^t - 1 \right)}{t} \left((A^{-1/2}BA^{-1/2})^t + 1 \right)^{-1} \\
&\leq \ln (A^{-1/2}BA^{-1/2}) \\
&\leq \frac{(A^{-1/2}BA^{-1/2})^t - 1}{t} (A^{-1/2}BA^{-1/2})^{-t/2} \\
&\leq \frac{\left((A^{-1/2}BA^{-1/2})^t - 1 \right)}{t} \left((A^{-1/2}BA^{-1/2})^t + 1 \right)^{-1} \\
&\quad + \frac{1}{2} \frac{(A^{-1/2}BA^{-1/2})^{2t} - 1}{2t} (A^{-1/2}BA^{-1/2})^{-t} \\
&\leq \frac{(A^{-1/2}BA^{-1/2})^{2t} - 1}{2t} (A^{-1/2}BA^{-1/2})^{-t} \\
&\leq \frac{(A^{-1/2}BA^{-1/2})^t - 1}{t}.
\end{aligned}$$

Now, by multiplying both sides of (2.7) with $A^{1/2}$, we get

$$\begin{aligned}
(2.8) \quad 0 &\leq A^{1/2} \frac{(A^{-1/2}BA^{-1/2})^t - 1}{t} (A^{-1/2}BA^{-1/2})^{-t} A^{1/2} \\
&\leq 2A^{1/2} \frac{\left((A^{-1/2}BA^{-1/2})^t - 1 \right)}{t} \left((A^{-1/2}BA^{-1/2})^t + 1 \right)^{-1} A^{1/2} \\
&\leq A^{1/2} (\ln (A^{-1/2}BA^{-1/2})) A^{1/2} \\
&\leq A^{1/2} \frac{(A^{-1/2}BA^{-1/2})^t - 1}{t} (A^{-1/2}BA^{-1/2})^{-t/2} A^{1/2} \\
&\leq A^{1/2} \frac{\left((A^{-1/2}BA^{-1/2})^t - 1 \right)}{t} \left((A^{-1/2}BA^{-1/2})^t + 1 \right)^{-1} A^{1/2} \\
&\quad + \frac{1}{2} A^{1/2} \frac{(A^{-1/2}BA^{-1/2})^{2t} - 1}{2t} (A^{-1/2}BA^{-1/2})^{-t} A^{1/2} \\
&\leq A^{1/2} \frac{(A^{-1/2}BA^{-1/2})^{2t} - 1}{2t} (A^{-1/2}BA^{-1/2})^{-t} A^{1/2} \\
&\leq A^{1/2} \frac{(A^{-1/2}BA^{-1/2})^t - 1}{t} A^{1/2}.
\end{aligned}$$

Observe that

$$\begin{aligned}
& A^{1/2} \frac{(A^{-1/2}BA^{-1/2})^t - 1}{t} (A^{-1/2}BA^{-1/2})^{-t} A^{1/2} \\
&= T_t(A|B) (A^{-1}\sharp_t B^{-1}) A = T_t(A|B) (A\sharp_t B)^{-1} A, \\
& A^{1/2} \frac{\left((A^{-1/2}BA^{-1/2})^t - 1\right)}{t} \left((A^{-1/2}BA^{-1/2})^t + 1\right)^{-1} A^{1/2} \\
&= A^{1/2} \frac{\left((A^{-1/2}BA^{-1/2})^t - 1\right)}{t} A^{1/2} A^{-1/2} \\
&= \left(A^{-1/2} \left(A^{1/2} (A^{-1/2}BA^{-1/2})^t A^{1/2} + A\right) A^{-1/2}\right)^{-1} A^{1/2} \\
&= A^{1/2} \frac{\left((A^{-1/2}BA^{-1/2})^t - 1\right)}{t} A^{1/2} A^{-1/2} \\
&= A^{1/2} \left(A^{1/2} (A^{-1/2}BA^{-1/2})^t A^{1/2} + A\right)^{-1} A^{1/2} A^{1/2} \\
&= T_t(A|B) (A\sharp_t B + A)^{-1} A, \\
& A^{1/2} \frac{(A^{-1/2}BA^{-1/2})^t - 1}{t} (A^{-1/2}BA^{-1/2})^{-t/2} A^{1/2} \\
&= A^{1/2} \frac{(A^{-1/2}BA^{-1/2})^t - 1}{t} A^{1/2} A^{-1/2} (A^{1/2}B^{-1}A^{1/2})^{t/2} A^{-1/2} A^{1/2} A^{1/2} \\
&= T_t(A|B) (A^{-1}\sharp_{t/2} B^{-1}) A = T_t(A|B) (A\sharp_{t/2} B)^{-1} A
\end{aligned}$$

and

$$\begin{aligned}
& A^{1/2} \frac{(A^{-1/2}BA^{-1/2})^{2t} - 1}{2t} (A^{-1/2}BA^{-1/2})^{-t} A^{1/2} \\
&= A^{1/2} \frac{(A^{-1/2}BA^{-1/2})^{2t} - 1}{2t} A^{1/2} A^{-1/2} (A^{1/2}B^{-1}A^{1/2})^t A^{-1/2} A^{1/2} A^{1/2} \\
&= T_{2t}(A|B) (A\sharp_{2t} B)^{-1} A.
\end{aligned}$$

By using the inequalities (2.8) we get the desired result (2.4). ■

If we take in (2.4) $t = \frac{1}{2}$, then we get the inequalities

$$\begin{aligned}
(2.9) \quad & 0 \leq 2(1_H - A(A\sharp B)^{-1})A \\
& \leq 4(A\sharp B - A)(A\sharp B + A)^{-1}A \\
& \leq S(A|B) \leq 2(A\sharp B - A)(A\sharp_{1/4} B)^{-1}A \\
& \leq 2(A\sharp B - A)(A\sharp B + A)^{-1}A + \frac{1}{2}(B - A)(A\sharp B)^{-1}A \\
& \leq (B - A)(A\sharp B)^{-1}A \leq 2(A\sharp B - A),
\end{aligned}$$

for any positive invertible operators with $B \geq A$. This provides a refinement of (1.8).

If we take in (2.4) $t = 1$, then we get

$$\begin{aligned}
 (2.10) \quad 0 &\leq (B - A) B^{-1} A \leq 2 (B - A) (B + A)^{-1} A \\
 &\leq S(A|B) \leq (B - A) (A \sharp B)^{-1} A \\
 &\leq (B - A) (B + A)^{-1} A + \frac{1}{4} (B - AB^{-1}A) \\
 &\leq \frac{1}{2} (B - AB^{-1}A) \leq B - A,
 \end{aligned}$$

for any positive invertible operators with $B \geq A$. This provides a refinement of (1.6).

If we take in (2.4) $t = 2$, then we get

$$\begin{aligned}
 (2.11) \quad 0 &\leq \frac{1}{2} (BA^{-1}B - A) (B^{-1}A)^2 \\
 &\leq (BA^{-1}B - A) (BA^{-1}B + A)^{-1} A \\
 &\leq S(A|B) \leq \frac{1}{2} (BA^{-1}B - A) B^{-1} A \\
 &\leq \frac{1}{2} (BA^{-1}B - A) (BA^{-1}B + A)^{-1} A + \frac{1}{8} \left((BA^{-1})^2 B - A \right) (B^{-1}A)^2 \\
 &\leq \frac{1}{4} \left((BA^{-1})^2 B - A \right) (B^{-1}A)^2 \leq \frac{1}{2} (BA^{-1}B - A),
 \end{aligned}$$

for any positive invertible operators with $B \geq A$. This provides a refinement of (1.7).

Corollary 1. *Let C be a positive invertible operator and $C \leq 1_H$, then for any $t > 0$ we have*

$$\begin{aligned}
 (2.12) \quad 0 &\leq \frac{1}{t} C (1_H - C^t) \\
 &\leq \frac{2}{t} C (1_H - C^t) (1_H + C^t)^{-1} \\
 &\leq \eta(C) \leq \frac{1}{t} (1_H - C^t) C^{1-\frac{t}{2}} \\
 &\leq \frac{1}{t} C (1_H - C^t) (1_H + C^t)^{-1} + \frac{1}{4t} (1_H - C^{2t}) C^{1-t} \\
 &\leq \frac{1}{2t} (1_H - C^{2t}) C^{1-t} \leq \frac{1}{t} C^{1-t} (1_H - C^t).
 \end{aligned}$$

If we take in (2.12) $t = \frac{1}{2}$, then we get

$$\begin{aligned}
 (2.13) \quad 0 &\leq 2C (1_H - C^{1/2}) \\
 &\leq 4C (1_H - C^{1/2}) (1_H + C^{1/2})^{-1} \\
 &\leq \eta(C) \leq 2 (1_H - C^{1/2}) C^{9/4} \\
 &\leq 2C (1_H - C^{1/2}) (1_H + C^{1/2})^{-1} + \frac{1}{2} (1_H - C) C^{1/2} \\
 &\leq (1_H - C) C^{1/2} \leq 2C^{1/2} (1_H - C^{1/2}),
 \end{aligned}$$

for any C be a positive invertible operator with $C \leq 1_H$, which is better than (1.12).

If we take in (2.12) $t = 1$, then we get

$$\begin{aligned}
 (2.14) \quad 0 &\leq C (1_H - C) \leq 2C (1_H - C) (1_H + C)^{-1} \\
 &\leq \eta(C) \leq (1_H - C) C^{3/2} \\
 &\leq C (1_H - C) (1_H + C)^{-1} + \frac{1}{4} (1_H - C^2) \\
 &\leq \frac{1}{2} (1_H - C^2) \leq 1_H - C,
 \end{aligned}$$

for any C be a positive invertible operator with $C \leq 1_H$, which is better than (1.10).

Finally, if we take in (2.12) $t = 2$, then we get

$$\begin{aligned}
 (2.15) \quad 0 &\leq \frac{1}{2} C (1_H - C^2) \\
 &\leq C (1_H - C^2) (1_H + C^2)^{-1} \\
 &\leq \eta(C) \leq \frac{1}{2} (1_H - C^2) \\
 &\leq \frac{1}{2} C (1_H - C^2) (1_H + C^2)^{-1} + \frac{1}{8} (1_H - C^4) C^{-1} \\
 &\leq \frac{1}{4} (1_H - C^4) C^{-1} \leq \frac{1}{2} C^{-1} (1_H - C^2),
 \end{aligned}$$

for any C be a positive invertible operator with $C \leq 1_H$, which is better than (1.11).

3. SOME REVERSES

We have:

Lemma 2. *For any $y \geq 1$ we have the inequalities*

$$(3.1) \quad 0 \leq \frac{y^2 - 1}{2y} - \ln y \leq \frac{1}{8} \frac{(y - 1)^3 (y + 1)}{y^2}$$

and

$$(3.2) \quad 0 \leq \ln y - \frac{2(y-1)}{y+1} \leq \frac{1}{8} \frac{(y-1)^3 (y+1)}{y^2}.$$

Proof. We use the following reverse of the second Hermite-Hadamard inequality obtained in [2]:

$$(3.3) \quad 0 \leq \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) dt \leq \frac{1}{8} (f'_-(b) - f'_+(a)) (b-a).$$

If we take in this inequality $f(t) = \frac{1}{t}$, then we get

$$(3.4) \quad 0 \leq \frac{a+b}{2ab} - \frac{\ln b - \ln a}{b-a} \leq \frac{1}{8} \frac{(b-a)^2 (b+a)}{a^2 b^2}$$

for any $a, b > 0$.

If in this inequality we take $a = 1$ and $b = y \geq 1$, then we get the desired result (3.1).

Further, we use the following reverse of the first Hermite-Hadamard inequality obtained in [1]:

$$(3.5) \quad 0 \leq \frac{1}{b-a} \int_a^b f(t) dt - f\left(\frac{a+b}{2}\right) \leq \frac{1}{8} (f'_-(b) - f'_+(a)) (b-a).$$

If we take in this inequality $f(t) = \frac{1}{t}$, then we get

$$(3.6) \quad 0 \leq \frac{\ln b - \ln a}{b-a} - \frac{2}{a+b} \leq \frac{1}{8} \frac{(b-a)^2 (b+a)}{a^2 b^2}$$

for any $a, b > 0$.

If in this inequality we take $a = 1$ and $b = y \geq 1$, then we get the desired result (3.2). ■

We also have:

Theorem 3. Let A, B be two positive invertible operators and $B \geq A$, then for any $t > 0$ we have

$$(3.7) \quad \begin{aligned} 0 &\leq T_{2t}(A|B) (A \sharp_t B)^{-1} A - S(A|B) \\ &\leq \frac{1}{8} T_t(A|B) (A^{-1} - (A \sharp_t B)^{-1}) A (A^{-1} - (A \sharp_t B)^{-1}) (A \sharp_t B + A) \end{aligned}$$

and

$$(3.8) \quad \begin{aligned} 0 &\leq S(A|B) - 2T_t(A|B) (A \sharp_t B + A)^{-1} A \\ &\leq \frac{1}{8} T_t(A|B) (A^{-1} - (A \sharp_t B)^{-1}) A (A^{-1} - (A \sharp_t B)^{-1}) (A \sharp_t B + A). \end{aligned}$$

Proof. From inequality (3.1) for $y = x^t$ with $x \geq 1$ and $t > 0$, we have

$$0 \leq \frac{x^{2t} - 1}{2x^t} - \ln x^t \leq \frac{1}{8} \frac{(x^t - 1)^3 (x^t + 1)}{x^{2t}},$$

that is equivalent to

$$0 \leq \frac{x^{2t} - 1}{2t} x^{-t} - \ln x \leq \frac{1}{8} \left(\frac{x^t - 1}{t} \right) (1 - x^{-t})^2 (x^t + 1),$$

for any $x \geq 1$ and $t > 0$.

By using the functional calculus, we have

$$\begin{aligned} (3.9) \quad 0 &\leq \frac{(A^{-1/2}BA^{-1/2})^{2t} - 1}{2t} (A^{-1/2}BA^{-1/2})^{-t} - \ln(A^{-1/2}BA^{-1/2}) \\ &\leq \frac{1}{8} \left(\frac{(A^{-1/2}BA^{-1/2})^t - 1}{t} \right) \left(1 - (A^{-1/2}BA^{-1/2})^{-t} \right)^2 \\ &\quad \left((A^{-1/2}BA^{-1/2})^t + 1 \right), \end{aligned}$$

for any A, B positive invertible operators with $B \geq A$ and for any $t > 0$.

If we multiply both sides with $A^{1/2}$, then we get

$$\begin{aligned} (3.10) \quad 0 &\leq A^{1/2} \frac{(A^{-1/2}BA^{-1/2})^{2t} - 1}{2t} (A^{-1/2}BA^{-1/2})^{-t} A^{1/2} \\ &\quad - A^{1/2} (\ln(A^{-1/2}BA^{-1/2})) A^{1/2} \\ &\leq \frac{1}{8} A^{1/2} \left(\frac{(A^{-1/2}BA^{-1/2})^t - 1}{t} \right) \left(1 - (A^{-1/2}BA^{-1/2})^{-t} \right)^2 \\ &\quad \left((A^{-1/2}BA^{-1/2})^t + 1 \right) A^{1/2}. \end{aligned}$$

Observe that

$$\begin{aligned}
& \frac{1}{8} A^{1/2} \left(\frac{(A^{-1/2} B A^{-1/2})^t - 1}{t} \right) A^{1/2} A^{-1/2} \\
& \left(A^{1/2} \left(A^{-1} - A^{-1/2} (A^{1/2} B^{-1} A^{1/2})^t A^{-1/2} \right) A^{1/2} \right)^2 \\
& \left(A^{-1/2} \left(A^{1/2} (A^{-1/2} B A^{-1/2})^t A^{1/2} + A \right) A^{-1/2} \right) A^{1/2} \\
& = \frac{1}{8} A^{1/2} \left(\frac{(A^{-1/2} B A^{-1/2})^t - 1}{t} \right) A^{1/2} \\
& A^{-1/2} A^{1/2} \left(A^{-1} - A^{-1/2} (A^{1/2} B^{-1} A^{1/2})^t A^{-1/2} \right) A^{1/2} \\
& A^{1/2} \left(A^{-1} - A^{-1/2} (A^{1/2} B^{-1} A^{1/2})^t A^{-1/2} \right) A^{1/2} \\
& A^{-1/2} \left(A^{1/2} (A^{-1/2} B A^{-1/2})^t A^{1/2} + A \right) A^{-1/2} A^{1/2} \\
& = \frac{1}{8} A^{1/2} \left(\frac{(A^{-1/2} B A^{-1/2})^t - 1}{t} \right) A^{1/2} \\
& \left(A^{-1} - A^{-1/2} (A^{1/2} B^{-1} A^{1/2})^t A^{-1/2} \right) A \\
& \left(A^{-1} - A^{-1/2} (A^{1/2} B^{-1} A^{1/2})^t A^{-1/2} \right) \\
& \left(A^{1/2} (A^{-1/2} B A^{-1/2})^t A^{1/2} + A \right) \\
& = \frac{1}{8} T_t(A|B) (A^{-1} - (A\sharp_t B)^{-1}) A (A^{-1} - (A\sharp_t B)^{-1}) (A\sharp_t B + A)
\end{aligned}$$

and by (3.10) we get the desired result (3.7).

The inequality (3.8) follows in a similar way and we omit the details. ■

If we take in (3.7) and (3.8) $t = \frac{1}{2}$, then we get

$$\begin{aligned}
(3.11) \quad 0 & \leq (B - A) (A\sharp B)^{-1} A - S(A|B) \\
& \leq \frac{1}{4} (A\sharp B - A) (A^{-1} - (A\sharp B)^{-1}) A (A^{-1} - (A\sharp B)^{-1}) (A\sharp B + A)
\end{aligned}$$

and

$$\begin{aligned}
(3.12) \quad 0 & \leq S(A|B) - 4 (A\sharp B - A) (A\sharp B + A)^{-1} A \\
& \leq \frac{1}{4} (A\sharp B - A) (A^{-1} - (A\sharp B)^{-1}) A (A^{-1} - (A\sharp B)^{-1}) (A\sharp B + A)
\end{aligned}$$

for any positive invertible operators with $B \geq A$.

If we take in (3.7) and (3.8) $t = 1$, then we get

$$(3.13) \quad \begin{aligned} 0 &\leq \frac{1}{2} (B - AB^{-1}A) - S(A|B) \\ &\leq \frac{1}{8} (B - A) (A^{-1} - B^{-1}) A (A^{-1} - B^{-1}) (B + A) \end{aligned}$$

and

$$(3.14) \quad \begin{aligned} 0 &\leq S(A|B) - 2(B - A) + (A \sharp_t B + A)^{-1} A \\ &\leq \frac{1}{8} (B - A) (A^{-1} - B^{-1}) A (A^{-1} - B^{-1}) (B + A) \end{aligned}$$

for any positive invertible operators with $B \geq A$.

Similar inequalities may be stated if we take $t = 2$ in Theorem 3, however the details are omitted.

Corollary 2. *Let C be a positive invertible operator and $C \leq 1_H$, then for any $t > 0$ we have*

$$(3.15) \quad 0 \leq \frac{1}{2t} (1_H - C^{2t}) C^{1-t} - \eta(C) \leq \frac{1}{8t} C^{1-2t} (1 - C^t)^3 (1 + C^t)$$

and

$$(3.16) \quad 0 \leq \eta(C) - \frac{2}{t} C (1_H - C^t) (1_H + C^t)^{-1} \leq \frac{1}{8t} C^{1-2t} (1 - C^t)^3 (1 + C^t).$$

If we take in this corollary $t = \frac{1}{2}$, then we get

$$(3.17) \quad 0 \leq (1_H - C) C^{1/2} - \eta(C) \leq \frac{1}{4} (1 - C^{1/2})^3 (1 + C^{1/2})$$

and

$$(3.18) \quad 0 \leq \eta(C) - 4C (1_H - C^{1/2}) (1_H + C^{1/2})^{-1} \leq \frac{1}{4} (1 - C^{1/2})^3 (1 + C^{1/2})$$

while, if we take $t = 1$, then we get

$$(3.19) \quad 0 \leq \frac{1}{2} (1_H - C^2) - \eta(C) \leq \frac{1}{8} C^{-1} (1 - C)^3 (1 + C)$$

and

$$(3.20) \quad 0 \leq \eta(C) - 2C (1_H - C) (1_H + C)^{-1} \leq \frac{1}{8} C^{-1} (1 - C)^3 (1 + C).$$

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