Mem. Gra. Sci. Eng. Shimane Univ.
Series B: Mathematics
49 (2016), pp. 1–13

QUASI-BOUNDED AND SINGULAR FUNCTIONS IN AN INFINITE NETWORK

NATHIYA NARAYANARAJU AND PREMALATHA KUMARESAN

Communicated by Toshihiro Nakanishi

(Received: February 19, 2015)

Abstract. In a hyperbolic network X, the properties of non-negative quasi-bounded harmonic functions and singular harmonic functions are considered by using the notion of reduced functions (balayage). Later some extensions of these results are given with respect to the class of non-negative quasi-bounded super-harmonic functions and the class of non-negative q-harmonic functions defined by discrete Schrödinger operators on X.

1. Introduction

The notions of quasi-bounded harmonic function and singular harmonic function were introduced by Parreau [4] in 1951. This concept was generalized by Arsove and Leutwiler [2] in 1974. In this paper we define a map on the family of non-negative functions having superharmonic majorants in X and this map is used to obtain necessary and sufficient conditions for a non-negative harmonic function to be a quasi-bounded harmonic function, singular harmonic function in an infinite network X. Further this map is used to define quasi-bounded superharmonic functions in X. Similar to the Parreau decomposition for non-negative harmonic functions, we give a unique decomposition for a non-negative superharmonic function in X as the sum of a quasi-bounded superharmonic function and a singular harmonic function.

We carry on the study to get a similar decomposition of Schrödinger harmonic functions defined as solutions of $\Delta u(x) = q(x)u(x)$ where $q(x)$ is a non-negative real-valued function on X such that $q \not\equiv 0$. We say that $u(x)$ is $q-$harmonic at $x \in X$ if $\Delta_q u(x) = \Delta u(x) - q(x)u(x) = 0$, where $q(x) \geq 0$, $q \not\equiv 0$ and give the Parreau decomposition as quasi-bounded $q-$harmonic function and singular $q-$harmonic function for any positive $q-$harmonic function. In this case, there always exist positive $q-$ superharmonic functions which are not $q-$harmonic. Consequently there will always exist positive $q-$harmonic functions on X. But it is possible that every such $q-$harmonic function is bounded so that there may not be any

2000 *Mathematics Subject Classification.* 30F15, 31C05, 31C20, 31C12.

Key words and phrases. Infinite networks, reduced functions, Quasi-bounded superharmonic functions, Singular harmonic functions, subordinate structures.

singular $q-$harmonic function on X. We give a number of sufficient conditions for the existence of positive singular $q-$harmonic functions on X.

2. Preliminaries

Let X be a countable set of vertices; a countable set Y of edges joining some pairs of nodes is given; the resulting graph is assumed to be connected, locally finite and without self-loops. Two vertices x and y are said to be neighbours, denoted by $x \sim y$, if and only if there is an edge joining x and y. Assume that for each pair of distinct vertices x and y in X is associated a number $t(x, y) \geq 0$ called conductance such that $t(x, y) > 0$ if and only if $x \sim y$; $t(x, y)$ and $t(y, x)$ may be different and $t(x) = \sum\limits_{y \sim x} t(x, y)$ for every $x \in X$. Then X is called an infinite network.

Given a subset E of X, the interior $\overset{\circ}{E}$ of E is defined as the set of vertices in E all of whose neighbours are in E. Write $\partial E = E \setminus \overset{\circ}{E}$. Let $d(e, x)$ denote the distance from the vertex e to x, that is the length of the path from e to x.

Let $s(x)$ be a real-valued function defined on X. Then the Laplacian of s at $x_0 \in X$ is defined by $\Delta s(x_0) = \sum\limits_{z \sim x_0} t(x_0, z)[s(z) - s(x_0)]$. A real valued function $s(x)$ on E is said to be superharmonic (respectively harmonic) on E if and only if $\Delta s(x) \leq 0$ (respectively $\Delta s(x) = 0$) for every $x \in \overset{\circ}{E}$; $s(x)$ is subharmonic if and only if $-s(x)$ is superharmonic on E. We recall some properties of superharmonic functions and statements of theorems which are given in [1].

Properties of superharmonic functions:

(1) Let s_1 and s_2 be two superharmonic functions on X. Then $s_1 + s_2$ is also superharmonic on X.

(2) If s is superharmonic on X and α is a non-negative constant then αs is also superharmonic on X.

(3) If s_1 and s_2 are two superharmonic functions on X, then $\inf(s_1, s_2)$ is also superharmonic on X .

(4) If $\{s_n\}$ is a sequence of superharmonic functions on X such that $s(x) = \lim\limits_{n \to \infty} s_n(x)$ is finite at every $x \in X$, then s is superharmonic on X.

(5) Let $\{u_i : i \in I\}$ be a family $\Im$ of upper directed superharmonic functions on X. That is, if $u_1, u_2 \in \Im$, there exists $u \in \Im$ such that $u \geq u_1$ and $u \geq u_2$. If $u = \sup\{u_i\}$ is finite-valued, then u is superharmonic on X.

(6) If $\{u_i : i \in I\}$ is a family of superharmonic functions on X such that $u_i \geq f$ for a real-valued function f, then $u = \inf\{u_i\}$ is superharmonic on X.

(7) Let s and t be real-valued functions in X. Let s be superharmonic and t be subharmonic on X such that $s \geq t$. Then there exists a harmonic function

h on X such that $s \geq h \geq t$ which is called the greatest harmonic minorant of s in X.

Definition 2.1. A non-negative superharmonic function p on X is said to be a potential if its greatest harmonic minorant is zero.

Theorem 2.2 (Riesz representation theorem)**.** *Any superharmonic function $s \geq 0$ on X can be written as the unique sum of a potential and a non-negative harmonic function on X.*

Definition 2.3. An infinite network X is said to be a hyperbolic network if and only if there exists a positive potential on X; otherwise, X is called a parabolic network.

We assume that X is a hyperbolic network throughout the paper.

Reduced functions: Let f be a real-valued function on X. Suppose there exists a superharmonic function s on X such that $s \geq f$ on X. Let $\mathfrak{F}$ be the family of all superharmonic functions u such that $u \geq f$ on X. The reduced function of f is defined as $Rf\,(x) = \inf\,\{u\,(x) : u \in \mathfrak{F}\}$.

Lemma 2.4. *Let f be a real-valued function majorised by a superharmonic function in X. Then Rf is superharmonic in X such that $f \leq Rf$. If f is subharmonic at a vertex x_0, then Rf is harmonic at x_0.*

Proof. The family $\mathfrak{F}$ is lower directed; for let $s_3 = \min(s_1, s_2)$ where $s_1, s_2 \in \mathfrak{F}$ then s_3 is a superharmonic function such that $f \leq s_3$ so that $s_3 \in \mathfrak{F}$. Hence there exists a decreasing sequence of functions $\{s_n\}$ in $\mathfrak{F}$ such that $Rf = \lim_{n\to\infty} s_n$. This implies Rf is superharmonic such that $f \leq Rf$.

Let f be subharmonic at x_0. Define

$$f_{x_0}(x) = \begin{cases} \sum\limits_{y\sim x_0} \frac{t(x_0,y)}{t(x_0)} Rf(y), & \text{if } x = x_0 \\ Rf\,(x)\,, & \text{if } x \neq x_0. \end{cases}$$

Then $f_{x_0}(x)$ is superharmonic on X and harmonic at $x = x_0$ such that $f \leq f_{x_0}$. Thus $f_{x_0} \in \mathfrak{F}$ and hence $Rf \leq f_{x_0}$. Clearly $f_{x_0} \leq Rf$. Hence $f_{x_0} = Rf$ and Rf is harmonic at x_0. □

Note:

(1) If f is a subharmonic function majorised by a superharmonic function in X, then Rf is the least harmonic majorant of f.

(2) If $f \geq 0$ is majorised by a superharmonic function in X, then $Rf(x)$ is harmonic at every vertex where $f(x) = 0$.

3. Quasi-bounded and Singular harmonic functions

Parreau [4] introduced the notions of non-negative quasi-bounded harmonic functions and non-negative singular harmonic functions. According to Parreau: A non-negative harmonic function u on a Riemann surface R is quasi-bounded if it is the limit of an increasing sequence of bounded non-negative harmonic functions on R; u is singular if the only non-negative bounded harmonic function on R majorised by u is the identically zero function. Every non-negative harmonic function on R has a unique representation as the sum of a non-negative quasi-bounded harmonic function on R and a non-negative singular harmonic function on R.

As Parreau defined, we define a non-negative quasi-bounded harmonic function and a non-negative singular harmonic function in an infinite network X below and give the unique representation for every non-negative harmonic function h in X as $h = q + s$, where q is quasi-bounded harmonic and s is singular harmonic in X.

Definition 3.1. A non-negative harmonic function u in an infinite network X is quasi-bounded if it is the limit of an increasing sequence of bounded non-negative harmonic functions in X; u is singular if the only non-negative bounded harmonic function in X majorised by u is the function identically zero.

Note: A non-negative harmonic function u is singular if and only if s is an upper bounded subharmonic function such that $s \leq u$ implies $s \leq 0$.

For, if s is an upper bounded subharmonic function on X, then s^+ is a bounded non-negative subharmonic function, $s^+ \leq u$. Then there exists a bounded harmonic function H on X such that $s^+ \leq H \leq u$. If u is a singular harmonic function, then $H = 0$; consequently $s \leq 0$. The converse is obvious.

Some properties of singular harmonic functions:

(1) Let v_1, v_2 be two non-negative harmonic functions such that $v_1 \leq v_2$. If v_2 is singular then v_1 is singular.

Proof. Let b be a bounded non-negative harmonic function in X such that $b \leq v_1$; then $b \leq v_2$ so that $b = 0$. □

(2) If u, v are singular harmonic, then $u + v$ is singular.

Proof. Let $b \geq 0$ be a bounded harmonic function such that $b \leq u + v$. Then $b - v \leq u$ which implies $(b - v)^+ \leq u$. Since $(b - v)^+ \leq b$, $(b - v)^+$ is a bounded subharmonic function such that $(b - v)^+ \leq u$. Since u is singular $(b - v)^+ = 0$ or $b \leq v$. Since v is singular $b = 0$. □

(3) Let $\{v_n\}$ be a sequence of singular harmonic functions in X. Let $v(x) = \sum\limits_{n=1}^{\infty} v_n(x)$. If $v(x_0) < \infty$ for some $x_0 \in X$, then v is singular harmonic.

Proof. Since $v(x_0) < \infty$, $v(x) < \infty$ for all $x \sim x_0$. For if $x \sim x_0$ and $v(x) = \infty$, then

$$v(x_0) = \sum_{n=1}^{\infty} v_n(x_0) = \sum_{n=1}^{\infty} \sum_{y \sim x_0} \frac{t(x_0, y)}{t(x_0)} v_n(y) = \sum_{y \sim x_0} \frac{t(x_0, y)}{t(x_0)} v(y) = \infty,$$

since $x \sim x_0$ and $v(x) = \infty$. This contradiction shows that $v(x) < \infty$ for all $x \sim x_0$ and hence harmonic at x_0. Since X is connected $v(x) < \infty$ for all $x \in X$ and harmonic in X. To show v is singular, let b be bounded harmonic function such that $0 \leq b \leq v$. Then $b - \sum\limits_{n=2}^{\infty} v_n \leq v_1$ implies $[b - \sum\limits_{n=2}^{\infty} v_n]^+ \leq v_1$. Since the left hand side is a bounded subharmonic function and v_1 is singular

$[b - \sum_{n=2}^{\infty} v_n]^+ = 0$. Therefore $b(x) \leq \sum_{n=2}^{\infty} v_n(x)$; proceeding like this we get for any integer $m > 0$, $b(x) \leq \sum_{n=m}^{\infty} v_n(x)$. Since $\sum_{n=1}^{\infty} v_n(x) < \infty$ for all x, for $y \in X$, there exists m sufficiently large such that $\sum_{n=m}^{\infty} v_n(y) < \epsilon$. Hence $b(y) < \epsilon$. Since ϵ is arbitrary, $b(y) = 0$. That is, $b = 0$. Hence v is singular. □

Theorem 3.2. *Let $h > 0$ be harmonic in X. Let $\wp$ be the set of all bounded non-negative harmonic functions b such that $b \leq h$. Then $u = \sup_{b \in \wp} b$ is the largest quasi-bounded harmonic function in X such that $u \leq h$ and $h - u$ is singular.*

Proof. Note that $\wp$ is upper directed. For, if $b_1, b_2 \in \wp$ then $\sup(b_1, b_2)$ is a bounded subharmonic function in X such that $\sup(b_1, b_2) \leq h$. This implies there exists a bounded harmonic function w such that $\sup(b_1, b_2) \leq w \leq h$, that is $w \in \wp$. Since X is countable, there exists $\{b_n\}$ an increasing sequence in $\wp$ such that $u = \lim_n b_n$. Hence u is a quasi-bounded harmonic function in X. Suppose q is quasi-bounded and $q \leq h$. Then $q = \lim_n q_n$, where $\{q_n\}$ is an increasing sequence of bounded harmonic functions such that $q_n \leq h$. Then $\{q_n\} \in \wp$. Hence $q \leq u$. Thus u is the greatest quasi-bounded harmonic function in X such that $u \leq h$. Let $v = h - u$. Suppose $b_0 \geq 0$ is a bounded harmonic function in X such that $b_0 \leq v$. Then $u + b_0 = \lim(b_n + b_0)$ is quasi-bounded such that $u + b_0 \leq h$; hence $u + b_0 \leq u$ which gives $b_0 = 0$. Thus v is singular. □

Corollary 3.3. *Any non-negative harmonic function h in X has a decomposition $h = q + s$ where q is the largest quasi-bounded harmonic function such that $q \leq h$ and s is a singular harmonic function in X.*

4. The operator T

Let $\Im$ be the class of all non-negative real-valued functions u on X admitting superharmonic majorants. Define a map $T : \Im \to \Im$ as below:

If $u \in \Im$, then there exists a non-negative superharmonic function φ on X such that $u \leq \varphi + n$, for any non-negative integer n. Then $u - n \leq \varphi$ implies $(u-n)^+ \leq \varphi$. Therefore $(u-n)^+ \in \Im$. Let us denote $(u-n)^+$ by u_n. Then $u_n = 0$ on $E_n = \{x \in X/u(x) \leq n\}$, where $E_n \subset E_{n+1}$ and $\cup E_n = X$. Consider the reduced function of u_n in X, that is, $Ru_n = \inf\{v : v \textit{ non-negative superharmonic such that } u_n \leq v \textit{ on } X\}$. Then

1. Ru_n is non-negative superharmonic in X and harmonic in E_n.
2. $u_n \leq Ru_n$ on X.
3. Since $\{u_n\}$ is a decreasing non-negative sequence in X, $\{Ru_n\}$ is also a decreasing non-negative superharmonic sequence in X.

If we define $Tu = \lim_{n\to\infty} Ru_n$, for every $u \in \Im$, then Tu is a non-negative harmonic function in X. For, let $x_0 \in X$. Then $x_0 \in E_n$ for some n. Then $Ru_n(x)$ is harmonic at $x = x_0$. Since $Ru_m(x)$ is harmonic at $x = x_0$ for every $m \geq n$ this implies Tu is harmonic at x_0. Since x_0 is arbitrary, Tu is harmonic on X admitting a superharmonic majorant. Hence $Tu \in \Im$, for every $u \in \Im$. Note that all non-negative superharmonic functions are in $\Im$.

Properties of the map T:

(1) If $u, v \in \Im$ and $u \leq v$ then $Tu \leq Tv$.

Proof. For any non-negative integer n, $u_n = (u-n)^+ \leq (v-n)^+ = v_n$ and $Ru_n \leq Rv_n$ which implies $Tu \leq Tv$. □

(2) If $u, v \in \Im$ then $T(\inf(u, v)) \leq \inf(Tu, Tv)$.

Proof. $\inf(u, v) \leq u$ and $\inf(u, v) \leq v$. Therefore by property (1) $T(\inf(u, v)) \leq Tu$ and $T(\inf(u, v)) \leq Tv$ and hence $T(\inf(u, v)) \leq \inf(Tu, Tv)$. □

(3) If $u, v \in \Im$ then $T(u + v) \leq Tu + Tv$.

Proof. Note that $(u + v - 2n)^+ \leq (u - n)^+ + (v - n)^+$. If $(u + v - 2n)^+ = (u+v)_{2n}$ then $(u+v)_{2n} \leq u_n + v_n \leq Ru_n + Rv_n$ which implies $R(u+v)_{2n} \leq Ru_n + Rv_n$. Taking limits on both sides we get $T(u + v) \leq Tu + Tv$. □

(4) Let $u \in \Im$ and h be a non-negative bounded function in X. Then $T(u+h) = Tu$.

Proof. Since h is a non-negative bounded function on X there exists a non-negative integer m such that $0 \leq h \leq m$ on X. Hence $h_n = 0$ for every $n \geq m$ so that $Rh_n = 0$ for every $n \geq m$ which implies $Th = 0$. Since $u \leq u + h$ we have $Tu \leq T(u + h) \leq Tu + Th = Tu$ and hence $Tu = T(u + h)$. □

(5) If $u \in \Im$ is superharmonic on X then $Tu \leq u$.

(6) Let $\{u_n\} \geq 0$ be a decreasing sequence of superharmonic functions in X such that $\lim_{n\to\infty} u_n = h$, where h is harmonic in X. Then $\lim_{n\to\infty} Tu_n = Th$ in X.

Proof. Since $h \leq u_n$, we have

$$Th \leq Tu_n \leq Th + T(u_n - h) \leq Th + u_n - h.$$

Allowing $n \to \infty$, we get $Th \leq \lim_{n\to\infty} Tu_n \leq Th$. Hence $\lim_{n\to\infty} Tu_n = Th$. □

(7) If $u \in \Im$ then $Tu = TRu$ in X.

Proof. Since $u \leq Ru$, $Tu \leq TRu$. Let φ be a non-negative superharmonic function in X such that $u \leq \varphi + n$, for any non-negative integer n. Then $Ru \leq \varphi + n$, implies $R(Ru)_n \leq \varphi$. Now taking infimum over all the admitting φ, we get $R(Ru)_n \leq Ru_n$, as limit $n \to \infty$ $TRu \leq Tu$. Hence $Tu = TRu$. □

Theorem 4.1. *Let h be a non-negative harmonic function in X. Then h is quasi-bounded if and only if $Th = 0$.*

Proof. Let h be a non-negative quasi-bounded harmonic function in X. That is, there exists an increasing sequence of non-negative bounded harmonic functions $\{h_n\}$ in X such that $\lim_{n\to\infty} h_n = h$. Write $h = (h - h_n) + h_n$. Since h_n is a non-negative bounded harmonic function by Properties 4 and 5 we have $Th = T((h-h_n)+h_n) = T(h-h_n) \leq h - h_n$. Taking limit on both sides we get $Th \leq 0$. That is $Th = 0$. Conversely let $Th = 0$. Since h is non-negative harmonic, $h_n = (h-n)^+$ is subharmonic so that Rh_n is non-negative harmonic in X and $h - Rh_n$ is harmonic in X. Now $Rh_n \geq h_n \geq h - n$ implies $h - Rh_n \leq n$. Thus $\{h_n^{'}\} = h - Rh_n$ is a sequence of non-negative bounded harmonic functions in X such that $\lim_{n\to\infty} h_n^{'} = h$. Hence h is quasi-bounded. □

Corollary 4.2. *If $u \in \Im$ and h is a quasi-bounded harmonic function in X then $T(u+h) = Tu$.*

Proof. Since $u \leq u + h$, we have $Tu \leq T(u+h) \leq Tu + Th = Tu$(by Theorem 4.1). Hence $T(u+h) = Tu$. □

Theorem 4.3. *Let h be a non-negative harmonic function in X. Then h is singular if and only if $Th = h$.*

Proof. Let h be a non-negative singular harmonic function in X. Then as in Theorem 4.1, $h - Rh_n$ is a non-negative bounded harmonic function in X such that $h - Rh_n \leq h$. Since h is singular we have $h - Rh_n = 0$ for any n. Thus $h = \lim_{n\to\infty} Rh_n = Th$. Conversely let $Th = h$ and u be a bounded non-negative harmonic function in X such that $u \leq h$. Then by Properties 4 and 5 we have $Th = T((h-u)+u) = T(h-u) \leq h - u$. Hence $u \leq h - Th = 0$ which implies $u = 0$. □

Theorem 4.4. *Let Ξ be an increasingly ordered family of non-negative singular harmonic functions in X. If $h = \sup\limits_{v\in\Xi} v$ is finite valued, then h is a non-negative singular harmonic function in X.*

Proof. We know that h is a non-negative harmonic function in X. Since $h \geq v$, $Th \geq Tv = v$ and we get $Th \geq h$. But $Th \leq h$ always. Hence $Th = h$, that is h is singular. □

5. Quasi-bounded superharmonic functions

Definition 5.1. (Arsove and Leutwiler [2])Extending the quasi-bounded harmonic and singular harmonic definition to non-negative superharmonic functions, we say that a non-negative superharmonic function v is quasi-bounded superharmonic if $Tv = 0$ and singular superharmonic if $Tv = v$ in X. Thus a singular superharmonic function is a singular harmonic function.

Example 5.2. (1) Any non-negative bounded superharmonic function is quasibounded superharmonic.
(2) Sum of two quasi-bounded superharmonic functions is quasibounded superharmonic.

(3) We know any positive superharmonic function on X is the increasing limit of potentials. For, $E_n = \{x \in X : d(e,x) \leq n\}$ are finite sets such that $E_n \subset E_{n+1}$ and $\cup E_n = X$. If $v > 0$ is a superharmonic function in X and $R_v^{E_n}$, the reduced function of v on E_n, then by (Corollary 3.2.8 [1]) $R_v^{E_n}$ is a potential in X. Note that the sequence $\{R_v^{E_n}\}$ is an increasing sequence of potentials in X converging to v. Also if p is a positive potential in X then Tp is harmonic and $Tp \leq p$ implies $Tp = 0$. Hence p is a quasibounded superharmonic function in X. Thus any superharmonic function $v > 0$ is the increasing limit of quasi-bounded superharmonic functions in X.

Theorem 5.3. *Let $\{u_n\}$ be a sequence of non-negative functions in X and $\{v_n\}$ be a sequence of superharmoic functions in X such that $u_n \leq v_n$ for every $n = 1, 2, 3, ...$ and $\sum_{n=1}^{\infty} v_n(x) \neq +\infty$ for each $x \in X$. Then $T(\sum_{n=1}^{\infty} u_n) \leq \sum_{n=1}^{\infty} Tu_n$.*

Proof. Let $u = \sum_{n=1}^{\infty} u_n$. If $z_0 \in X$ is any arbitrary vertex and ϵ is any positive number, then there exists $\{\lambda_n\}$, a sequence of positive integers such that

$$Ru_{n\lambda_n}(z_0) \leq Tu_n(z_0) + \frac{\epsilon}{2^n} \ (n = 1, 2, 3, ...).$$

where $Ru_{n\lambda_n}$ is the reduced function of $u_{n\lambda_n} = (u_n - \lambda_n)^+$ on X. Also,

$$u_n \leq Ru_{n\lambda_n} + \lambda_n \ (n = 1, 2, 3, ...).$$

For every $(m = 1, 2, 3, ...)$

$$u = \sum_{n=1}^{m} u_n + \sum_{n=m+1}^{\infty} u_n \leq \sum_{n=1}^{m}(Ru_{n\lambda_n} + \lambda_n) + \sum_{n=m+1}^{\infty} v_n.$$

Set $\sigma_m = \sum_{n=1}^{m} \lambda_n \ (m = 1, 2, 3, ...)$. Then

$$\begin{aligned} u &\leq \sum_{n=1}^{m} Ru_{n\lambda_n} + \sigma_m + \sum_{n=m+1}^{\infty} v_n \\ \textit{That is, } (u - \sigma_m)^+ = u_{\sigma_m} &\leq \sum_{n=1}^{m} Ru_{n\lambda_n} + \sum_{n=m+1}^{\infty} v_n \\ Ru_{\sigma_m} &\leq \sum_{n=1}^{m} Ru_{n\lambda_n} + \sum_{n=m+1}^{\infty} v_n. \end{aligned}$$

Allowing $m \to \infty$, we get

$$\begin{aligned} Tu &\leq \sum_{n=1}^{\infty} Ru_{n\lambda_n} \\ \textit{Hence } Tu(z_0) &\leq \sum_{n=1}^{\infty} Ru_{n\lambda_n}(z_0) \\ &\leq \sum_{n=1}^{\infty} Tu_n(z_0) + \sum_{n=1}^{\infty} \frac{\epsilon}{2^n}. \end{aligned}$$

ϵ being arbitrary, we get

$$Tu(z_0) \leq \sum_{n=1}^{\infty} Tu_n(z_0).$$

Since z_0 is arbitrary, we have $T(\sum_{n=1}^{\infty} u_n) \leq \sum_{n=1}^{\infty} Tu_n$. □

Corollary 5.4. *Let $\{u_n\}$ be a sequence of quasi-bounded superharmonic functions in X. If $u = \sum_{n=1}^{\infty} u_n < \infty$, then u is a quasi-bounded superharmonic function in X.*

Proof. By Theorem 5.3, $T(\sum_{n=1}^{\infty} u_n) \leq \sum_{n=1}^{\infty} Tu_n$. Since each u_n is quasi-bounded superharmonic, $Tu_n = 0$ which implies $Tu = 0$. That is, u is a quasi-bounded superharmonic function in X. □

Theorem 5.5. *Any non-negative superharmonic function v can be uniquely written as $v = v_b + v_s$ in X, where v_b is a quasi-bounded superharmonic function and v_s is a singular harmonic function in X.*

Proof. By Riesz representation theorem v can be uniquely written as $v = p + h$ in X, where p is a potential and h is the greatest harmonic minorant of v in X. Now by Corollary 3.3, h can be uniquely written as $h = q + v_s$ where q is a quasi-bounded harmonic function and v_s is a singular harmonic function in X. Hence $v = v_b + v_s$ where $v_b = p + q$ is quasi-bounded superharmonic and v_s is a singular harmonic function in X.

Uniqueness: Let $v = v_b^{'} + v_s^{'}$ be another such decomposition. Since $v_s^{'} \leq v_b + v_s$, $v_s^{'} = Tv_s^{'} \leq T(v_b + v_s) \leq Tv_b + Tv_s = Tv_s = v_s$. Similarly, we can prove $v_s \leq v_s^{'}$. Thus the uniqueness. □

Corollary 5.6. *Let v be a non-negative superharmonic function in X. Then $Tv = v_s$, where v_s is the singular harmonic part of v.*

Proof. By Theorem 5.5 $v = v_b + v_s$, where v_b is a quasi-bounded superharmonic function and v_s is a singular harmonic function in X. $Tv = T(v_b + v_s) \leq Tv_b + Tv_s = 0 + v_s$. Now $v_s \leq v$ implies $Tv_s \leq Tv$; that is, $v_s \leq Tv$. Hence $Tv = v_s$. □

Remark 5.7. (1) Let v_1, v_2 be two non-negative superharmonic functions such that $v_1 \leq v_2$. Then singular harmonic part of $v_1 \leq$ singular harmonic part of v_2.

(2) Let v_1, v_2 be two non-negative superharmonic functions in X. Then $T(v_1 + v_2) = Tv_1 + Tv_2$.

Proof. By Theorem 5.5 v_1, v_2 can be uniquely written as $v_1 = q_1 + s_1$ and $v_2 = q_2 + s_2$, where q_1, q_2 are quasi bounded superharmonic and s_1, s_2 are singular harmonic functions in X. If $v_1 + v_2 = v_3$ then $v_3 = q_3 + s_3$, where q_3 is quasi-bounded superharmonic and s_3 is singular harmonic in X. Since the representation is unique $s_3 = s_1 + s_2$. Now by Corollary 5.6, $T(v_1 + v_2) = Tv_1 + Tv_2$. □

(3) Let $\{v_n\}$ be a sequence of non-negative superharmonic functions in X such that $\sum\limits_{n=1}^{\infty} v_n(x) < \infty$ for some $x \in X$. Then $T(\sum\limits_{n=1}^{\infty} v_n) = \sum\limits_{n=1}^{\infty} Tv_n$.

Proof. If $v(x) = \sum\limits_{n=1}^{\infty} v_n(x)$, then $v(x)$ is superharmonic in X. $\sum\limits_{n=1}^{\infty} v_n = \sum\limits_{n=1}^{\infty} (q_n + s_n)$, where q_n's and s_n's are quasi-bounded superharmonic and singular harmonic parts of v_n's respectively. By Property 3 of singular harmonic functions and by Corollary 5.4, we get $\sum\limits_{n=1}^{\infty} s_n$ is singular harmonic in X and $\sum\limits_{n=1}^{\infty} q_n$ is quasi-bounded superharmonic in X. By Corollary 5.6, $T(v) = \sum\limits_{n=1}^{\infty} s_n = \sum\limits_{n=1}^{\infty} Tv_n$. □

Theorem 5.8. *Let $v = v_b + v_s$ be the unique decomposition of a non-negative superharmonic function v in X, where v_b is a quasi-bounded superharmonic function and v_s is a singular harmonic function in X. Then* $\inf(v_b, v_s)$ *is a potential in X.*

Proof. Let h be a harmonic function in X such that $0 \leq h \leq \inf(v_b, v_s)$. Since v_b is quasi-bounded superharmonic, by Riesz representation theorem $v_b = p + h_1$ where p is a potential and h_1 is the greatest harmonic minorant of v_b. Then $h - h_1 \leq p$ so that $h \leq h_1$. Since $Th_1 = 0, Th = 0$. Hence $h = \sup b_n$, where $\{b_n\}$ is a sequence of non-negative bounded harmonic functions in X. Now $h \leq v_s$ implies $b_n \leq v_s$. Since v_s is singular and b_n is a non-negative bounded harmonic function, we get $b_n = 0$ for every n. Therefore $h = 0$. Hence $\inf(v_b, v_s)$ is a potential. □

Theorem 5.9. *Let v be a non-negative superharmonic function in X. Then Tv is the greatest singular harmonic minorant of v. In fact $Tv = \sup\limits_{h \in \mathscr{S}} h$, where $\mathscr{S}$ is the family of all singular harmonic functions h in X such that $h \leq v$.*

Proof. $\mathscr{S}$ is increasingly ordered. For, $\sup(h_1, h_2)$ is a non-negative subharmonic function majorized by v. Hence there exists a harmonic function $u \geq 0$ such that $\sup(h_1, h_2) \leq u \leq v$. Since $h_1 \leq u$, $h_1 = Th_1 \leq Tu$; similarly $h_2 \leq Tu$. If $h = Tu$ then by Corollary 5.6, h is a singular harmonic function such that $\sup(h_1, h_2) \leq h \leq u \leq v$. Hence $h \in \mathscr{S}$. Since X has a countable number of vertices, we can extract an increasing sequence $\{u_n\} \in \mathscr{S}$ such that $\sup\limits_{h \in \mathscr{S}} h(x) = \lim_n u_n(x)$

for each vertex $x \in X$. Now $H(x) = \lim_n u_n(x)$ is a harmonic function such that $u_n(x) \leq H(x)$ for each n so that $u_n(x) = Tu_n(x) \leq TH(x)$; when $n \to \infty$, $\lim_n u_n(x) \leq TH(x)$, that is $H(x) \leq TH(x)$. But $TH(x) \leq H(x)$ always so that $H(x) = TH(x)$ for all $x \in X$. That is $H(x)$ is singular harmonic. $H(x) \leq v(x)$ implies $H = TH \leq Tv$ on X. But $Tv \in \mathscr{S}$ so that $H \geq Tv$. Consequently $H = Tv$. That is, $Tv(x) = \sup_{h \in \mathscr{S}} h(x)$ for each $x \in X$. □

Corollary 5.10. *If $u \in \Im$, the class of all non-negative real-valued functions u on X admitting superharmonic majorants, then $T^2u = Tu$. In particular if $v > 0$ is a superharmonic function in X, then $T^2v = v$.*

Proof. If $u \in \Im$, then Ru is superharmonic, TRu is singular harmonic by Theorem 5.9 and by the Property 7 of T, $Tu = TRu$. Therefore Tu is singular harmonic in X. Hence $T^2u = Tu$. □

Corollary 5.11. *A non-negative superharmonic function v is quasi-bounded if and only if 0 is the only singular harmonic minorant of v.*

6. Quasi-bounded harmonic and singular harmonic functions in subordinate structures.

In an infinite network X, we have defined that a function u is superharmonic in X if and only if $t(x)u(x) \geq \sum_{y\sim x} t(x,y)u(y)$ at every vertex x in X. Let us say that $\{t(x,y)\}$ defines a Δ-structure in X. Define $t' \geq 0$ a real valued function on $X \times X$ such that:

(1) $0 \leq t'(x,y) \leq t(x,y)$ for any pair x and y in X.
(2) $t'(x,y) > 0$ if and only if $x \sim y$.
(3) $t'(x,y) < t(x,y)$ for atleast one pair x and y.

Then this system Δ' of $t'(x,y)$ is said to be subordinate to Δ. Write $\Delta' u(x) = -u(x) + \sum_{y\sim x} \frac{t'(x,y)}{t(x)} u(y)$.

Example 6.1. The harmonic structure defined by the Schrödinger equation $\Delta_q u(x) = \Delta u(x) - q(x)u(x)$ where $q(x) \geq 0$, but $q \not\equiv 0$ in X is subordinate to the harmonic structure defined by the Laplace equation $\Delta u(x) = 0$ in X. For if we take $t'(x,y) = t(x)\frac{t(x,y)}{t(x)+q(x)}$, where $q(x) \geq 0$ and $q(z) > 0$ for at least one vertex $z \in X$, then $\Delta' u(x) = -u(x) + \sum_{y\sim x} \frac{t(x,y)}{t(x)+q(x)} u(y)$ which has the same sign as $\Delta_q u(x)$. A function $u(x)$ defined on X is said to be $q-$superharmonic(q-harmonic) in X if and only if $\Delta_q u(x) \leq 0 (\Delta_q u(x) = 0)$ for every vextex $x \in X$.

A function $u(x)$ defined on X is said to be $\Delta'-$superharmonic(Δ'-harmonic) in X if and only if $\Delta' u(x) \leq 0 (\Delta' u(x) = 0)$ for every vextex $x \in X$. If a function $u(x)$ is $\Delta-$superharmonic in X, then $t(x)u(x) \geq \sum_{y\sim x} t(x,y)u(y) \geq \sum_{y\sim x} t'(x,y)u(y)$, implies $u(x)$ is $\Delta'-$superharmonic in X. A positive $\Delta'-$superharmonic function v is a $\Delta'-$potential if its greatest $\Delta'-$harmonic minorant is zero. The constant function 1 is $\Delta'-$superharmonic but not $\Delta'-$harmonic in X. Hence there always exists a $\Delta'-$potential in X.

Now as in Section 4 define a map T' on the family of non-negative functions having $\Delta'-$superharmonic majorants on X as $T'u = \lim_{n\to\infty} R'u_n$, where u is a non-negative function having a $\Delta'-$superharmonic majorant on X and $R'u_n$ is infimum of all $\Delta'-$superharmonic functions dominating $u_n = (u-n)^+$ on X. Then all the results in Sections 4 and 5 are true in this subordinate structure also. Hence we say that a non-negative $\Delta'-$superharmonic function u is $\Delta'-$quasi-bounded superharmonic if and only if $T'u = 0$ in X and u is $\Delta'-$singular superharmonic if and only if $T'u = u$ in X. Any non-negative $\Delta'-$singular superharmonic function is $\Delta'-$singular harmonic in X. Constants are $\Delta'-$quasi-bounded superharmonic functions in X.

Theorem 6.2. *Every $\Delta-$quasi-bounded superharmonic function is a $\Delta'-$quasi-bounded superharmonic function in X.*

Proof. Every non-negative $\Delta-$superharmonic function is $\Delta'-$superharmonic in X. Since for any $u \in \Im$, $R'u_n \leq Ru_n$, we get $T'u \leq Tu$. Hence if u is a $\Delta-$quasi-bounded superharmonic function in X, by Definition 5.1, $Tu = 0$ and hence $T'u = 0$. Thus u is $\Delta'-$quasi-bounded superharmonic in X. $\square$

Existence of $\Delta^{'}-$singular harmonic functions:

Since there are $\Delta^{'}-$superharmonic functions in X that are not harmonic, we can always construct positive $\Delta^{'}-$harmonic functions on X. But there may not be any positive unbounded $\Delta^{'}-$harmonic functions in X in which case there can be no non-trivial singular $\Delta^{'}-$harmonic functions in X. We give a few sufficient conditions for the existence of non-trivial singular $\Delta^{'}-$harmonic functions in X.

An infinite network X is said to be $\Delta^{'}-$parahyperbolic [5] if and only if 1 is a $\Delta'-$potential in X. Consequently, an infinite network X is $\Delta^{'}-$parahyperbolic if and only if the only bounded $\Delta'-$harmonic function in X is zero.

Remark 6.3.
(1) In a $\Delta^{'}-$parahyperbolic network the only $\Delta'-$quasi-bounded harmonic function is zero. Hence in a $\Delta^{'}-$parahyperbolic network any non-negative $\Delta'-$harmonic function is a $\Delta'-$singular harmonic function.
(2) Any non-trivial $\Delta^{'}-$ singular harmonic function in X is an unbounded $\Delta^{'}-$ non-negative harmonic function.
(3) If there is no non-negative $\Delta^{'}-$bounded harmonic function in X, then any non-negative $\Delta^{'}-$ harmonic function is a non-trivial $\Delta^{'}-$ singular harmonic function in X.
(4) Consider the minimum principle given in Theorem 3.5, [5]. Suppose this minimum principle is valid in X; then there exists a $\Delta^{'}-$ singular harmonic function in X.
(5) If the $\Delta^{'}-$ harmonic measure of the point at infinity (see [5]) is zero, then by Theorem 3.6 [5] X is a $\Delta^{'}-$parahyperbolic network and hence there exists a $\Delta^{'}-$singular harmonic function in X.

(6) If there exists a $\Delta^{'}-$ potential p in X such that for some constant α, $p \geq \alpha$ outside a finite set, then there exists a $\Delta^{'}-$singular harmonic function in X.
For, if $p \geq \alpha$ outside a finite set A, then for some constant $\beta > 0, p \geq \beta$ on X. This implies β is a $\Delta^{'}-$potential and hence 1 is a $\Delta^{'}-$potential in X. Therefore a $\Delta^{'}-$ singular harmonic function exists in X.

Example 6.4. Example of a network X in which any positive harmonic function is unbounded. Consider the infinite ray $X = \{0, 1, 2, 3, ...\}$ with transition indices $t(n, n+1) = t(n+1, n) = \frac{1}{2}$, where $n = 0, 1, 2, ...$ Let $q(n)$ be a real valued function defined on X such that $q(0) = 1$ and $q(n) = 0$ if $n \geq 1$. In this infinite network the only positive $\Delta^{'}-$ harmonic function is proportional to $h(n) = 2n + 1$.

Proposition 6.5 ([6])**.** *Let $\{b_i\}_{1\leq i\leq m}$ be a collection of bounded positive $\Delta^{'}-$ harmonic functions in X such that any positive $\Delta^{'}-$ harmonic function is of the form $b = \sum_{i=1}^{m} \alpha_i b_i$ for some $\alpha_i \geq 0$. If there is any unbounded non-negative $\Delta^{'}-$ harmonic function h in X, then there are $\Delta^{'}-$ singular harmonic functions in X.*

Proof. Let $h = q + s$ where q is a $\Delta^{'}-$quasi-bounded harmonic function and s is a $\Delta^{'}-$ singular harmonic function in X. Now $q = \sup_n u_n$ where $\{u_n\}$ is a sequence of non-negative harmonic functions on X. Now for any $n \geq 1$, $u_n = \sum_{i=1}^{m} \alpha_{in} b_i$. Suppose $\overline{\lim_n} \alpha_{in} = \infty$ for some i, then $q \geq u_n \geq \alpha_{in} b_i$ would imply that $q = \infty$. Hence for each i, there is a constant $\lambda_i > 0$ such that $\alpha_{in} \leq \lambda_i$. Hence $q \leq \sum_{i=1}^{m} \lambda_i b_i$ is bounded. Consequently, the unbounded $\Delta^{'}-$ harmonic function $h = q + s, s \not\equiv 0$. That is, a positive $\Delta^{'}-$ singular harmonic function exists in X. □

References

[1] V. Anandam, *Harmonic Functions and Potentials on Finite or Infinite Networks*, UMI Lecture Notes, Springer, 2011.
[2] M. Arsove and H. Leutwiler, *Quasi-bounded and singular functions*, Trans. Amer. Math. Soc. **189** (1974), 276–302.
[3] P.Cartier, *Fonctions harmoniques sur un arbre*, Sympos. Math. **9** (1972), 203–270.
[4] M. Parreau, *Sur les moyennes des fonctions harmoniques et analytiques et la classification des surfaces de Riemann,* Ann. Inst. Fourier **3** (1951), 103–197.
[5] S. Sivan and M. Venkataraman, *Parahyperbolic Networks,* Mem. Fac. Sci. Eng. Shimane University **44** (2011), 1–16.
[6] Wellington H. Ow, *Quasi-bounded and singular solutions of $\Delta u = pu$ on open Riemann surfaces,* Journal d'Analyse Mathématique **28** (1975), 262–272.

N. Narayanaraju: Ramanujan Institute for advanced study in mathematics, University of Madras, Chennai-600 005, India
E-mail address: nadhiyan@gmail.com

P. Kumaresan: Ramanujan Institute for advanced study in mathematics, University of Madras, Chennai-600 005, India
E-mail address: premalathakumaresan@gmail.com