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# DISCRETE $q$-GREEN POTENTIALS WITH FINITE ENERGY

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Abstract. Discrete $q$-Green potentials related to the equation $\Delta u - qu = 0$ on an infinite network were studied in [12] as a discrete analogue to [9]. We study some properties of $q$-Green potentials with finite $q$-Green energy. The $q$-Dirichlet energy plays an important role instead of the Dirichlet sum. Our aim is to show that results obtained in [7] in case $q = 0$ hold similarly even in case $q \geq 0$. We show that every $q$-Dirichlet potential can be expressed as a difference of two $q$-Green potentials with finite $q$-Green energy.

## 1. Introduction with preliminaries

Discrete potential theory on infinite networks related to the discrete Laplacian $\Delta$ has been studied by many authors; for example, Anandam [1], Ayadi [2], Kasue [3], Kumaresan and Narayanaraju [4], Lyons and Peres [8], and Yamasaki [11].

Many potential theoretic results related to the equation $\Delta_q u := \Delta u - qu = 0$ on a Riemann surface were given in [9]. The $q$-harmonic Green function ($q$-Green function, for short) implies the Green function related to $\Delta_q$. As for the $q$-Green function of an infinite network, some results which have counterparts in [9] were shown in [12]. Our aim of this paper is to show that every $q$-Dirichlet potential can be expressed as a difference of two $q$-Green potentials with finite $q$-Green energy. We proved in [7] that this property holds in case $q = 0$.

More precisely, let $\mathcal{N} = \langle V, E, K, r \rangle$ be an infinite network which is connected and locally finite and has no self-loop, where $V$ is the set of nodes, $E$ is the set of arcs, and the resistance $r$ is a strictly positive function on $E$. For $x \in V$ and for $e \in E$ the node-arc incidence matrix $K$ is defined by $K(x, e) = 1$ if $x$ is the initial node of $e$; $K(x, e) = -1$ if $x$ is the terminal node of $e$; $K(x, e) = 0$ otherwise. Let $L(V)$ be the set of all real valued functions on $V$, $L^+(V)$ the set of all non-negative real valued functions on $V$, and $L_0(V)$ the set of all $u \in L(V)$ with finite support. We similarly define $L(E)$, $L^+(E)$, and $L_0(E)$. Let $q$ be a non-negative function on

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$V$ with $q \neq 0$. For $u \in L(V)$ we define the *discrete derivative* $\nabla u \in L(E)$, the *Laplacian* $\Delta u \in L(V)$, and the *q-Laplacian* $\Delta_q u \in L(V)$ as

$$\nabla u(e) = -r(e)^{-1} \sum_{x \in V} K(x,e) u(x),$$
$$\Delta u(x) = \sum_{e \in E} K(x,e) \nabla u(e),$$
$$\Delta_q u(x) = \Delta u(x) - q(x) u(x).$$

For convenience we give specific forms. For $e \in E$ let $x^+ \in V$ be the initial node of $e$ and $x^- \in V$ the terminal node of $e$. Then

$$\nabla u(e) = \frac{u(x^-) - u(x^+)}{r(e)}.$$

For $x \in V$ let $\{e_1, \dots, e_d\}$ be the set of arcs adjacent to $x$ and let $y_j$ be the other node of $e_j$ for each $j$. Then

$$\Delta u(x) = \sum_{j=1}^{d} \frac{u(y_j) - u(x)}{r(e_j)},$$

$$\Delta_q u(x) = \sum_{j=1}^{d} \frac{u(y_j) - u(x)}{r(e_j)} - q(x) u(x).$$

For $u, v \in L(V)$, we put

$$(u,v)_{\mathbf{D}} = \sum_{e \in E} r(e) \nabla u(e) \nabla v(e),$$
$$\|u\|_{\mathbf{D}} = (u,u)_{\mathbf{D}}^{1/2} \quad \text{(Dirichlet sum)},$$
$$(u,v)_{\mathbf{E}} = \sum_{e \in E} r(e) \nabla u(e) \nabla v(e) + \sum_{x \in V} q(x) u(x) v(x),$$
$$\|u\|_{\mathbf{E}} = (u,u)_{\mathbf{E}}^{1/2} \quad \text{($q$-Dirichlet energy)}.$$

We define some classes of functions on $V$ as

$$\mathbf{D} = \{u \in L(V) \mid \|u\|_{\mathbf{D}} < \infty\},$$
$$\mathbf{E} = \{u \in L(V) \mid \|u\|_{\mathbf{E}} < \infty\},$$
$$\mathbf{H}_q = \{u \in L(V) \mid \Delta_q u = 0\}.$$

It is easy to see that $\mathbf{E}$ is a Hilbert space with respect to the inner product $(u,v)_{\mathbf{E}}$. On the other hand, $(u,v)_{\mathbf{D}}$ is a degenerate bilinear form in $\mathbf{D}$; for example, $(1,u)_{\mathbf{D}} = 0$ and $\|u+1\|_{\mathbf{D}} = \|u\|_{\mathbf{D}}$ for $u \in \mathbf{D}$. It was shown in [11, Theorem 1.1] that $\mathbf{D}$ is a Hilbert space with respect to the inner product $(u,v)_{\mathbf{D}} + u(o)v(o)$ for a fixed node $o \in V$. We easily verify that a sequence $\{u_n\}_n \subset \mathbf{D}$ converges to $u$ in $\mathbf{D}$ if and only if $\lim_{n\to\infty} \|u_n - u\|_{\mathbf{D}} = 0$ and $\{u_n\}_n$ converges pointwise to $u$. Denote by $\mathbf{D}_0$ and $\mathbf{E}_0$ the closure of $L_0(V)$ in $\mathbf{D}$ and in $\mathbf{E}$ respectively. We call a function in $\mathbf{D}$, in $\mathbf{D}_0$, in $\mathbf{E}$, and in $\mathbf{E}_0$ a *Dirichlet function*, a *Dirichlet potential*, a *q-Dirichlet function*, and a *q-Dirichlet potential*, respectively.

It was shown in [7] that the space $\mathbf{D}_0$ is equal to the space of the differences of Green potentials with finite energy provided that conditions (LD) and (CLD) are fulfilled. As an application, we showed a Riesz decomposition of a function whose Laplacian is a Dirichlet function. Our aim is to verify that similar results for $q$-Green potentials are also valid by replacing conditions (LD) and (CLD) by $(\mathrm{LD})_q$ and $(\mathrm{CLD})_q$, which are defined in Section 3. In contrast with (LD) and (CLD), our modified conditions contain some barriers caused by the term $qu$. We shall discuss in Section 4 some relations among these conditions.

## 2. The $q$-Green function

Let us recall some fundamental results related to the $q$-Dirichlet functions established in [12].

**Lemma 2.1** ([12, Theorem 3.1])**.** $\mathbf{E}_0 = \mathbf{D}_0 \cap \mathbf{E}$.

**Lemma 2.2** ([12, Lemma 3.1])**.** $(u, h)_{\mathbf{E}} = 0$ *for every* $u \in \mathbf{E}_0$ *and* $h \in \mathbf{H}_q \cap \mathbf{E}$.

**Lemma 2.3** ([12, Theorem 3.2])**.** *Every* $u \in \mathbf{E}$ *is decomposed uniquely into the form* $u = v + h$ *with* $v \in \mathbf{E}_0$ *and* $h \in \mathbf{H}_q \cap \mathbf{E}$.

We give a fundamental property of the norm in $\mathbf{E}$, which is used repeatedly in the following.

**Lemma 2.4.** *If* $\{u_n\}_n \subset \mathbf{E}$ *converges to* $u \in \mathbf{E}$ *in the norm of* $\mathbf{E}$*, then* $\{u_n\}_n$ *converges pointwise to* $u$.

*Proof.* Let $v_n = u_n - u$ and assume that $\|v_n\|_{\mathbf{E}} \to 0$ as $n \to \infty$. There exists $x_0 \in V$ such that $q(x_0) > 0$. The fact $q(x_0)|v_n(x_0)|^2 \le \|v_n\|_{\mathbf{E}}^2$ shows that $v_n(x_0) \to 0$ as $n \to \infty$. Since $\|v_n\|_{\mathbf{D}} \le \|v_n\|_{\mathbf{E}} \to 0$ as $n \to \infty$, by [10, Corollary 2 of Lemma 1] it follows that $\{v_n\}_n$ converges pointwise to 0. □

We call a function $T$ defined on $\mathbb{R}$ into $\mathbb{R}$ a normal contraction of $\mathbb{R}$ if $T0 = 0$ and $|Ts_1 - Ts_2| \le |s_1 - s_2|$ for $s_1, s_2 \in \mathbb{R}$. For example, $Ts = \max\{s, 0\}$ is a normal contraction of $\mathbb{R}$.

**Lemma 2.5** ([12, Lemma 4.2 and before it])**.** *Let* $T$ *be a normal contraction of* $\mathbb{R}$*. Then* $\|T \circ u\|_{\mathbf{E}} \le \|u\|_{\mathbf{E}}$ *for* $u \in \mathbf{E}$*. Moreover,* $T \circ u \in \mathbf{E}_0$ *if* $u \in \mathbf{E}_0$.

**Lemma 2.6.** *Let* $f \in L_0(V)$ *and* $u \in \mathbf{E}$*. Then*

$$(u, f)_{\mathbf{E}} = -\sum_{x \in V} (\Delta_q u(x)) f(x).$$

*Proof.* Since $(u, f)_{\mathbf{D}} = -\sum_{x \in V} (\Delta u(x)) f(x)$ by [10, Lemma 3], we have

$$\begin{aligned} (u, f)_{\mathbf{E}} &= -\sum_{x \in V} (\Delta u(x)) f(x) + \sum_{x \in V} q(x) u(x) f(x) \\ &= -\sum_{x \in V} (\Delta_q u(x)) f(x) \end{aligned}$$

as required. □

We say that $u \in L(V)$ is *$q$-superharmonic* or *$q$-harmonic* on $V$ if $\Delta_q u \le 0$ or $\Delta_q u = 0$ respectively. Recall that the (harmonic) Green function $g_a \in \mathbf{D}_0$ of $\mathcal{N}$ with pole at $a \in V$ is defined as the unique solution of the boundary value problem:

$$\Delta g_a(x) = -\delta_a(x) \quad \text{for } x \in V,$$

where $\delta_a(a) = 1$ and $\delta_a(x) = 0$ for $x \neq a$. See [11] for details.

The *$q$-Green function* $\tilde{g}_a \in \mathbf{E}_0$ of $\mathcal{N}$ with pole at $a \in V$ is defined similarly by

$$\Delta_q \tilde{g}_a(x) = -\delta_a(x) \quad \text{for } x \in V.$$

Note that $q$-Green functions always exist and satisfy that $\tilde{g}_a(x) = \tilde{g}_x(a)$ for $a, x \in V$ and that $0 < \tilde{g}_a(x) \le \tilde{g}_a(a)$ for $a, x \in V$. See [12, Theorems 4.1, 4.2, and 4.3].

## 3. Representation of the space $\mathbf{E}_0$

Let $\mu, \nu \in L^+(V)$. Recall that the Green potential $G\mu \in L(V)$ and the mutual Green energy $G(\mu, \nu)$ are defined by

$$G\mu(x) = \sum_{y \in V} g_x(y)\mu(y), \quad G(\mu, \nu) = \sum_{x \in V} (G\mu(x))\nu(x).$$

Similarly we define the *$q$-Green potential* $G_q\mu \in L(V)$ and the *mutual $q$-Green energy* $G_q(\mu, \nu)$ by

$$G_q\mu(x) = \sum_{y \in V} \tilde{g}_x(y)\mu(y), \quad G_q(\mu, \nu) = \sum_{x \in V} (G_q\mu(x))\nu(x).$$

We call $G_q(\mu, \mu)$ the *$q$-Green energy* of $\mu$. Let us put

$$\mathcal{M}_q = \{\mu \in L^+(V) \mid G_q\mu(x) < \infty \text{ for each } x \in V\},$$
$$\mathcal{E}_q = \{\mu \in \mathcal{M}_q \mid G_q(\mu, \mu) < \infty\}.$$

**Lemma 3.1** ([12, Lemma 7.1])**.** *$\Delta_q G_q \mu = -\mu$ for $\mu \in \mathcal{M}_q$.*

**Lemma 3.2** ([12, Theorem 7.2])**.** *If $\mu \in \mathcal{E}_q$, then $G_q\mu \in \mathbf{E}_0$ and $\Delta_q G_q \mu \le 0$. Conversely, if $u \in \mathbf{E}_0$ satisfies $\Delta_q u \le 0$, then $u = G_q\mu$ for some $\mu \in \mathcal{E}_q$.*

We show some results for the $q$-Green potential and the mutual $q$-Green energy, which are similar to those considered in [7].

**Lemma 3.3.** *For $\mu, \nu \in L_0(V) \cap L^+(V)$ we have*

$$(G_q\mu, G_q\nu)_{\mathbf{E}} = G_q(\mu, \nu).$$

*Proof.* Let $\mu, \nu \in L_0(V) \cap L^+(V)$. Lemma 3.2 shows that $G_q\mu \in \mathbf{E}_0$, so that there exists a sequence $\{f_n\}_n \subset L_0(V)$ which converges to $G_q\mu$ in the norm of $\mathbf{E}$. Especially $\{f_n\}_n$ converges pointwise to $G_q\mu$. Lemmas 2.6 and 3.1 imply that

$$(f_n, G_q\nu)_{\mathbf{E}} = -\sum_{x \in V} f_n(x)(\Delta_q G_q \nu(x)) = \sum_{x \in V} f_n(x)\nu(x).$$

Letting $n \to \infty$, we have the assertion. □

**Lemma 3.4.** *For $\mu \in \mathcal{E}_q$, there exists $\{\mu_n\}_n \subset L_0(V) \cap L^+(V)$ such that $\{G_q\mu_n\}_n$ converges to $G_q\mu$ in the norm of $\mathbf{E}$ and that $\{\mu_n\}_n$ converges pointwise to $\mu$.*

*Proof.* let $\mu \in \mathcal{E}_q$. Let $\{\mathcal{N}_n\}_n$ be an exhaustion of $\mathcal{N}$ with $\mathcal{N}_n = \langle V_n, E_n \rangle$. We put $\mu_n = \mu$ on $V_n$ and $\mu_n = 0$ on $V \setminus V_n$. Clearly, $\{\mu_n\}_n$ increases monotonically and converges pointwise to $\mu$. Fatou's lemma shows that

$$G_q\mu(x) \le \liminf_{n\to\infty} G_q\mu_n(x) = \lim_{n\to\infty} G_q\mu_n(x) \le G_q\mu(x),$$

so that $\{G_q\mu_n\}_n$ converges pointwise to $G_q\mu$.

For $m < n$, the monotonicity of $\{\mu_n\}_n$ implies that $\{\|G_q\mu_n\|_{\mathbf{E}}\}$ converges and, together with Lemma 3.3, that

$$(G_q\mu_m, G_q\mu_n)_{\mathbf{E}} = G_q(\mu_m, \mu_n) \ge G_q(\mu_m, \mu_m) = \|G_q\mu_m\|_{\mathbf{E}}^2.$$

Consequently

$$\begin{aligned}\|G_q\mu_n - G_q\mu_m\|_{\mathbf{E}}^2 &= \|G_q\mu_n\|_{\mathbf{E}}^2 - 2(G_q\mu_n, G_q\mu_m)_{\mathbf{E}} + \|G_q\mu_m\|_{\mathbf{E}}^2 \\ &\le \|G_q\mu_n\|_{\mathbf{E}}^2 - \|G_q\mu_m\|_{\mathbf{E}}^2.\end{aligned}$$

Since $G_q\mu_n \in \mathbf{E}_0$ by Lemma 3.2, it follows that $\{G_q\mu_n\}_n$ converges to some $v \in \mathbf{E}_0$ in the norm of $\mathbf{E}$. This means that $v = G_q\mu$, and that $\{G_q\mu_n\}_n$ converges to $G_q\mu$ in the norm of $\mathbf{E}$. □

**Proposition 3.5.** *Let $\{\mu_n\}_n \subset \mathcal{E}_q$. If $\{G_q\mu_n\}_n$ converges to some $u \in \mathbf{E}$ in the norm of $\mathbf{E}$, then $u = G_q\mu$ for some $\mu \in \mathcal{E}_q$.*

*Proof.* Let $\{\mu_n\}_n \subset \mathcal{E}_q$. Lemma 3.2 implies that $G_q\mu_n \in \mathbf{E}_0$, so that $u \in \mathbf{E}_0$. Lemma 3.1 shows

$$\Delta_q u(x) = \lim_{n\to\infty} \Delta_q G_q\mu_n(x) = -\lim_{n\to\infty} \mu_n(x) \le 0.$$

Again by Lemma 3.2 we have that $u = G_q\mu$ for some $\mu \in \mathcal{E}_q$. □

Now we introduce two conditions which are similar to conditions (LD) and (CLD) considered in [7]. We say that $\mathcal{N}$ satisfies condition $(\mathrm{LD})_q$ if there exists a constant $c > 0$ such that

$$(\mathrm{LD})_q \qquad \|\Delta_q f\|_{\mathbf{E}} \le c\|f\|_{\mathbf{E}} \quad \text{for all } f \in L_0(V).$$

We say that $\mathcal{N}$ satisfies condition $(\mathrm{CLD})_q$ if there exists a constant $c > 0$ such that

$$(\mathrm{CLD})_q \qquad \|f\|_{\mathbf{E}} \le c\|\Delta_q f\|_{\mathbf{E}} \quad \text{for all } f \in L_0(V).$$

**Lemma 3.6.** *Assume* $(\mathrm{LD})_q$. *Then there exists a constant $c > 0$ such that $\|\Delta_q u\|_{\mathbf{E}} \le c\|u\|_{\mathbf{E}}$ for all $u \in \mathbf{E}$.*

*Proof.* Let $u \in \mathbf{E}$. By Lemma 2.3 we find $v \in \mathbf{E}_0$ and $h \in \mathbf{H}_q \cap \mathbf{E}$ such that $u = v + h$. Lemma 2.2 shows that

$$\begin{aligned}\|u\|_{\mathbf{E}}^2 &= \|v\|_{\mathbf{E}}^2 + 2(v,h)_{\mathbf{E}} + \|h\|_{\mathbf{E}}^2 \\ &= \|v\|_{\mathbf{E}}^2 + \|h\|_{\mathbf{E}}^2 \ge \|v\|_{\mathbf{E}}^2.\end{aligned}$$

Let $\{f_n\}_n$ be a sequence in $L_0(V)$ which converges to $v$ in the norm of $\mathbf{E}$. Then $(\mathrm{LD})_q$ implies that $\|\Delta_q f_n\|_{\mathbf{E}} \le c\|f_n\|_{\mathbf{E}}$ for all $n$. Since $\{\Delta_q f_n\}_n$ converges pointwise to $\Delta_q v$, Fatou's lemma gives

$$\begin{aligned}\|\Delta_q u\|_{\mathbf{E}} = \|\Delta_q v\|_{\mathbf{E}} &\le \liminf_{n\to\infty} \|\Delta_q f_n\|_{\mathbf{E}} \\ &\le c \liminf_{n\to\infty} \|f_n\|_{\mathbf{E}} = c\|v\|_{\mathbf{E}} \le c\|u\|_{\mathbf{E}}\end{aligned}$$

as required. □

**Lemma 3.7.** *Assume* $(\mathrm{LD})_q$. *Then* $\Delta_q u \in \mathbf{E}_0$ *for* $u \in \mathbf{E}_0$.

*Proof.* Let $u \in \mathbf{E}_0$ and $\{f_n\}_n$ a sequence in $L_0(V)$ which converges to $u$ in the norm of $\mathbf{E}$. Then $\|f_n - f_m\|_{\mathbf{E}} \to 0$ as $n, m \to \infty$. Condition $(\mathrm{LD})_q$ implies that

$$\|\Delta_q f_n - \Delta_q f_m\|_{\mathbf{E}} \le c\|f_n - f_m\|_{\mathbf{E}} \to 0$$

as $n, m \to \infty$. Thus $\{\Delta_q f_n\}_n$ is a Cauchy sequence in $\mathbf{E}$ and converges to some $v \in \mathbf{E}_0$ in the norm of $\mathbf{E}$. Since $\{\Delta_q f_n\}_n$ converges pointwise to $\Delta_q u$, we see that $\Delta_q u = v \in \mathbf{E}_0$. □

**Proposition 3.8.** *Assume both* $(\mathrm{LD})_q$ *and* $(\mathrm{CLD})_q$. *Then there exists a constant* $c > 0$ *such that*

$$\|u\|_{\mathbf{E}} \le c\|\Delta_q u\|_{\mathbf{E}} \quad \textit{for all } u \in \mathbf{E}_0.$$

*Proof.* Let $u \in \mathbf{E}_0$. There exists a sequence $\{f_n\}_n \subset L_0(V)$ which converges to $u$ in the norm of $\mathbf{E}$. Lemma 3.6 shows that there exists $c_1 > 0$ such that $\|\Delta_q u - \Delta_q f_n\|_{\mathbf{E}} \le c_1\|u - f_n\|_{\mathbf{E}}$ for all $n$, so that $\|\Delta_q f_n\|_{\mathbf{E}} \to \|\Delta_q u\|_{\mathbf{E}}$ as $n \to \infty$. By $(\mathrm{CLD})_q$, there exists $c_2 > 0$ such that $\|f_n\|_{\mathbf{E}} \le c_2\|\Delta_q f_n\|_{\mathbf{E}}$ for all $n$. We have

$$\|u\|_{\mathbf{E}} = \lim_{n\to\infty} \|f_n\|_{\mathbf{E}} \le c_2 \lim_{n\to\infty} \|\Delta_q f_n\|_{\mathbf{E}} = c_2\|\Delta_q u\|_{\mathbf{E}},$$

as required. □

**Lemma 3.9.** *Let* $\{u_n\}_n$ *be a sequence in* $\mathbf{E}_0$ *such that* $\{\|u_n\|_{\mathbf{E}}\}_n$ *is bounded and that* $\{u_n\}_n$ *converges pointwise to a function* $u \in \mathbf{E}$. *Then* $\lim_{n\to\infty}(u_n, v)_{\mathbf{E}} = (u, v)_{\mathbf{E}}$ *for* $v \in \mathbf{E}_0$.

*Proof.* Let $v \in \mathbf{E}_0$. For any $\varepsilon > 0$, there exists $f \in L_0(V)$ such that $\|v - f\|_{\mathbf{E}} < \varepsilon$. We take $M$ with $\|u_n\|_{\mathbf{E}} \le M$ for all $n$. Fatou's lemma shows that $\|u\|_{\mathbf{E}} \le M$. It is easy to see that $|(u_n - u, f)_{\mathbf{E}}| < \varepsilon$ for sufficiently large $n$. We have

$$\begin{aligned}|(u_n - u, v)_{\mathbf{E}}| &\le |(u_n - u, v - f)_{\mathbf{E}}| + |(u_n - u, f)_{\mathbf{E}}| \\ &\le \|u_n - u\|_{\mathbf{E}}\|v - f\|_{\mathbf{E}} + \varepsilon < (2M + 1)\varepsilon,\end{aligned}$$

and the assertion. □

**Lemma 3.10.** *If* $\mu \in \mathbf{E}_0 \cap L^+(V)$, *then there exists* $\{\mu_n\}_n \subset L_0(V) \cap L^+(V)$ *which converges to* $\mu$ *in the norm of* $\mathbf{E}$.

*Proof.* Let $\mu \in \mathbf{E}_0 \cap L^+(V)$. There exists a sequence $\{f_n\}_n$ in $L_0(V)$ which converges to $\mu$ in the norm of $\mathbf{E}$. Let $\mu_n = \max\{f_n, 0\}$. Then $\|\mu_n\|_{\mathbf{E}} \le \|f_n\|_{\mathbf{E}}$ by Lemma 2.5. Since $\mu \ge 0$, $\{\mu_n\}_n$ converges pointwise to $\mu$. Fatou's lemma gives

$$\begin{aligned}\|\mu\|_{\mathbf{E}} &\le \liminf_{n\to\infty} \|\mu_n\|_{\mathbf{E}} \le \limsup_{n\to\infty} \|\mu_n\|_{\mathbf{E}} \\ &\le \lim_{n\to\infty} \|f_n\|_{\mathbf{E}} = \|\mu\|_{\mathbf{E}},\end{aligned}$$

or $\lim_{n\to\infty}\|\mu_n\|_{\mathbf{E}} = \|\mu\|_{\mathbf{E}}$. Since $\{\|f_n\|_{\mathbf{E}}\}_n$ is bounded, so is $\{\|\mu_n\|_{\mathbf{E}}\}_n$. By Lemma 3.9, $(\mu_n,\mu)_{\mathbf{E}} \to (\mu,\mu)_{\mathbf{E}} = \|\mu\|^2_{\mathbf{E}}$ as $n\to\infty$. Thus we have

$$\|\mu-\mu_n\|^2_{\mathbf{E}} = \|\mu\|^2_{\mathbf{E}} - 2(\mu,\mu_n)_{\mathbf{E}} + \|\mu_n\|^2_{\mathbf{E}} \to 0$$

as $n\to\infty$. □

**Theorem 3.11.** *$\mathcal{E}_q = \mathbf{E}_0 \cap L^+(V)$ if both* $(\mathrm{LD})_q$ *and* $(\mathrm{CLD})_q$ *are fulfilled.*

*Proof.* Let $\mu\in\mathcal{E}_q$. By Lemma 3.4, there exists $\{\mu_n\}_n \subset L_0(V)\cap L^+(V)$ such that $\{G_q\mu_n\}_n$ converges to $G_q\mu$ in the norm of $\mathbf{E}$ and that $\{\mu_n\}_n$ converges pointwise to $\mu$. Lemma 3.2 shows that $G_q\mu\in\mathbf{E}_0$ and $G_q\mu_n\in\mathbf{E}_0$ for each $n$. By Lemmas 3.1 and 3.6

$$\|\mu-\mu_n\|_{\mathbf{E}} = \|\Delta_q G_q\mu_n - \Delta_q G_q\mu\|_{\mathbf{E}} \le c\|G_q\mu_n - G_q\mu\|_{\mathbf{E}} \to 0$$

as $n\to\infty$. Thus $\mu\in\mathbf{E}_0$.

We show the converse. Let $\mu\in\mathbf{E}_0\cap L^+(V)$. By Lemma 3.10, there exists $\{\mu_n\}_n \subset L_0(V)\cap L^+(V)$ which converges to $\mu$ in the norm of $\mathbf{E}$. Lemma 3.2 implies $G_q\mu_n\in\mathbf{E}_0$ for each $n$. Proposition 3.8 and Lemma 3.1 show that

$$\|G_q\mu_n - G_q\mu_m\|_{\mathbf{E}} \le c\|\Delta_q(G_q\mu_n - G_q\mu_m)\|_{\mathbf{E}} = c\|\mu_m-\mu_n\|_{\mathbf{E}} \to 0$$

as $n,m\to\infty$. Therefore $\{G_q\mu_n\}_n$ converges to some $u\in\mathbf{E}_0$ in the norm of $\mathbf{E}$. Fatou's lemma and Lemma 3.3 give

$$G_q(\mu,\mu) \le \liminf_{n\to\infty} G_q(\mu_n,\mu_n) = \lim_{n\to\infty}\|G_q\mu_n\|^2_{\mathbf{E}} = \|u\|^2_{\mathbf{E}} < \infty.$$

Namely $\mu\in\mathcal{E}_q$. □

For any $u\in L(V)$, we define $G_qu$ by $G_qu = G_qu^+ - G_qu^-$ if both $u^+ = \max\{u,0\}$ and $u^- = -\min\{u,0\}$ belong to $\mathcal{M}_q$.

**Theorem 3.12.** *$\mathbf{E}_0 = \mathcal{E}_q - \mathcal{E}_q$ if both* $(\mathrm{LD})_q$ *and* $(\mathrm{CLD})_q$ *are fulfilled. In this case, $u^+, u^- \in \mathcal{E}_q$ for $u\in\mathbf{E}_0$.*

*Proof.* By Theorem 3.11, $\mathcal{E}_q - \mathcal{E}_q \subset \mathbf{E}_0$. Conversely, for $u\in\mathbf{E}_0$, Lemma 2.5 and Theorem 3.11 imply that $u^+,u^-\in\mathbf{E}_0\cap L^+(V) = \mathcal{E}_q$, so that $\mathbf{E}_0\subset\mathcal{E}_q-\mathcal{E}_q$. □

**Theorem 3.13.** *$G_qu\in\mathbf{E}_0$ and $\Delta_qG_qu = -u$ for $u\in\mathbf{E}_0$ if both* $(\mathrm{LD})_q$ *and* $(\mathrm{CLD})_q$ *are fulfilled.*

*Proof.* Let $u\in\mathbf{E}_0$. Theorem 3.12 shows that $u^+,u^-\in\mathcal{E}_q$. Lemma 3.2 implies $G_qu = G_qu^+ - G_qu^- \in\mathbf{E}_0$. By Lemma 3.1 we have

$$\Delta_qG_qu = \Delta_qG_qu^+ - \Delta_qG_qu^- = -u^+ + u^- = -u$$

as required. □

**Corollary 3.14.** *$\{G_qu \mid u\in\mathbf{E}_0\}\subset\mathbf{E}_0$ if both* $(\mathrm{LD})_q$ *and* $(\mathrm{CLD})_q$ *are fulfilled.*

**Theorem 3.15.** *$G_q\Delta_qu = -u$ for $u\in\mathbf{E}_0$ if both* $(\mathrm{LD})_q$ *and* $(\mathrm{CLD})_q$ *are fulfilled.*

*Proof.* Let $u\in\mathbf{E}_0$. Then $v := \Delta_qu\in\mathbf{E}_0$ by Lemma 3.7. Theorem 3.13 shows that $G_qv\in\mathbf{E}_0$ and that $\Delta_q(u+G_qv) = v - v = 0$. Therefore $u+G_qv\in\mathbf{E}_0\cap\mathbf{H}_q$. Thus $u+G_qv=0$ by Lemma 2.2. □

We arrive at the following main result.

**Theorem 3.16.** $\mathbf{E}_0 = \{G_q\mu - G_q\nu \mid \mu, \nu \in \mathcal{E}_q\}$ *if both* $(\mathrm{LD})_q$ *and* $(\mathrm{CLD})_q$ *are fulfilled.*

*Proof.* Lemma 3.2 implies that $\{G_q\mu - G_q\nu \mid \mu, \nu \in \mathcal{E}_q\} \subset \mathbf{E}_0$. We show the converse. Let $u \in \mathbf{E}_0$. We have $v := -\Delta_q u \in \mathbf{E}_0$ by Lemma 3.7. Theorem 3.15 shows that $u = G_q v = G_q v^+ - G_q v^-$. Theorem 3.12 implies that $v^+, v^- \in \mathcal{E}_q$, and that $u \in \{G_q\mu - G_q\nu \mid \mu, \nu \in \mathcal{E}_q\}$. $\square$

As an application of our results, we shall give a version of Riesz decomposition of $u \in \mathbf{E}^{(2)} = \{u \in L(V) \mid \Delta_q u \in \mathbf{E}\}$ as follows. Let us put

$$\mathbf{E}_0^{(2)} = \{u \in L(V) \mid \Delta_q u \in \mathbf{E}_0\},$$
$$\mathbf{H}_q^{(2)} = \{u \in L(V) \mid \Delta_q u \in \mathbf{H}_q\}.$$

**Theorem 3.17.** *If both* $(\mathrm{LD})_q$ *and* $(\mathrm{CLD})_q$ *are fulfilled, then for every* $u \in \mathbf{E}^{(2)}$*, there exist a unique* $v \in \mathbf{E}_0$ *and a unique* $w \in \mathbf{H}_q^{(2)} \cap \mathbf{E}^{(2)}$ *such that* $u = G_q v + w$.

*Proof.* Let $u \in \mathbf{E}^{(2)}$. Applying Lemma 2.3 to $\Delta_q u \in \mathbf{E}$ yields

$$\Delta_q u = -v + h \quad \text{with } v \in \mathbf{E}_0 \text{ and } h \in \mathbf{H}_q \cap \mathbf{E}.$$

Theorem 3.13 shows that $\Delta_q G_q v = -v \in \mathbf{E}_0$. Hence $G_q v \in \mathbf{E}_0^{(2)}$. Let $w = u - G_q v$. Then $w \in \mathbf{E}^{(2)}$ and

$$\Delta_q w = \Delta_q u - \Delta_q G_q v = (-v + h) + v = h \in \mathbf{H}_q,$$

so that $w \in \mathbf{H}_q^{(2)}$.

To show the uniqueness, we assume that $u = G_q v_1 + w_1 = G_q v_2 + w_2$ with $v_1, v_2 \in \mathbf{E}_0$ and $w_1, w_2 \in \mathbf{H}_q^{(2)} \cap \mathbf{E}^{(2)}$. Theorem 3.13 shows that $w_1 - w_2 = G_q v_2 - G_q v_1 \in \mathbf{E}_0$. Lemma 3.7 implies $\Delta_q(w_1 - w_2) \in \mathbf{E}_0$. Since $w_1 - w_2 \in \mathbf{H}_q^{(2)}$, it follows that $\Delta_q(w_1 - w_2) \in \mathbf{H}_q$. Lemma 2.2 shows that $\Delta_q(w_1 - w_2) = 0$, so that $w_1 - w_2 \in \mathbf{H}_q \cap \mathbf{E}_0$. Again by Lemma 2.2 we have $w_1 = w_2$, so that $G_q v_1 = G_q v_2$. Theorem 3.13 gives $v_1 = -\Delta_q G_q v_1 = -\Delta_q G_q v_2 = v_2$. $\square$

**Corollary 3.18.** $\mathbf{E}^{(2)} = \mathbf{E}_0^{(2)} + \mathbf{H}_q^{(2)} \cap \mathbf{E}^{(2)}$ *if both* $(\mathrm{LD})_q$ *and* $(\mathrm{CLD})_q$ *are fulfilled.*

*Proof.* Clearly $\mathbf{E}_0^{(2)} + \mathbf{H}_q^{(2)} \cap \mathbf{E}^{(2)} \subset \mathbf{E}^{(2)}$. We show the converse. Let $u \in \mathbf{E}^{(2)}$. By Theorem 3.17 we take $v \in \mathbf{E}_0$ and $w \in \mathbf{H}_q^{(2)} \cap \mathbf{E}^{(2)}$ such that $u = G_q v + w$. Theorem 3.13 shows that $\Delta_q G_q v = -v \in \mathbf{E}_0$, so that $G_q v \in \mathbf{E}_0^{(2)}$. $\square$

## 4. Conditions $(\mathrm{LD})_q$ and $(\mathrm{CLD})_q$

We considered in [7] the following conditions:

(LD) There exists a constant $c > 0$ such that $\|\Delta f\|_{\mathbf{D}} \le c\|f\|_{\mathbf{D}}$ for all $f \in L_0(V)$;
(CLD) There exists a constant $c > 0$ such that $\|f\|_{\mathbf{D}} \le c\|\Delta f\|_{\mathbf{D}}$ for all $f \in L_0(V)$.

Note that $(\mathrm{LD})_q$ and $(\mathrm{CLD})_q$ in Section 3 are obtained by replacing $\mathbf{D}$ by $\mathbf{E}$ and $\Delta$ by $\Delta_q$ in (LD) and (CLD).

We recall

**Lemma 4.1** ([6, Lemma 3.2])**.** *Assume* (LD)*. Then there exists a constant $c > 0$ such that $\|\Delta u\|_{\mathbf{D}} \le c\|u\|_{\mathbf{D}}$ for all $u \in \mathbf{D}$.*

First of all, we note that $\|\Delta u\|_{\mathbf{D}} < \infty$ does not imply $\|\Delta_q u\|_{\mathbf{D}} < \infty$. In fact, let $u = 1$ on $V$ and $q \in L^+(V) \setminus \mathbf{D}$. Then $\|\Delta u\|_{\mathbf{D}} = 0$ and $\|\Delta_q u\|_{\mathbf{D}} = \|q\|_{\mathbf{D}} = \infty$.

Let us define $t(x,y)$ and $t(x)$ for $x, y \in V$ by

$$t(x,y) = \sum_{e\in E} |K(x,e)K(y,e)|r(e)^{-1} \quad \text{if } x \ne y,$$

$$t(x,x) = 0,$$

$$t(x) = \sum_{e\in E} |K(x,e)|r(e)^{-1} = \sum_{y\in V} t(x,y).$$

Then we have

$$\Delta u(x) = -t(x)u(x) + \sum_{y\in V} t(x,y)u(y).$$

For convenience sake, we introduce the following conditions:

($q$B) $q(x)$ is bounded on $V$;

($t$B) $t(x)$ is bounded on $V$.

**Lemma 4.2.** *Assume both* ($q$B) *and* ($t$B)*. Then there exists a constant $c > 0$ such that $\|qu\|_{\mathbf{D}} \le c\big(\sum_{x\in V} u(x)^2\big)^{1/2}$ and $\|qu\|_{\mathbf{D}} \le c\|u\|_{\mathbf{E}}$ for all $u \in \mathbf{E}$.*

*Proof.* Let $\gamma$ satisfy $t(x) \le \gamma$ and $q(x) \le \gamma$ for all $x \in V$. Let $u \in \mathbf{E}$. For $e \in E$, let $x_1$ and $x_2 \in V$ be the initial node and the terminal node of $e$. Then

$$\begin{aligned}(\nabla(qu)(e))^2 &= r(e)^{-2}\left(q(x_2)u(x_2) - q(x_1)u(x_1)\right)^2 \\ &\le r(e)^{-2} \times 2\left(q(x_2)^2u(x_2)^2 + q(x_1)^2u(x_1)^2\right) \\ &\le 2r(e)^{-2} \times \gamma\left(q(x_1)u(x_1)^2 + q(x_2)u(x_2)^2\right) \\ &= 2\gamma r(e)^{-2} \sum_{x\in V} |K(x,e)|q(x)u(x)^2.\end{aligned}$$

We have

$$\begin{aligned}\|qu\|_{\mathbf{D}}^2 &= \sum_{e\in E} r(e)(\nabla(qu)(e))^2 \le 2\gamma \sum_{e\in E} r(e)^{-1} \sum_{x\in V} |K(x,e)|q(x)u(x)^2 \\ &= 2\gamma \sum_{x\in V} t(x)q(x)u(x)^2 \le 2\gamma^2 \sum_{x\in V} q(x)u(x)^2,\end{aligned}$$

which implies $\|qu\|_{\mathbf{D}}^2 \le 2\gamma^3 \sum_{x\in V} u(x)^2$ and $\|qu\|_{\mathbf{D}}^2 \le 2\gamma^2\|u\|_{\mathbf{E}}^2$. □

**Proposition 4.3.** $(\mathrm{LD})_q$ *implies both* ($q$B) *and* ($t$B)*.*

*Proof.* Condition $(\mathrm{LD})_q$ shows that there exists $c > 0$ such that $\|\Delta\delta_a\|_{\mathbf{E}} \le c\|\delta_a\|_{\mathbf{E}}$ for all $a \in V$, where $\delta_a$ is the characteristic function of $\{a\}$. We shall show that $t(a) + q(a) \le c$.

Let $\{e_j\}_{j=1}^d \subset E$ be the arcs adjacent to $a$ and let $b_j \in V$ be the other node of $e_j$. For $e \in E$

$$\nabla\delta_a(e) = -r(e)^{-1}\sum_{x\in V} K(x,e)\delta_a(x) = -r(e)^{-1}K(a,e).$$

Since $K(x,e)^2 = |K(x,e)|$ in general,

$$\begin{aligned}\|\delta_a\|_{\mathbf{E}}^2 &= \sum_{e\in E} r(e)^{-1}K(a,e)^2 + \sum_{x\in V} q(x)\delta_a(x)^2 \\ &= \sum_{e\in E} r(e)^{-1}|K(a,e)| + q(a) = t(a) + q(a).\end{aligned}$$

On the other hand

$$\begin{aligned}\Delta_q\delta_a(x) &= \sum_{e\in E} K(x,e)\nabla\delta_a(e) - q(x)\delta_a(x) \\ &= -\sum_{e\in E} K(x,e)r(e)^{-1}K(a,e) - q(x)\delta_a(x) \\ &= -\sum_{i=1}^{d} K(x,e_i)r(e_i)^{-1}K(a,e_i) - q(x)\delta_a(x).\end{aligned}$$

Especially

$$\Delta_q\delta_a(a) = -t(a) - q(a).$$

Since $K(x,e_i)K(a,e_i) = 0$ unless $x = a$ or $x = b_i$ and $K(b_i,e_i)K(a,e_i) = -1$, it follows that

$$\begin{aligned}\nabla(\Delta_q\delta_a)(e) &= -r(e)^{-1}\sum_{x\in V} K(x,e)\Delta_q\delta_a(x) \\ &= r(e)^{-1}\sum_{x\in V} K(x,e)\Big(\sum_{i=1}^{d} K(x,e_i)r(e_i)^{-1}K(a,e_i) + q(x)\delta_a(x)\Big) \\ &= r(e)^{-1}\Big(K(a,e)t(a) - \sum_{i=1}^{d} K(b_i,e)r(e_i)^{-1} + K(a,e)q(a)\Big).\end{aligned}$$

If $e = e_j$, then, by $K(b_j,e_j) = -K(a,e_j)$,

$$\begin{aligned}\nabla(\Delta_q\delta_a)(e_j) &= r(e_j)^{-1}\Big(K(a,e_j)t(a) - K(b_j,e_j)r(e_j)^{-1} + K(a,e_j)q(a)\Big) \\ &= r(e_j)^{-1}K(a,e_j)\Big(t(a) + r(e_j)^{-1} + q(a)\Big).\end{aligned}$$

Consequently

$$\begin{aligned}
\|\Delta_q\delta_a\|_{\mathbf{E}}^2 &\ge \sum_{j=1}^{d} r(e_j)|\nabla(\Delta\delta_a)(e_j)|^2 + q(a)(\Delta_q\delta_a(a))^2 \\
&= \sum_{j=1}^{d} r(e_j)^{-1}\Big(t(a) + r(e_j)^{-1} + q(a)\Big)^2 + q(a)(-t(a)-q(a))^2 \\
&\ge \sum_{j=1}^{d} r(e_j)^{-1}\Big(t(a) + q(a)\Big)^2 + q(a)(t(a)+q(a))^2 \\
&= \Big(t(a)+q(a)\Big)^3.
\end{aligned}$$

Combining these we have $\Big(t(a)+q(a)\Big)^3 \le c^2\Big(t(a)+q(a)\Big)$, or $t(a)+q(a) \le c$. $\square$

Assuming $q = 0$ in the proposition above, we have

**Corollary 4.4.** (LD) *implies* ($t$B).

**Proposition 4.5.** *If both* (LD) *and* ($q$B) *are fulfilled, then there exists a constant* $c > 0$ *such that* $\|\Delta_q u\|_{\mathbf{D}} \le c\|u\|_{\mathbf{E}}$ *for all* $u \in \mathbf{E}$.

*Proof.* Let $u \in \mathbf{E}$. Note that Corollary 4.4 implies ($t$B). Lemmas 4.1 and 4.2 show that there exist constants $c_1 > 0$ and $c_2 > 0$ such that $\|\Delta u\|_{\mathbf{D}} \le c_1\|u\|_{\mathbf{D}}$ and $\|qu\|_{\mathbf{D}} \le c_2\|u\|_{\mathbf{E}}$. We have

$$\|\Delta_q u\|_{\mathbf{D}} \le \|\Delta u\|_{\mathbf{D}} + \|qu\|_{\mathbf{D}} \le (c_1+c_2)\|u\|_{\mathbf{E}}$$

as required. $\square$

Denote by $\mathbf{S}_q^+$ the set of $u \in L^+(V)$ such that $\Delta_q u \le 0$.

**Lemma 4.6.** *Assume both* ($q$B) *and* ($t$B). *Then there exists a constant* $c > 0$ *such that* $|\Delta_q u(x)| \le cu(x)$ *on* $V$ *for all* $u \in \mathbf{S}_q^+$.

*Proof.* Let $u \in \mathbf{S}_q^+$. If we set $\Delta^* u(x) = \sum_{y\in V} t(x,y)u(y)$, then, since $\Delta_q u(x) = \Delta^* u(x) - (t(x)+q(x))u(x)$, it follows that

$$(t(x)+q(x))u(x) \ge \Delta^* u(x) \ge 0,$$

so that

$$|\Delta_q u(x)| \le |\Delta^* u(x)| + |(t(x)+q(x))u(x)| \le 2(t(x)+q(x))u(x).$$

We may take $c = 2\sup_{x\in V}(t(x)+q(x))$. $\square$

**Theorem 4.7.** *If both* (LD) *and* ($q$B) *are fulfilled, then there exists a constant* $c > 0$ *such that*

$$\|\Delta_q u\|_{\mathbf{E}} \le c\|u\|_{\mathbf{E}} \quad \textit{for all } u \in \mathbf{E}_0 \cap \mathbf{S}_q^+.$$

*Proof.* Let $u \in \mathbf{E}_0 \cap \mathbf{S}_q^+$. Note that Corollary 4.4 implies ($t$B). Proposition 4.5 and Lemma 4.6 show that there exist constants $c_1 > 0$ and $c_2 > 0$ such that $\|\Delta_q u\|_{\mathbf{D}} \le c_1 \|u\|_{\mathbf{E}}$ and $|\Delta_q u(x)| \le c_2 u(x)$ on $V$. We have

$$\begin{aligned}\|\Delta_q u\|_{\mathbf{E}}^2 &= \|\Delta_q u\|_{\mathbf{D}}^2 + \sum_{x\in V} q(x)(\Delta_q u(x))^2 \le c_1^2 \|u\|_{\mathbf{E}}^2 + c_2^2 \sum_{x\in V} q(x)u(x)^2 \\ &\le (c_1^2 + c_2^2)\|u\|_{\mathbf{E}}^2,\end{aligned}$$

as required. □

**Proposition 4.8.** *If both* ($q$B) *and* ($t$B) *are fulfilled and if $q$ is superharmonic on $V$, i.e., $\Delta q \le 0$ on $V$, then there exists a constant $c > 0$ such that*

$$\sum_{x\in V} q(x)(\Delta_q u(x))^2 \le c \sum_{x\in V} q(x)u(x)^2$$

*for all $u \in L(V)$.*

*Proof.* Let $\gamma$ satisfy $t(x) \le \gamma$ and $q(x) \le \gamma$ for all $x \in V$. We set $\Delta^* u(x) = \sum_{y\in V} t(x,y)u(y)$. Schwarz's inequality implies that

$$\begin{aligned}(\Delta^* u(x))^2 &\le \Big(\sum_{y\in V} t(x,y)\Big)\Big(\sum_{y\in V} t(x,y)u(y)^2\Big) = t(x)\sum_{y\in V} t(x,y)u(y)^2 \\ &\le \gamma \sum_{y\in V} t(x,y)u(y)^2.\end{aligned}$$

Since $q$ is superharmonic on $V$, i.e., $\Delta^* q(x) \le t(x)q(x)$ on $V$, it follows that

$$\begin{aligned}\sum_{x\in V} q(x)(\Delta^* u(x))^2 &\le \gamma \sum_{x\in V} q(x) \sum_{y\in V} t(x,y)u(y)^2 \\ &= \gamma \sum_{y\in V} u(y)^2 \sum_{x\in V} t(x,y)q(x) \\ &= \gamma \sum_{y\in V} u(y)^2 \Delta^* q(y) \\ &\le \gamma \sum_{y\in V} u(y)^2 t(y)q(y) \le \gamma^2 \sum_{y\in V} q(y)u(y)^2.\end{aligned}$$

We have

$$\begin{aligned}(\Delta_q u(x))^2 &= \Big(\Delta^* u(x) - (t(x)+q(x))u(x)\Big)^2 \\ &\le 2(\Delta^* u(x))^2 + 2(t(x)+q(x))^2 u(x)^2 \\ &\le 2(\Delta^* u(x))^2 + 8\gamma^2 u(x)^2,\end{aligned}$$

so that

$$\begin{aligned}\sum_{x\in V} q(x)(\Delta_q u(x))^2 &\le 2\sum_{x\in V} q(x)(\Delta^* u(x))^2 + 8\gamma^2 \sum_{x\in V} q(x)u(x)^2 \\ &\le 10\gamma^2 \sum_{x\in V} q(x)u(x)^2.\end{aligned}$$

This completes the proof. $\square$

**Theorem 4.9.** *If $q$ is superharmonic on $V$, then* $(\mathrm{LD})_q$ *follows from* (LD) *and* ($q$B).

*Proof.* Let $f \in L_0(V)$ and assume (LD) and ($q$B). Proposition 4.5 shows that there exists a constant $c_1 > 0$ such that $\|\Delta_q f\|_{\mathbf{D}} \le c_1 \|f\|_{\mathbf{E}}$. Since ($t$B) is fulfilled by Corollary 4.4, there exists a constant $c_2 > 0$ such that

$$\sum_{x \in V} q(x)(\Delta_q f(x))^2 \le c_2 \sum_{x \in V} q(x) f(x)^2 \le c_2 \|f\|_{\mathbf{E}}^2$$

by Proposition 4.8. Thus we have $\|\Delta_q f\|_{\mathbf{E}}^2 \le (c_1^2 + c_2)\|f\|_{\mathbf{E}}^2$, so that $(\mathrm{LD})_q$ is fulfilled. $\square$

As a generalized version of Poincaré-Sobolev's inequality, we introduced in [7] the following condition (SPS): There exists a constant $c > 0$ such that

$$\text{(SPS)} \qquad \sum_{x \in V} f(x)^2 \le c\|f\|_{\mathbf{D}}^2 \quad \text{for all } f \in L_0(V).$$

**Lemma 4.10** ([7, Lemma 2.1])**.** *Assume* (SPS)*. Then there exists a constant $c > 0$ such that*

$$\sum_{x \in V} u(x)^2 \le c\|u\|_{\mathbf{D}}^2 \quad \textit{for all } u \in \mathbf{D}_0.$$

**Proposition 4.11.** *If both* (SPS) *and* ($q$B) *are fulfilled, then there exists a constant $c > 0$ such that $\|u\|_{\mathbf{E}} \le c\|u\|_{\mathbf{D}}$ for all $u \in \mathbf{D}_0$.*

*Proof.* Let $\gamma$ be such that $q(x) \le \gamma$ for all $x \in V$. By Lemma 4.10, there exists a constant $c_1 > 0$ such that

$$\|u\|_{\mathbf{E}}^2 = \|u\|_{\mathbf{D}}^2 + \sum_{x \in V} q(x)u(x)^2 \le \|u\|_{\mathbf{D}}^2 + \gamma \sum_{x \in V} u(x)^2 \le (1 + c_1\gamma)\|u\|_{\mathbf{D}}^2,$$

which shows the assertion. $\square$

**Corollary 4.12.** $\mathbf{E}_0 = \mathbf{D}_0$ *if both* (SPS) *and* ($q$B) *are fulfilled.*

*Proof.* Since $\mathbf{D}_0 \subset \mathbf{E}$ by Proposition 4.11, we have $\mathbf{E}_0 = \mathbf{D}_0 \cap \mathbf{E} = \mathbf{D}_0$ by Lemma 2.1. $\square$

**Lemma 4.13.** *Assume all of* (SPS), ($q$B), *and* ($t$B)*. Then there exists a constant $c > 0$ such that $\|qu\|_{\mathbf{D}} \le c\|u\|_{\mathbf{D}}$ for all $u \in \mathbf{D}_0$.*

*Proof.* Let $u \in \mathbf{D}_0$. Then $u \in \mathbf{E}_0$ by Corollary 4.12. Lemmas 4.2 and 4.10 show that $\|qu\|_{\mathbf{D}} \le c_1(\sum_{x \in V} u(x)^2)^{1/2}$ and $\sum_{x \in V} u(x)^2 \le c_2\|u\|_{\mathbf{D}}^2$. Combining these, we have $\|qu\|_{\mathbf{D}}^2 \le c_1^2 c_2 \|u\|_{\mathbf{D}}^2$. $\square$

**Lemma 4.14.** $\{\Delta_q u \mid u \in \mathbf{D}_0\} \subset \mathbf{D}_0$ *if all of* (LD), (SPS), *and* ($q$B) *are fulfilled.*

*Proof.* Let $u \in \mathbf{D}_0$. Then $\Delta u \in \mathbf{D}_0$ by [5, Lemma 6.1]. Let $\{f_n\}_n$ be a sequence in $L_0(V)$ such that $\|u - f_n\|_{\mathbf{D}} \to 0$ as $n \to \infty$. There exists a constant $c_1 > 0$ such that $\|qu - qf_n\|_{\mathbf{D}} \le c_1\|u - f_n\|_{\mathbf{D}}$ by Lemma 4.13. Since $qf_n \in L_0(V)$, we see that $qu \in \mathbf{D}_0$. Therefore $\Delta_q u = \Delta u - qu \in \mathbf{D}_0$. $\square$

**Theorem 4.15.** (LD)$_q$ *follows from all of* (LD), (SPS), *and* ($q$B).

*Proof.* Assume all of (LD), (SPS), and ($q$B). Let $\gamma$ be a number such that $q(x) \le \gamma$ for all $x \in V$. Let $f \in L_0(V)$. There exists a constant $c_1 > 0$ such that $\|\Delta_q f\|_{\mathbf{D}} \le c_1\|f\|_{\mathbf{E}}$ by Proposition 4.5. Since $\Delta_q f \in L_0(V)$, we have $\sum_{x\in V}(\Delta_q f(x))^2 \le c_2\|\Delta_q f\|_{\mathbf{D}}^2$ by Lemma 4.10. We have

$$\begin{aligned}\|\Delta_q f\|_{\mathbf{E}}^2 &\le c_1^2\|f\|_{\mathbf{E}}^2 + \sum_{x\in V} q(x)(\Delta_q f(x))^2 \le c_1^2\|f\|_{\mathbf{E}}^2 + \gamma c_2\|\Delta_q f\|_{\mathbf{D}}^2 \\ &\le c_1^2(1+\gamma c_2)\|f\|_{\mathbf{E}}^2,\end{aligned}$$

which shows (LD)$_q$. □

**Theorem 4.16.** (SPS) *implies* (CLD)$_q$.

*Proof.* Let $f \in L_0(V)$. Since $\Delta_q f \in L_0(V)$, there exists a constant $c_1 > 0$ by (SPS) such that

$$\sum_{x\in V}(\Delta_q f(x))^2 \le c_1\|\Delta_q f\|_{\mathbf{D}}^2 \quad \text{and} \quad \sum_{x\in V} f(x)^2 \le c_1\|f\|_{\mathbf{D}}^2.$$

Lemma 2.6 shows that

$$\begin{aligned}\|f\|_{\mathbf{E}}^2 &= -\sum_{x\in V}(\Delta_q f(x))f(x) \le \Big(\sum_{x\in V}(\Delta_q f(x))^2\Big)^{1/2}\Big(\sum_{x\in V} f(x)^2\Big)^{1/2} \\ &\le c_1\|\Delta_q f\|_{\mathbf{D}}\|f\|_{\mathbf{D}} \le c_1\|\Delta_q f\|_{\mathbf{E}}\|f\|_{\mathbf{E}},\end{aligned}$$

or $\|f\|_{\mathbf{E}} \le c_1\|\Delta_q f\|_{\mathbf{E}}$. □

Finally we give an example to show that (LD) does not imply (LD)$_q$.

**Example 4.17.** Let $\mathcal{N} = \langle V, E, K, r\rangle$ be a linear network, where $V = \{x_n\}_{n=0}^{\infty}$, $E = \{e_n\}_{n=1}^{\infty}$, and $r(e_n) = 1$ for each $n \ge 1$. Let $K(x_{n-1}, e_n) = 1$ and $K(x_n, e_n) = -1$ for each $n \ge 1$, and let $K(x, e) = 0$ for any other pairs. We showed in [6, Corollary 2.3] that $\mathcal{N}$ satisfies (LD).

To prove that (LD)$_q$ is not satisfied, we choose $q(x_k) = k$. Consider the function $f_n$ defined by $f_n(x_k) = 1$ if $k < n$ and $f_n(x_k) = 0$ otherwise. Then $\nabla f_n(e_k) = -\delta_{n,k}$, where $\delta_{n,k}$ is Kronecker's delta. Therefore

$$\|f_n\|_{\mathbf{E}}^2 = \sum_{k=1}^{\infty}(-\delta_{n,k})^2 + \sum_{k=0}^{n-1} k\cdot 1^2 = 1 + \frac{1}{2}n(n-1).$$

On the other hand, $\Delta_q f_n(x_k) = -k$ for $k \le n-2$, so that

$$\|\Delta_q f_n\|_{\mathbf{E}}^2 \ge \sum_{k=0}^{n-2} q(x_k)(\Delta_q f_n(x_k))^2 = \sum_{k=0}^{n-2} k^3 = \frac{1}{4}(n-1)^2(n-2)^2.$$

Consequently

$$\lim_{n\to\infty}\frac{\|\Delta_q f_n\|_{\mathbf{E}}}{\|f_n\|_{\mathbf{E}}} = \infty,$$

which means that $\mathcal{N}$ does not satisfy (LD)$_q$.

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