DISCRETE $q$-GREEN POTENTIALS WITH FINITE ENERGY

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ABSTRACT. Discrete $q$-Green potentials related to the equation $\Delta u - qu = 0$ on an infinite network were studied in [12] as a discrete analogue to [9]. We study some properties of $q$-Green potentials with finite $q$-Green energy. The $q$-Dirichlet energy plays an important role instead of the Dirichlet sum. Our aim is to show that results obtained in [7] in case $q = 0$ hold similarly even in case $q \geq 0$. We show that every $q$-Dirichlet potential can be expressed as a difference of two $q$-Green potentials with finite $q$-Green energy.

1. INTRODUCTION WITH PRELIMINARIES

Discrete potential theory on infinite networks related to the discrete Laplacian $\Delta$ has been studied by many authors; for example, Anandam [1], Ayadi [2], Kasue [3], Kumaresan and Narayanaraju [4], Lyons and Peres [8], and Yamasaki [11].

Many potential theoretic results related to the equation $\Delta_q u := \Delta u - qu = 0$ on a Riemann surface were given in [9]. The $q$-harmonic Green function ($q$-Green function, for short) implies the Green function related to $\Delta_q$. As for the $q$-Green function of an infinite network, some results which have counterparts in [9] were shown in [12]. Our aim of this paper is to show that every $q$-Dirichlet potential can be expressed as a difference of two $q$-Green potentials with finite $q$-Green energy. We proved in [7] that this property holds in case $q = 0$.

More precisely, let $\mathcal{N} = \langle V, E, K, r \rangle$ be an infinite network which is connected and locally finite and has no self-loop, where $V$ is the set of nodes, $E$ is the set of arcs, and the resistance $r$ is a strictly positive function on $E$. For $x \in V$ and for $e \in E$ the node-arc incidence matrix $K$ is defined by $K(x, e) = 1$ if $x$ is the initial node of $e$; $K(x, e) = -1$ if $x$ is the terminal node of $e$; $K(x, e) = 0$ otherwise. Let $L(V)$ be the set of all real valued functions on $V$, $L^+(V)$ the set of all non-negative real valued functions on $V$, and $L_0(V)$ the set of all $u \in L(V)$ with finite support. We similarly define $L(E)$, $L^+(E)$, and $L_0(E)$. Let $q$ be a non-negative function on $E$.
V with \( q \neq 0 \). For \( u \in L(V) \) we define the discrete derivative \( \nabla u \in L(E) \), the Laplacian \( \Delta u \in L(V) \), and the \( q \)-Laplacian \( \Delta_q u \in L(V) \) as

\[
\nabla u(e) = -r(e)^{-1} \sum_{x \in V} K(x, e) u(x),
\]

\[
\Delta u(x) = \sum_{e \in E} K(x, e) \nabla u(e),
\]

\[
\Delta_q u(x) = \Delta u(x) - q(x) u(x).
\]

For convenience we give specific forms. For \( e \in E \) let \( x^+ \in V \) be the initial node of \( e \) and \( x^- \in V \) the terminal node of \( e \). Then

\[
\nabla u(e) = \frac{u(x^-) - u(x^+)}{r(e)}.
\]

For \( x \in V \) let \( \{e_1, \ldots, e_d\} \) be the set of arcs adjacent to \( x \) and let \( y_j \) be the other node of \( e_j \) for each \( j \). Then

\[
\Delta u(x) = \sum_{j=1}^d \frac{u(y_j) - u(x)}{r(e_j)},
\]

\[
\Delta_q u(x) = \sum_{j=1}^d \frac{u(y_j) - u(x)}{r(e_j)} - q(x) u(x).
\]

For \( u, v \in L(V) \), we put

\[
(u, v)_D = \sum_{e \in E} r(e) \nabla u(e) \nabla v(e),
\]

\[
\|u\|_D = (u, u)_D^{1/2} \quad \text{(Dirichlet sum),}
\]

\[
(u, v)_E = \sum_{e \in E} r(e) \nabla u(e) \nabla v(e) + \sum_{x \in V} q(x) u(x) v(x),
\]

\[
\|u\|_E = (u, u)_E^{1/2} \quad \text{(q-Dirichlet energy)}.
\]

We define some classes of functions on \( V \) as

\[
D = \{ u \in L(V) \mid \|u\|_D < \infty \},
\]

\[
E = \{ u \in L(V) \mid \|u\|_E < \infty \},
\]

\[
H_q = \{ u \in L(V) \mid \Delta_q u = 0 \}.
\]

It is easy to see that \( E \) is a Hilbert space with respect to the inner product \( (u, v)_E \). On the other hand, \( (u, v)_D \) is a degenerate bilinear form in \( D \); for example, \( (1, u)_D = 0 \) and \( \|u + 1\|_D = \|u\|_D \) for \( u \in D \). It was shown in [11, Theorem 1.1] that \( D \) is a Hilbert space with respect to the inner product \( (u, v)_D + u(o)v(o) \) for a fixed node \( o \in V \). We easily verify that a sequence \( \{u_n\} \subseteq D \) converges to \( u \) in \( D \) if and only if \( \lim_{n \to \infty} \|u_n - u\|_D = 0 \) and \( \{u_n\} \) converges pointwise to \( u \). Denote by \( D_0 \) and \( E_0 \) the closure of \( L_0(V) \) in \( D \) and in \( E \) respectively. We call a function in \( D \), in \( D_0 \), in \( E \), and in \( E_0 \) a Dirichlet function, a Dirichlet potential, a q-Dirichlet function, and a q-Dirichlet potential, respectively.
It was shown in [7] that the space $D_0$ is equal to the space of the differences of Green potentials with finite energy provided that conditions (LD) and (CLD) are fulfilled. As an application, we showed a Riesz decomposition of a function whose Laplacian is a Dirichlet function. Our aim is to verify that similar results for $q$-Green potentials are also valid by replacing conditions (LD) and (CLD) by (LD)$_q$ and (CLD)$_q$, which are defined in Section 3. In contrast with (LD) and (CLD), our modified conditions contain some barriers caused by the term $q u$. We shall discuss in Section 4 some relations among these conditions.

2. The $q$-Green function

Let us recall some fundamental results related to the $q$-Dirichlet functions established in [12].

Lemma 2.1 ([12, Theorem 3.1]). $E_0 = D_0 \cap E$.

Lemma 2.2 ([12, Lemma 3.1]). $(u, h)_E = 0$ for every $u \in E_0$ and $h \in H_q \cap E$.

Lemma 2.3 ([12, Theorem 3.2]). Every $u \in E$ is decomposed uniquely into the form $u = v + h$ with $v \in E_0$ and $h \in H_q \cap E$.

We give a fundamental property of the norm in $E$, which is used repeatedly in the following.

Lemma 2.4. If $\{u_n\}_n \subset E$ converges to $u \in E$ in the norm of $E$, then $\{u_n\}_n$ converges pointwise to $u$.

Proof. Let $v_n = u_n - u$ and assume that $\|v_n\|_E \to 0$ as $n \to \infty$. There exists $x_0 \in V$ such that $q(x_0) > 0$. The fact $q(x_0)|v_n(x_0)|^2 \leq \|v_n\|^2_E$ shows that $v_n(x_0) \to 0$ as $n \to \infty$. Since $\|v_n\|_D \leq \|v_n\|_E \to 0$ as $n \to \infty$, by [10, Corollary 2 of Lemma 1] it follows that $\{v_n\}_n$ converges pointwise to 0.

We call a function $T$ defined on $\mathbb{R}$ into $\mathbb{R}$ a normal contraction of $\mathbb{R}$ if $T0 = 0$ and $|Ts_1 - Ts_2| \leq |s_1 - s_2|$ for $s_1, s_2 \in \mathbb{R}$. For example, $Ts = \max\{s, 0\}$ is a normal contraction of $\mathbb{R}$.

Lemma 2.5 ([12, Lemma 4.2 and before it]). Let $T$ be a normal contraction of $\mathbb{R}$. Then $\|T \circ u\|_E \leq \|u\|_E$ for $u \in E$. Moreover, $T \circ u \in E_0$ if $u \in E_0$.

Lemma 2.6. Let $f \in L_0(V)$ and $u \in E$. Then

$$(u, f)_E = -\sum_{x \in V} (\Delta_q u(x)) f(x).$$

Proof. Since $(u, f)_D = -\sum_{x \in V} (\Delta u(x)) f(x)$ by [10, Lemma 3], we have

$$
(u, f)_E = -\sum_{x \in V} (\Delta u(x)) f(x) + \sum_{x \in V} q(x) u(x) f(x)
$$

$$= -\sum_{x \in V} (\Delta_q u(x)) f(x)
$$
as required. □
Lemma 3.4. Letting

\[ f \]

\[ \text{especially converges to} \ G \]

which are similar to those considered in [7].

Conversely, if \( u \) is [12, Theorem 7.2]...Lemma 3.1.

We call \( \text{Green energy} \ G \)

\[ q \]

Similarly we define the \( q \)-Green energy \( G_q \)

\[ \mu \]

\[ \nu \]

\[ \text{for each} \ x \in V \}, \]

\[ \text{for some} \ q \in \mathbb{N} \text{.} \]

Lemma 3.1 ([12, Lemma 7.1]). \( \Delta q \mu = -\mu \) for \( \mu \in \mathcal{M}_q \).

Lemma 3.2 ([12, Theorem 7.2]). If \( \mu \in \mathcal{E}_q \), then \( G_q \mu \in \mathcal{E}_0 \) and \( \Delta q G_q \mu \leq 0 \). Conversely, if \( u \in \mathcal{E}_0 \) satisfies \( \Delta q u \leq 0 \), then \( u = G_q \mu \) for some \( \mu \in \mathcal{E}_q \).

We show some results for the \( q \)-Green potential and the mutual \( q \)-Green energy, which are similar to those considered in [7].

Lemma 3.3. For \( \mu, \nu \in L_0(V) \cap L^+(V) \) we have

\[ (G_q \mu, G_q \nu)_E = G_q(\mu, \nu). \]

Proof. Let \( \mu, \nu \in L_0(V) \cap L^+(V) \). Lemma 3.2 shows that \( G_q \mu \in \mathcal{E}_0 \), so that there exists a sequence \( \{f_n\}_n \subset L_0(V) \) which converges to \( G_q \mu \) in the norm of \( \mathcal{E} \). Especially \( \{f_n\}_n \) converges pointwise to \( G_q \mu \). Lemmas 2.6 and 3.1 imply that

\[ (f_n, G_q \nu)_E = -\sum_{x \in V} f_n(x)(\Delta q G_q \nu(x)) = \sum_{x \in V} f_n(x)\nu(x). \]

Letting \( n \to \infty \), we have the assertion.

Lemma 3.4. For \( \mu \in \mathcal{E}_q \), there exists \( \{\mu_n\}_n \subset L_0(V) \cap L^+(V) \) such that \( G_q \mu_n \)

converges to \( G_q \mu \) in the norm of \( \mathcal{E} \) and that \( \{\mu_n\}_n \) converges pointwise to \( \mu \).
Proof. Let $\mu \in \mathcal{E}_q$. Let $\{\mathcal{N}_n\}_n$ be an exhaustion of $\mathcal{N}$ with $\mathcal{N}_n = (V_n, E_n)$. We put $\mu_n = \mu$ on $V_n$ and $\mu_n = 0$ on $V \setminus V_n$. Clearly, $\{\mu_n\}_n$ increases monotonically and converges pointwise to $\mu$. Fatou’s lemma shows that
\[ G_q\mu(x) \leq \liminf_{n \to \infty} G_q\mu_n(x) = \lim_{n \to \infty} G_q\mu_n(x) \leq G_q\mu(x), \]
so that $\{G_q\mu_n\}_n$ converges pointwise to $G_q\mu$.

For $m < n$, the monotonicity of $\{\mu_n\}_n$ implies that $\|G_q\mu_n\|_E$ converges and, together with Lemma 3.3, that
\[ (G_q\mu_m, G_q\mu_n)_E = G_q(\mu_m, \mu_n) \geq G_q(\mu_m, \mu_m) = \|G_q\mu_m\|_E^2. \]
Consequently
\[ \|G_q\mu_n - G_q\mu_m\|_E^2 = \|G_q\mu_n\|_E^2 - 2(G_q\mu_n, G_q\mu_m)_E + \|G_q\mu_m\|_E^2 \]
\[ \leq \|G_q\mu_n\|_E^2 - \|G_q\mu_m\|_E^2. \]
Since $G_q\mu_n \in \mathcal{E}_0$ by Lemma 3.2, it follows that $\{G_q\mu_n\}_n$ converges to some $v \in \mathcal{E}_0$ in the norm of $E$. This means that $v = G_q\mu$, and that $\{G_q\mu_n\}_n$ converges to $G_q\mu$ in the norm of $E$. \hfill \square

**Proposition 3.5.** Let $\{\mu_n\}_n \subset \mathcal{E}_q$. If $\{G_q\mu_n\}_n$ converges to some $u \in E$ in the norm of $E$, then $u = G_q\mu$ for some $\mu \in \mathcal{E}_q$.

Proof. Let $\{\mu_n\}_n \subset \mathcal{E}_q$. Lemma 3.2 implies that $G_q\mu_n \in \mathcal{E}_0$, so that $u \in \mathcal{E}_0$. Lemma 3.1 shows
\[ \Delta_q u(x) = \lim_{n \to \infty} \Delta_q G_q\mu_n(x) = - \lim_{n \to \infty} \mu_n(x) \leq 0. \]
Again by Lemma 3.2 we have that $u = G_q\mu$ for some $\mu \in \mathcal{E}_q$. \hfill \square

Now we introduce two conditions which are similar to conditions (LD) and (CLD) considered in [7]. We say that $\mathcal{N}$ satisfies condition (LD)$_q$ if there exists a constant $c > 0$ such that
\[ (LD)_q \quad \|\Delta_q f\|_E \leq c\|f\|_E \quad \text{for all } f \in L_0(V). \]
We say that $\mathcal{N}$ satisfies condition (CLD)$_q$ if there exists a constant $c > 0$ such that
\[ (CLD)_q \quad \|f\|_E \leq c\|\Delta_q f\|_E \quad \text{for all } f \in L_0(V). \]

**Lemma 3.6.** Assume (LD)$_q$. Then there exists a constant $c > 0$ such that $\|\Delta_q u\|_E \leq c\|u\|_E$ for all $u \in E$.

Proof. Let $u \in E$. By Lemma 2.3 we find $v \in \mathcal{E}_0$ and $h \in H_q \cap E$ such that $u = v + h$. Lemma 2.2 shows that
\[ \|u\|_E^2 = \|v\|_E^2 + 2(v, h)_E + \|h\|_E^2 \]
\[ = \|v\|_E^2 + \|h\|_E^2 \geq \|v\|_E^2. \]
Let $\{f_n\}_n$ be a sequence in $L_0(V)$ which converges to $v$ in the norm of $E$. Then (LD)$_q$ implies that $\|\Delta_q f_n\|_E \leq c\|f_n\|_E$ for all $n$. Since $\{\Delta_q f_n\}_n$ converges pointwise
to $\Delta_q v$, Fatou’s lemma gives
\[
\|\Delta_q u\|_E = \|\Delta_q v\|_E \leq \liminf_{n \to \infty} \|\Delta_q f_n\|_E \\
\leq c \liminf_{n \to \infty} \|f_n\|_E = \|v\|_E \leq c \|u\|_E
\]
as required. \qed

Lemma 3.7. Assume (LD)$_q$. Then $\Delta_q u \in E_0$ for $u \in E_0$.

Proof. Let $u \in E_0$ and \( \{f_n\} \) a sequence in $L_0(V)$ which converges to $u$ in the norm of $E$. Then $\|f_n - f_m\|_E \to 0$ as $n, m \to \infty$. Condition (LD)$_q$ implies that
\[
\|\Delta_q f_n - \Delta_q f_m\|_E \leq c \|f_n - f_m\|_E \to 0
\]
as $n, m \to \infty$. Thus $\{\Delta_q f_n\}$ is a Cauchy sequence in $E$ and converges to some $v \in E_0$ in the norm of $E$. Since $\{\Delta_q f_n\}$ converges pointwise to $\Delta_q u$, we see that $\Delta_q u = v \in E_0$. \qed

Proposition 3.8. Assume both (LD)$_q$ and (CLD)$_q$. Then there exists a constant $c > 0$ such that
\[
\|u\|_E \leq c \|\Delta_q u\|_E \quad \text{for all } u \in E_0.
\]

Proof. Let $u \in E_0$. There exists a sequence $\{f_n\} \subset L_0(V)$ which converges to $u$ in the norm of $E$. Lemma 3.6 shows that there exists $c_1 > 0$ such that $\|\Delta_q u - \Delta_q f_n\|_E \leq c_1 \|u - f_n\|_E$ for all $n$, so that $\|\Delta_q f_n\|_E \to \|\Delta_q u\|_E$ as $n \to \infty$. By (CLD)$_q$, there exists $c_2 > 0$ such that $\|f_n\|_E \leq c_2 \|\Delta_q f_n\|_E$ for all $n$. We have
\[
\|u\|_E = \lim_{n \to \infty} \|f_n\|_E \leq c_2 \lim_{n \to \infty} \|\Delta_q f_n\|_E = c_2 \|\Delta_q u\|_E,
\]
as required. \qed

Lemma 3.9. Let $\{u_n\}$ be a sequence in $E_0$ such that $\|u_n\|_E$ is bounded and that $\{u_n\}$ converges pointwise to a function $u \in E$. Then $\lim_{n \to \infty} (u_n, v)_E = (u, v)_E$ for $v \in E_0$.

Proof. Let $v \in E_0$. For any $\varepsilon > 0$, there exists $f \in L_0(V)$ such that $\|v - f\|_E < \varepsilon$. We take $M$ with $\|u_n\|_E \leq M$ for all $n$. Fatou’s lemma shows that $\|u\|_E \leq M$. It is easy to see that $|(u_n - u, f)_E| < \varepsilon$ for sufficiently large $n$. We have
\[
|(u_n - u, v)_E| \leq |(u_n - u, v - f)_E| + |(u_n - u, f)_E| \\
\leq \|u_n - u\|_E \|v - f\|_E + \varepsilon < (2M + 1)\varepsilon,
\]
and the assertion. \qed

Lemma 3.10. If $\mu \in E_0 \cap L^+(V)$, then there exists $\{\mu_n\} \subset L_0(V) \cap L^+(V)$ which converges to $\mu$ in the norm of $E$.

Proof. Let $\mu \in E_0 \cap L^+(V)$. There exists a sequence $\{f_n\}$ in $L_0(V)$ which converges to $\mu$ in the norm of $E$. Let $\mu_n = \max\{f_n, 0\}$. Then $\|\mu_n\|_E \leq \|f_n\|_E$ by Lemma 2.5. Since $\mu \geq 0$, $\{\mu_n\}$ converges pointwise to $\mu$. Fatou’s lemma gives
\[
\|\mu\|_E \leq \liminf_{n \to \infty} \|\mu_n\|_E \leq \limsup_{n \to \infty} \|\mu_n\|_E \\
\leq \lim_{n \to \infty} \|f_n\|_E = \|\mu\|_E,
\]

or \( \lim_{n \to \infty} \| \mu_n \|_E = \| \mu \|_E \). Since \( \{ \| f_n \|_E \}_n \) is bounded, so is \( \{ \| \mu_n \|_E \}_n \). By Lemma 3.9, \( (\mu_n, \mu)_E \to (\mu, \mu)_E = \| \mu \|_E^2 \) as \( n \to \infty \). Thus we have

\[
\| \mu - \mu_n \|_E^2 = \| \mu \|_E^2 - 2(\mu, \mu_n)_E + \| \mu_n \|_E^2 \to 0
\]
as \( n \to \infty \).

**Theorem 3.11.** \( \mathcal{E}_q = E_0 \cap L^+(V) \) if both (LD)\(_q\) and (CLD)\(_q\) are fulfilled.

**Proof.** Let \( \mu \in \mathcal{E}_q \). By Lemma 3.4, there exists \( \{ \mu_n \}_n \subset L_0(V) \cap L^+(V) \) such that \( \{ G_q \mu_n \}_n \) converges to \( G_q \mu \) in the norm of \( E \) and that \( \{ \mu_n \}_n \) converges pointwise to \( \mu \). Lemma 3.2 shows that \( G_q \mu_n \in E_0 \) and \( G_q \mu_n \in E_0 \) for each \( n \). By Lemmas 3.1 and 3.6

\[
\| \mu - \mu_n \|_E = \| \Delta_q G_q \mu_n - \Delta_q G_q \mu \|_E \leq c \| G_q \mu_n - G_q \mu \|_E \to 0
\]
as \( n \to \infty \). Thus \( \mu \in E_0 \).

We show the converse. Let \( \mu \in E_0 \cap L^+(V) \). By Lemma 3.10, there exists \( \{ \mu_n \}_n \subset L_0(V) \cap L^+(V) \) which converges to \( \mu \) in the norm of \( E \). Lemma 3.2 implies \( G_q \mu_n \in E_0 \) for each \( n \). Proposition 3.8 and Lemma 3.1 show that

\[
\| G_q \mu_n - G_q \mu_m \|_E \leq c \| \Delta_q (G_q \mu_n - G_q \mu_m) \|_E = c \| \mu_n - \mu_m \|_E \to 0
\]
as \( n, m \to \infty \). Therefore \( \{ G_q \mu_n \}_n \) converges to some \( u \in E_0 \) in the norm of \( E \). Fatou’s lemma and Lemma 3.3 give

\[
G_q(\mu, \mu) \leq \liminf_{n \to \infty} G_q(\mu_n, \mu_n) = \lim_{n \to \infty} \| G_q \mu_n \|_E^2 = \| u \|_E^2 < \infty.
\]

Namely \( \mu \in \mathcal{E}_q \).

For any \( u \in L(V) \), we define \( G_q u \) by \( G_q u = G_q u^+ - G_q u^- \) if both \( u^+ = \max\{u, 0\} \) and \( u^- = -\min\{u, 0\} \) belong to \( M_q \).

**Theorem 3.12.** \( E_0 = \mathcal{E}_q - \mathcal{E}_q \) if both (LD)\(_q\) and (CLD)\(_q\) are fulfilled. In this case, \( u^+, u^- \in \mathcal{E}_q \) for \( u \in E_0 \).

**Proof.** By Theorem 3.11, \( \mathcal{E}_q - \mathcal{E}_q \subset E_0 \). Conversely, for \( u \in E_0 \), Lemma 2.5 and Theorem 3.11 imply that \( u^+, u^- \in E_0 \cap L^+(V) = \mathcal{E}_q \), so that \( E_0 \subset \mathcal{E}_q - \mathcal{E}_q \).

**Theorem 3.13.** \( G_q u \in E_0 \) and \( \Delta_q G_q u = -u \) for \( u \in E_0 \) if both (LD)\(_q\) and (CLD)\(_q\) are fulfilled.

**Proof.** Let \( u \in E_0 \). Theorem 3.12 shows that \( u^+, u^- \in \mathcal{E}_q \). Lemma 3.2 implies \( G_q u = G_q u^+ - G_q u^- \in E_0 \). By Lemma 3.1 we have

\[
\Delta_q G_q u = \Delta_q G_q u^+ - \Delta_q G_q u^- = -u^+ + u^- = -u
\]
as required.

**Corollary 3.14.** \( \{ G_q u \mid u \in E_0 \} \subset E_0 \) if both (LD)\(_q\) and (CLD)\(_q\) are fulfilled.

**Theorem 3.15.** \( G_q \Delta_q u = -u \) for \( u \in E_0 \) if both (LD)\(_q\) and (CLD)\(_q\) are fulfilled.

**Proof.** Let \( u \in E_0 \). Then \( v := \Delta_q u \in E_0 \) by Lemma 3.7. Theorem 3.13 shows that \( G_q v \in E_0 \) and that \( \Delta_q (u + G_q v) = v - v = 0 \). Therefore \( u + G_q v \in E_0 \cap H_q \). Thus \( u + G_q v = 0 \) by Lemma 2.2. □
We arrive at the following main result.

**Theorem 3.16.** \( E_0 = \{ G_q u - G_q v \mid u, v \in E_q \} \) if both (LD)\(_q\) and (CLD)\(_q\) are fulfilled.

**Proof.** Lemma 3.2 implies that \( \{ G_q u - G_q v \mid u, v \in E_q \} \subset E_0 \). We show the converse. Let \( u \in E_0 \). We have \( v := -\Delta_q u \in E_0 \) by Lemma 3.7. Theorem 3.15 shows that \( u = G_q v = G_q v^+ - G_q v^- \). Theorem 3.12 implies that \( v^+, v^- \in E_q \), and that \( u \in \{ G_q u - G_q v \mid u, v \in E_q \} \).

As an application of our results, we shall give a version of Riesz decomposition of \( u \in E^{(2)} = \{ u \in L(V) \mid \Delta_q u \in E \} \) as follows. Let us put

\[
\begin{align*}
E^{(2)}_0 &= \{ u \in L(V) \mid \Delta_q u \in E_0 \}, \\
H^{(2)}_q &= \{ u \in L(V) \mid \Delta_q u \in H_q \}.
\end{align*}
\]

**Theorem 3.17.** If both (LD)\(_q\) and (CLD)\(_q\) are fulfilled, then for every \( u \in E^{(2)} \), there exist a unique \( v \in E_0 \) and a unique \( w \in H^{(2)}_q \cap E^{(2)} \) such that \( u = G_q v + w \).

**Proof.** Let \( u \in E^{(2)} \). Applying Lemma 2.3 to \( \Delta_q u \in E \) yields

\[
\Delta_q u = -v + h \quad \text{with} \quad v \in E_0 \quad \text{and} \quad h \in H_q \cap E.
\]

Theorem 3.13 shows that \( \Delta_q G_q v = -v \in E_0 \). Hence \( G_q v \in E^{(2)}_0 \). Let \( w = u - G_q v \). Then \( w \in E^{(2)} \) and

\[
\Delta_q w = \Delta_q u - \Delta_q G_q v = (-v + h) + v = h \in H_q,
\]

so that \( w \in H^{(2)}_q \).

To show the uniqueness, we assume that \( u = G_q v_1 + w_1 = G_q v_2 + w_2 \) with \( v_1, v_2 \in E_0 \) and \( w_1, w_2 \in H^{(2)}_q \cap E^{(2)} \). Theorem 3.13 shows that \( w_1 - w_2 = G_q v_2 - G_q v_1 \in E_0 \). Lemma 3.7 implies \( \Delta_q (w_1 - w_2) \in E_0 \). Since \( w_1 - w_2 \in H^{(2)}_q \), it follows that \( \Delta_q (w_1 - w_2) \in H_q \). Lemma 2.2 shows that \( \Delta_q (w_1 - w_2) = 0 \), so that \( w_1 - w_2 \in H_q \cap E_0 \). Again by Lemma 2.2 we have \( w_1 = w_2 \), so that \( G_q v_1 = G_q v_2 \). Theorem 3.13 gives \( v_1 = -\Delta_q G_q v_1 = -\Delta_q G_q v_2 = v_2 \). □

**Corollary 3.18.** \( E^{(2)} = E^{(2)}_0 + H^{(2)}_q \cap E^{(2)} \) if both (LD)\(_q\) and (CLD)\(_q\) are fulfilled.

**Proof.** Clearly \( E^{(2)}_0 + H^{(2)}_q \cap E^{(2)} \subset E^{(2)} \). By Theorem 3.17 we take \( v \in E_0 \) and \( w \in H^{(2)}_q \cap E^{(2)} \) such that \( u = G_q v + w \). Theorem 3.13 shows that \( \Delta_q G_q v = -v \in E_0 \), so that \( G_q v \in E^{(2)}_0 \). □

4. **Conditions (LD)\(_q\) AND (CLD)\(_q\)**

We considered in [7] the following conditions:

(\text{LD}) \ There exists a constant \( c > 0 \) such that \( \| \Delta f \|_D \leq c \| f \|_D \) for all \( f \in L_0(V) \);

(\text{CLD}) \ There exists a constant \( c > 0 \) such that \( \| f \|_D \leq c \| \Delta f \|_D \) for all \( f \in L_0(V) \).

Note that (LD)\(_q\) and (CLD)\(_q\) in Section 3 are obtained by replacing \( D \) by \( E \) and \( \Delta \) by \( \Delta_q \) in (LD) and (CLD).

We recall...
Lemma 4.1 ([6, Lemma 3.2]). Assume (LD). Then there exists a constant \( c > 0 \) such that \( \|\Delta u\|_D \leq c\|u\|_D \) for all \( u \in D \).

First of all, we note that \( \|\Delta u\|_D < \infty \) does not imply \( \|\Delta_q u\|_D < \infty \). In fact, let \( u = 1 \) on \( V \) and \( q \in L^+ (V) \setminus D \). Then \( \|\Delta u\|_D = 0 \) and \( \|\Delta_q u\|_D = \|q\|_D = \infty \).

Let us define \( t(x, y) \) and \( t(x) \) for \( x, y \in V \) by

\[
t(x, y) = \sum_{e \in E} |K(x, e)K(y, e)|r(e)^{-1} \quad \text{if } x \neq y,
\]

\[
t(x) = \sum_{e \in E} |K(x, e)|r(e)^{-1} = \sum_{y \in V} t(x, y).
\]

Then we have

\[
\Delta u(x) = -t(x)u(x) + \sum_{y \in V} t(x, y)u(y).
\]

For convenience sake, we introduce the following conditions:

(qB) \( q(x) \) is bounded on \( V \);
(tB) \( t(x) \) is bounded on \( V \).

Lemma 4.2. Assume both (qB) and (tB). Then there exists a constant \( c > 0 \) such that \( \|qu\|_D \leq c(\sum_{x \in V} u(x)^2)^{1/2} \) and \( \|qu\|_D \leq c\|u\|_E \) for all \( u \in E \).

Proof. Let \( \gamma \) satisfy \( t(x) \leq \gamma \) and \( q(x) \leq \gamma \) for all \( x \in V \). Let \( u \in E \). For \( e \in E \), let \( x_1 \) and \( x_2 \) be the initial node and the terminal node of \( e \). Then

\[
(\nabla (qu)(e))^2 = r(e)^{-2} (q(x_2)u(x_2) - q(x_1)u(x_1))^2
\]

\[
\leq r(e)^{-2} \times 2 (q(x_2)^2u(x_2)^2 + q(x_1)^2u(x_1)^2)
\]

\[
\leq 2r(e)^{-2} \times \gamma (q(x_1)u(x_1)^2 + q(x_2)u(x_2)^2)
\]

\[
= 2\gamma r(e)^{-2} \sum_{x \in V} |K(x, e)|q(x)u(x)^2.
\]

We have

\[
\|qu\|_D^2 = \sum_{e \in E} r(e)(\nabla (qu)(e))^2 \leq 2\gamma \sum_{e \in E} r(e)^{-1} \sum_{x \in V} |K(x, e)|q(x)u(x)^2
\]

\[
= 2\gamma \sum_{x \in V} t(x)q(x)u(x)^2 \leq 2\gamma^2 \sum_{x \in V} q(x)u(x)^2,
\]

which implies \( \|qu\|_D^2 \leq 2\gamma^2 \sum_{x \in V} u(x)^2 \) and \( \|qu\|_D^2 \leq 2\gamma^2 \|u\|_E^2 \).

Proposition 4.3. (LD)$_q$ implies both (qB) and (tB).

Proof. Condition (LD)$_q$ shows that there exists \( c > 0 \) such that \( \|\Delta \delta_a\|_E \leq c\|\delta_a\|_E \) for all \( a \in V \), where \( \delta_a \) is the characteristic function of \( \{a\} \). We shall show that \( t(a) + q(a) \leq c \).
Let \( \{e_j\}_{j=1}^d \subset E \) be the arcs adjacent to \( a \) and let \( b_j \in V \) be the other node of \( e_j \). For \( e \in E \)

\[
\nabla \delta_a(e) = -r(e)^{-1} \sum_{x \in V} K(x, e) \delta_a(x) = -r(e)^{-1} K(a, e).
\]

Since \( K(x, e)^2 = |K(x, e)| \) in general,

\[
\|\delta_a\|_E^2 = \sum_{e \in E} r(e)^{-1} K(a, e)^2 + \sum_{x \in V} q(x) \delta_a(x)^2
\]

\[
= \sum_{e \in E} r(e)^{-1} |K(a, e)| + q(a) = t(a) + q(a).
\]

On the other hand

\[
\Delta_q \delta_a(x) = \sum_{e \in E} K(x, e) \nabla \delta_a(e) - q(x) \delta_a(x)
\]

\[
= - \sum_{e \in E} K(x, e) r(e)^{-1} K(a, e) - q(x) \delta_a(x)
\]

\[
= - \sum_{i=1}^d K(x, e_i) r(e_i)^{-1} K(a, e_i) - q(x) \delta_a(x).
\]

Especially

\[
\Delta_q \delta_a(a) = -t(a) - q(a).
\]

Since \( K(x, e_i)K(a, e_i) = 0 \) unless \( x = a \) or \( x = b_i \) and \( K(b_i, e_i)K(a, e_i) = -1 \), it follows that

\[
\nabla(\Delta_q \delta_a)(e) = -r(e)^{-1} \sum_{x \in V} K(x, e) \Delta_q \delta_a(x)
\]

\[
= r(e)^{-1} \sum_{x \in V} K(x, e) \left( \sum_{i=1}^d K(x, e_i) r(e_i)^{-1} K(a, e_i) + q(x) \delta_a(x) \right)
\]

\[
= r(e)^{-1} \left( K(a, e) t(a) - \sum_{i=1}^d K(b_i, e) r(e_i)^{-1} + K(a, e) q(a) \right).
\]

If \( e = e_j \), then, by \( K(b_j, e_j) = -K(a, e_j) \),

\[
\nabla(\Delta_q \delta_a)(e_j) = r(e_j)^{-1} \left( K(a, e_j) t(a) - K(b_j, e_j) r(e_j)^{-1} + K(a, e_j) q(a) \right)
\]

\[
= r(e_j)^{-1} K(a, e_j) \left( t(a) + r(e_j)^{-1} + q(a) \right).
\]
Consequently
\[
\|\Delta_q \delta_a\|_E^2 \geq \sum_{j=1}^d r(e_j)|\nabla(\Delta \delta_a)(e_j)|^2 + q(a)(\Delta_q \delta_a(a))^2
\]
\[
= \sum_{j=1}^d r(e_j)^{-1}\left(t(a) + r(e_j)^{-1} + q(a)\right)^2 + q(a)(-t(a) - q(a))^2
\]
\[
\geq \sum_{j=1}^d r(e_j)^{-1}\left(t(a) + q(a)\right)^2 + q(a)(t(a) + q(a))^2
\]
\[
= \left(t(a) + q(a)\right)^3.
\]
Combining these we have \(\left(t(a) + q(a)\right)^3 \leq c^2(t(a) + q(a))\), or \(t(a) + q(a) \leq c\). \(\square\)

Assuming \(q = 0\) in the proposition above, we have

**Corollary 4.4.** (LD) implies (tB).

**Proposition 4.5.** If both (LD) and (qB) are fulfilled, then there exists a constant \(c > 0\) such that \(\|\Delta_q u\|_D \leq c\|u\|_E\) for all \(u \in E\).

**Proof.** Let \(u \in E\). Note that Corollary 4.4 implies (tB). Lemmas 4.1 and 4.2 show that there exist constants \(c_1 > 0\) and \(c_2 > 0\) such that \(\|\Delta u\|_D \leq c_1\|u\|_D\) and \(\|qu\|_D \leq c_2\|u\|_E\). We have
\[
\|\Delta_q u\|_D \leq \|\Delta u\|_D + \|qu\|_D \leq (c_1 + c_2)\|u\|_E
\]
as required. \(\square\)

Denote by \(S_q^+\) the set of \(u \in L^+(V)\) such that \(\Delta_q u \leq 0\).

**Lemma 4.6.** Assume both (qB) and (tB). Then there exists a constant \(c > 0\) such that \(|\Delta_q u(x)| \leq cu(x)\) on \(V\) for all \(u \in S_q^+\).

**Proof.** Let \(u \in S_q^+\). If we set \(\Delta^* u(x) = \sum_{y \in V} t(x, y)u(y)\), then, since \(\Delta_q u(x) = \Delta^* u(x) - (t(x) + q(x))u(x)\), it follows that
\[
(t(x) + q(x))u(x) \geq \Delta^* u(x) \geq 0,
\]
so that
\[
|\Delta_q u(x)| \leq |\Delta^* u(x)| + |(t(x) + q(x))u(x)| \leq 2(t(x) + q(x))u(x).
\]
We may take \(c = 2\sup_{x \in V}(t(x) + q(x))\). \(\square\)

**Theorem 4.7.** If both (LD) and (qB) are fulfilled, then there exists a constant \(c > 0\) such that
\[
\|\Delta_q u\|_E \leq c\|u\|_E \quad \text{for all } u \in E_0 \cap S_q^+.
\]
Proof. Let \( u \in E_0 \cap S^+_q \). Note that Corollary 4.4 implies (tB). Proposition 4.5 and Lemma 4.6 show that there exist constants \( c_1 > 0 \) and \( c_2 > 0 \) such that \( \|\Delta_q u\|^2_D \leq c_1 \|u\|^2_E \) and \( |\Delta_q u(x)| \leq c_2 u(x) \) on \( V \). We have
\[
\|\Delta_q u\|^2_E = \|\Delta_q u\|^2_D + \sum_{x \in V} q(x)(\Delta_q u(x))^2 \leq c_1 \|u\|^2_E + c_2 \sum_{x \in V} q(x)u(x)^2
\]
\[
\leq (c_1^2 + c_2^2) \|u\|^2_E,
\]
as required. \( \square \)

**Proposition 4.8.** If both (qB) and (tB) are fulfilled and if \( q \) is superharmonic on \( V \), i.e., \( \Delta q \leq 0 \) on \( V \), then there exists a constant \( c > 0 \) such that
\[
\sum_{x \in V} q(x)(\Delta_q u(x))^2 \leq c \sum_{x \in V} q(x)u(x)^2
\]
for all \( u \in L(V) \).

Proof. Let \( \gamma \) satisfy \( t(x) \leq \gamma \) and \( q(x) \leq \gamma \) for all \( x \in V \). We set \( \Delta^* u(x) = \sum_{y \in V} t(x,y)u(y) \). Schwarz’s inequality implies that
\[
(\Delta^* u(x))^2 \leq \left( \sum_{y \in V} t(x,y) \right)^2 \left( \sum_{y \in V} t(x,y)u(y)^2 \right) = t(x) \sum_{y \in V} t(x,y)u(y)^2
\]
\[
\leq \gamma \sum_{y \in V} t(x,y)u(y)^2.
\]
Since \( q \) is superharmonic on \( V \), i.e., \( \Delta^* q(x) \leq t(x)q(x) \) on \( V \), it follows that
\[
\sum_{x \in V} q(x)(\Delta^* u(x))^2 \leq \gamma \sum_{x \in V} q(x) \sum_{y \in V} t(x,y)u(y)^2
\]
\[
= \gamma \sum_{y \in V} u(y)^2 \sum_{x \in V} t(x,y)q(x)
\]
\[
= \gamma \sum_{y \in V} u(y)^2 \Delta^* q(y)
\]
\[
\leq \gamma \sum_{y \in V} u(y)^2 t(y)q(y) \leq \gamma^2 \sum_{y \in V} q(y)u(y)^2.
\]
We have
\[
(\Delta_q u(x))^2 = \left( \Delta^* u(x) - (t(x) + q(x))u(x) \right)^2
\]
\[
\leq 2(\Delta^* u(x))^2 + 2(t(x) + q(x))^2u(x)^2
\]
\[
\leq 2(\Delta^* u(x))^2 + 8\gamma^2 u(x)^2,
\]
so that
\[
\sum_{x \in V} q(x)(\Delta_q u(x))^2 \leq 2 \sum_{x \in V} q(x)(\Delta^* u(x))^2 + 8\gamma^2 \sum_{x \in V} q(x)u(x)^2
\]
\[
\leq 10\gamma^2 \sum_{x \in V} q(x)u(x)^2.
\]
This completes the proof. \qed

**Theorem 4.9.** If $q$ is superharmonic on $V$, then $(LD)_q$ follows from $(LD)$ and (qB).

**Proof.** Let $f \in L_0(V)$ and assume $(LD)$ and (qB). Proposition 4.5 shows that there exists a constant $c_1 > 0$ such that $\|\Delta_q f\|_D \leq c_1 \|f\|_E$. Since (tB) is fulfilled by Corollary 4.4, there exists a constant $c_2 > 0$ such that

$$\sum_{x \in V} q(x) (\Delta_q f(x))^2 \leq c_2 \sum_{x \in V} q(x) f(x)^2 \leq c_2 \|f\|_E^2$$

by Proposition 4.8. Thus we have $\|\Delta_q f\|_D^2 \leq (c_1^2 + c_2 c_2) \|f\|_E^2$, so that $(LD)_q$ is fulfilled. \qed

As a generalized version of Poincaré-Sobolev’s inequality, we introduced in [7] the following condition (SPS): There exists a constant $c > 0$ such that

(SPS) \quad $\sum_{x \in V} f(x)^2 \leq c \|f\|_D^2$ for all $f \in L_0(V)$.

**Lemma 4.10** ([7, Lemma 2.1]). Assume (SPS). Then there exists a constant $c > 0$ such that

$$\sum_{x \in V} u(x)^2 \leq c \|u\|_D^2 \quad \text{for all } u \in D_0.$$

**Proposition 4.11.** If both (SPS) and (qB) are fulfilled, then there exists a constant $c > 0$ such that $\|u\|_E \leq c \|u\|_D$ for all $u \in D_0$.

**Proof.** Let $\gamma$ be such that $q(x) \leq \gamma$ for all $x \in V$. By Lemma 4.10, there exists a constant $c_1 > 0$ such that

$$\|u\|_E^2 = \|u\|_D^2 + \sum_{x \in V} q(x) u(x)^2 \leq \|u\|_D^2 + \gamma \|u\|_E^2 \leq (1 + c_1 \gamma) \|u\|_D^2,$$

which shows the assertion. \qed

**Corollary 4.12.** $E_0 = D_0$ if both (SPS) and (qB) are fulfilled.

**Proof.** Since $D_0 \subseteq E$ by Proposition 4.11, we have $E_0 = D_0 \cap E = D_0$ by Lemma 2.1. \qed

**Lemma 4.13.** Assume all of (SPS), (qB), and (tB). Then there exists a constant $c > 0$ such that $\|qu\|_D \leq c \|u\|_D$ for all $u \in D_0$.

**Proof.** Let $u \in D_0$. Then $u \in E_0$ by Corollary 4.12. Lemmas 4.2 and 4.10 show that $\|qu\|_D \leq c_1 (\sum_{x \in V} u(x)^2)^{1/2}$ and $\sum_{x \in V} u(x)^2 \leq c_2 \|u\|_E^2$. Combining these, we have $\|qu\|_D \leq c_1 c_2 \|u\|_D^2$. \qed

**Lemma 4.14.** $\{\Delta_q u \mid u \in D_0\} \subseteq D_0$ if all of $(LD)$, (SPS), and (qB) are fulfilled.

**Proof.** Let $u \in D_0$. Then $\Delta u \in D_0$ by [5, Lemma 6.1]. Let $\{f_n\}_n$ be a sequence in $L_0(V)$ such that $\|u - f_n\|_D \to 0$ as $n \to \infty$. There exists a constant $c_1 > 0$ such that $\|qu - qf_n\|_D \leq c_1 \|u - f_n\|_D$ by Lemma 4.13. Since $qf_n \in L_0(V)$, we see that $qu \in D_0$. Therefore $\Delta_q u = \Delta u - qu \in D_0$. \qed
Theorem 4.15. \((LD)_q\) follows from all of \((LD)\), \((SPS)\), and \((qB)\).

Proof. Assume all of \((LD)\), \((SPS)\), and \((qB)\). Let \(\gamma\) be a number such that \(q(x) \leq \gamma\) for all \(x \in V\). Let \(f \in L_0(V)\). There exists a constant \(c_1 > 0\) such that \(\|\Delta_q f\|_E^2 \leq c_1 \|f\|_E^2\) by Proposition 4.5. Since \(\Delta_q f \in L_0(V)\), we have \(\sum_{x \in V} (\Delta_q f(x))^2 \leq c_2 \|\Delta_q f\|_D^2\) by Lemma 4.10. We have

\[
\|\Delta_q f\|_E^2 \leq c_1 \|f\|_E^2 + \sum_{x \in V} q(x)(\Delta_q f(x))^2 \leq c_1 \|f\|_E^2 + \gamma c_2 \|\Delta_q f\|_D^2 \\
\leq c_1(1 + \gamma c_2) \|f\|_E^2,
\]

which shows \((LD)_q\). \(\square\)

Theorem 4.16. \((SPS)\) implies \((CLD)_q\).

Proof. Let \(f \in L_0(V)\). Since \(\Delta_q f \in L_0(V)\), there exists a constant \(c_1 > 0\) by \((SPS)\) such that

\[
\sum_{x \in V} (\Delta_q f(x))^2 \leq c_1 \|\Delta_q f\|_D^2 \quad \text{and} \quad \sum_{x \in V} (f(x))^2 \leq c_1 \|f\|_D^2.
\]

Lemma 2.6 shows that

\[
\|f\|_E^2 = -\sum_{x \in V} (\Delta_q f(x))^2 f(x) \leq \left(\sum_{x \in V} (\Delta_q f(x))^2\right)^{1/2} \left(\sum_{x \in V} (f(x))^2\right)^{1/2} \\
\leq c_1 \|\Delta_q f\|_D \|f\|_D \leq c_1 \|\Delta_q f\|_E \|f\|_E,
\]

or \(\|f\|_E \leq c_1 \|\Delta_q f\|_E\). \(\square\)

Finally we give an example to show that \((LD)\) does not imply \((LD)_q\).

Example 4.17. Let \(\mathcal{N} = \langle V, E, K, r \rangle\) be a linear network, where \(V = \{x_n\}_{n=0}^\infty\), \(E = \{e_n\}_{n=1}^\infty\), and \(r(e_n) = 1\) for each \(n \geq 1\). Let \(K(x_{n-1}, e_n) = 1\) and \(K(x_n, e_n) = -1\) for each \(n \geq 1\), and let \(K(x, e) = 0\) for any other pairs. We showed in [6, Corollary 2.3] that \(\mathcal{N}\) satisfies \((LD)\).

To prove that \((LD)_q\) is not satisfied, we choose \(q(x_k) = k\). Consider the function \(f_n\) defined by \(f_n(x_k) = 1\) if \(k < n\) and \(f_n(x_k) = 0\) otherwise. Then \(\nabla f_n(e_k) = -\delta_{n,k}\), where \(\delta_{n,k}\) is Kronecker’s delta. Therefore

\[
\|f_n\|_E^2 = \sum_{k=1}^\infty (-\delta_{n,k})^2 + \sum_{k=0}^{n-1} k \cdot 1^2 = 1 + \frac{1}{2} n(n - 1).
\]

On the other hand, \(\Delta_q f_n(x_k) = -k\) for \(k \leq n - 2\), so that

\[
\|\Delta_q f_n\|_E^2 \geq \sum_{k=0}^{n-2} q(x_k)(\Delta_q f_n(x_k))^2 = \sum_{k=0}^{n-2} k^3 = \frac{1}{4} (n - 1)^2 (n - 2)^2.
\]

Consequently

\[
\lim_{n \to \infty} \frac{\|\Delta_q f_n\|_E}{\|f_n\|_E} = \infty,
\]

which means that \(\mathcal{N}\) does not satisfy \((LD)_q\).
REFERENCES


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