THE DIRICHLET PROBLEM FOR $\infty$-HARMONIC FUNCTIONS ON A NETWORK

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Abstract. We define the notion of $\infty$-harmonic functions on a network as a discrete version of that on a euclidean domain, and show some properties of such functions. We discuss the Dirichlet problems for discrete $\infty$-harmonic functions. We also show that limits of discrete $p$-harmonic functions as $p \to \infty$ are in fact discrete $\infty$-harmonic.

1. INTRODUCTION

An $\infty$-harmonic function in a euclidean domain $D \subset \mathbb{R}^d$ ($d \geq 2$) is defined to be a viscosity solution of the equation

$$
\Delta_\infty u := \frac{1}{2} \nabla u \cdot \nabla |\nabla u|^2 = 0
$$

in $D$ (see [1, 2, 3]). For $1 < p < \infty$, a $p$-harmonic function in $D$ is a continuous weak solution to the $p$-Laplace equation

$$
\Delta_p u := \text{div}(|\nabla u|^{p-2} \nabla u) = 0
$$

in $D$. If $u_n$ is $p_n$-harmonic in $D$ with $p_n \to \infty$ and $u_n \to u$, then $u$ is $\infty$-harmonic in $D$ (see [1]). This fact shows that (1) is the limiting equation of (2) as $p \to \infty$, and explains the terminology $\infty$-harmonic.

The purpose of this paper is to define the notion of $\infty$-harmonic functions on a network as a discrete version of that on a euclidean domain and obtain some properties related to such functions. A discrete analogue of the $p$-Laplacian $\Delta_p$ can be readily defined on a network (see, e.g., [6, 7, 5]). However, there seems to be no appropriate discrete version of the $\infty$-Laplacian. One may define $\infty$-harmonic functions on a network as limits of $p$-harmonic functions as $p \to \infty$; but this definition is somewhat indirect and not so appropriate to handle with to obtain local properties.

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An \( \infty \)-harmonic function in a euclidean domain \( D \) also arises in the Lipschitz extension problem; \( u \in W^{1,\infty}(D) \) is called an absolutely minimal Lipschitz extension in \( D \) if
\[
\| \nabla u \|_{L^\infty(V)} \leq \| \nabla v \|_{L^\infty(V)}
\]
for any domain \( V \subset D \) and any \( v \) with \( u - v \in W^{1,\infty}_0(V) \). It is known that an absolutely minimal Lipschitz extension is \( \infty \)-harmonic (see [2]).

This suggests our definition of a discrete \( \infty \)-harmonicity on a network. We define the \( \infty \)-harmonicity of a function on vertices by means of its \( \infty \)-mean value around a vertex as in [4]. By using discrete derivative of a function, we obtain a useful criterion as Theorem 3.1 for \( \infty \)-harmonicity. Most of properties of classical discrete harmonic functions hold. We discuss the Dirichlet problem for \( \infty \)-harmonic functions on a network. We shall introduce in Section 4 an ideal boundary of a network. Roughly speaking, this ideal boundary is the set of infinite paths. Given a function on the ideal boundary, we shall show in Theorem 5.7 the existence of \( \infty \)-harmonic functions satisfying the boundary condition. As in the classical theory, a set of \( \infty \)-superharmonic functions and a set of \( \infty \)-subharmonic functions give the upper solution and the lower solution of our Dirichlet problem. It is shown that these solutions take the given boundary value if the boundary value is a bounded Lipschitz function. We show in Theorem 5.9 that the solutions to the Dirichlet problem give optimal solutions to an \( \infty \)-variational problem. We show a boundary maximum principle for the sum of two \( \infty \)-subharmonic functions in Lemma 5.11. With the aid of this result, we show in Theorem 5.12 that the solution to the Dirichlet problem is unique. Finally we show in Section 6 that the limit of \( p \)-harmonic functions as \( p \to \infty \) is \( \infty \)-harmonic.

2. Preliminaries

Let \((V, E)\) be a locally finite and connected infinite graph without self-loops, where \( V \) is the set of vertices and \( E \) is the set of edges. This means that \( V \) is a countable set and that an element of \( E \) is an ordered pair \((x, y)\) of vertices \( x, y \in V \). We assume that \((y, x) \notin E\) for \((x, y) \in E\). Let
\[
\partial x = \{ y \in V; (x, y) \in E \}, \quad N x = \partial x \cup \{ x \}, \quad D = D \cup \{ x \in V; (x, y) \in E \text{ for some } y \in D \}.
\]

From our assumptions
\begin{enumerate}
  \item \((x, x) \notin E\) for \( x \in V \);
  \item \(\partial x\) is a finite set for each \( x \in V \);
  \item for each \( x, y \in V \), there is a sequence \( \{x_i\}_{i=0}^l \) of distinct vertices such that \( x = x_0, y = x_l \) and \((x_{j-1}, x_j) \in E\) for \( j = 1, 2, \ldots, l \).
\end{enumerate}

A sequence in (3) is called a path from \( x \) to \( y \).

A resistance \( r \) is a positive function on \( E \). We assume that \( r(y, x) = r(x, y) \) for each edge \((x, y) \in E\). A network is a triplet \((V, E, r)\), where \((V, E)\) is a graph and \( r \) is a resistance.
Let $L(D)$ be the set of real valued functions in a subset $D \subset V$. For $u \in L(D)$ and $(x,y) \in E$ with $x,y \in D$ we define the discrete derivative $\nabla u$ at $(x,y)$ as

$$\nabla u(x, y) = \frac{u(y) - u(x)}{r(x, y)}.$$

We define the $\infty$-Dirichlet seminorm $D_\infty[u]$ of $u \in L(V)$ by

$$D_\infty[u] = \sup_{(x,y) \in E} |\nabla u(x, y)|.$$

Let $D^{(\infty)}$ be the set of functions in $V$ with finite $\infty$-Dirichlet seminorms.

### 3. Local $\infty$-variational problem

For $x \in V$ and a function $u \in L(Nx)$ let

$$M_u(x) = \max_{y \in \partial x} |\nabla u(x, y)|,$$

$$\mu_\infty^{x,u}(t) = \max_{y \in \partial x} \frac{|u(y) - t|}{r(x, y)}$$

for $t \in \mathbb{R}$. Note that $M_u(x) = \mu_\infty^{x,u}(u(x))$. Since $\mu_\infty^{x,u}$ is a convex function such that $\lim_{t \to \pm \infty} \mu_\infty^{x,u}(t) = \infty$ and that it is not constant on any open interval, it follows that there exists a unique $\infty$-mean value $H_\infty^x u$ such that $\mu_\infty^{x,u}(t) \geq \mu_\infty^{x,u}(H_\infty^x u)$ for any $t \in \mathbb{R}$.

Let $x \in V$ and $u \in L(Nx)$. If $u$ satisfies $u(x) \leq H_\infty^x u$ ($u(x) \geq H_\infty^x u$, $u(x) = H_\infty^x u$, resp.), then $u$ is said to be $\infty$-subharmonic ($\infty$-superharmonic, $\infty$-harmonic, resp.) at $x$. Let $D \subset V$ and $u \in L(\partial D)$. If $u$ is $\infty$-subharmonic ($\infty$-superharmonic, $\infty$-harmonic, resp.) at each $x \in D$, then $u$ is said to be $\infty$-subharmonic ($\infty$-superharmonic, $\infty$-harmonic, resp.) in $D$. Note that $u$ is $\infty$-superharmonic if and only if $-u$ is $\infty$-subharmonic.

We repeatedly use the next theorem, which characterizes $\infty$-superharmonic functions and $\infty$-subharmonic functions.

**Theorem 3.1.** Let $x \in V$ and $u$ a function on $Nx$.

1. $u$ is $\infty$-superharmonic at $x$ if and only if there is a vertex $y \in \partial x$ such that $\nabla u(x, y) = -M_u(x)$.
2. $u$ is $\infty$-subharmonic at $x$ if and only if there is a vertex $y \in \partial x$ such that $\nabla u(x, y) = M_u(x)$.

**Proof.** Let $t_0 = H_\infty^x u$. Note that there is $y \in \partial x$ such that either $\nabla u(x, y) = M_u(x)$ or $\nabla u(x, y) = -M_u(x)$.

Case 1: $u(x) < t_0$. Using $M_u(x) = \mu_\infty^{x,u}(u(x)) \geq \mu_\infty^{x,u}(t_0)$ we have

$$\nabla u(x, z) = \frac{u(z) - u(x)}{r(x, z)} > \frac{u(z) - t_0}{r(x, z)} \geq -\mu_\infty^{x,u}(t_0) \geq -M_u(x).$$

for $z \in \partial x$. This means that $\nabla u(x, z) \neq -M_u(x)$ for $z \in \partial x$, and that there is $y_1 \in \partial x$ such that $\nabla u(x, y_1) = M_u(x)$.

Case 2: $u(x) > t_0$. It follows from an argument similar to Case 1 that $\nabla u(x, z) \neq M_u(x)$ for each $z \in \partial x$ and that there is $y_2 \in \partial x$ such that $\nabla u(x, y_2) = -M_u(x)$. 

Case 3: $u(x) = t_0$. We show that there is $y_1 \in \partial x$ such that $\nabla u(x, y_1) = M_u(x)$. On the contrary we assume that $\nabla u(x, z) < M_u(x)$ for each $z \in \partial x$. Then

$$\frac{u(z) - t_0}{r(x, z)} = \frac{u(z) - u(x)}{r(x, z)} < M_u(x).$$

There is $\varepsilon > 0$ such that $r(x, z)^{-1}(u(z) - t_0 + \varepsilon) < M_u(x)$ for any $z \in \partial x$. Since

$$M_u(x) > \frac{u(z) - (t_0 - \varepsilon)}{r(x, z)} = \frac{u(z) - u(x)}{r(x, z)} + \varepsilon \frac{r(x, z)}{r(x, z)} \geq -M_u(x) + \varepsilon \frac{r(x, z)}{r(x, z)} > -M_u(x),$$

it follows that $\mu_{x,u}^\infty(t_0 - \varepsilon) < M_u(x) = \mu_{x,u}^\infty(u(x)) = \mu_{x,u}^\infty(t_0)$, which contradicts the definition of $t_0$. This means that there is $y_1 \in \partial x$ such that $\nabla u(x, y_1) = M_u(x)$. Similarly there is $y_2 \in \partial x$ such that $\nabla u(x, y_2) = -M_u(x)$.

Now suppose that $u$ is $\infty$-superharmonic at $x$. Then either Case 2 or Case 3 holds. There is $y_2 \in \partial x$ such that $\nabla u(x, y_2) = -M_u(x)$. Conversely, we assume that $\nabla u(x, y_2) = -M_u(x)$ for some $y_2 \in \partial x$. Then Case 1 cannot hold, so that $u(x) \geq t_0$. This means that $u$ is $\infty$-superharmonic at $x$. Therefore (1) holds. We can similarly prove (2).

Next proposition implies the Harnack inequality.

**Proposition 3.2.** Let $x \in V$ and let $u$ be a function on $N x$. Let $c_x = \max_{y, z \in \partial x} r(x, y)/r(x, z)$.

1. If $u$ is $\infty$-superharmonic at $x$ and $u \geq 0$ on $N x$, then $u(y) \leq (1 + c_x)u(x)$ for $y \in \partial x$.

2. If $u$ is $\infty$-subharmonic at $x$ and $u \leq 0$ on $N x$, then $u(y) \geq (1 + c_x)u(x)$ for $y \in \partial x$.

**Proof.** We shall prove (1) only. Theorem 3.1 shows that there is $z \in \partial x$ such that $\nabla u(x, z) = -M_u(x)$. Since $\nabla u(x, y) \leq M_u(x)$ for $y \in \partial x$ and $u(z) \geq 0$, it follows that

$$\frac{u(y) - u(x)}{r(x, y)} = \nabla u(x, y) \leq -\nabla u(x, z) = \frac{u(x) - u(z)}{r(x, z)} \leq \frac{u(x)}{r(x, z)}.$$ 

This implies that $u(y) \leq (1 + r(x, y)/r(x, z))u(x)$ and the assertion. □

**Lemma 3.3.** Let $x \in V$. Let $u$ and $v$ be functions on $\partial x$ with $u \leq v$. Then $H_x^\infty u \leq H_x^\infty v$.

**Proof.** On the contrary we assume that $H_x^\infty u > H_x^\infty v$. Then $u(y) - H_x^\infty u < v(y) - H_x^\infty v$ for each $y \in \partial x$. Let $\tilde{v}$ be the function with $\tilde{v}(x) = H_x^\infty v$ and $\tilde{v} = v$ on $\partial x$. Let $\tilde{u}$ be the function with $\tilde{u}(x) = H_x^\infty u$ and $\tilde{u} = u$ on $\partial x$. Since $\tilde{u}$ is $\infty$-harmonic at $x$, Theorem 3.1 implies that there is $y_1 \in \partial x$ such that

$$M_{\tilde{u}}(x) = \nabla \tilde{u}(x, y_1) = \frac{\tilde{u}(y_1) - \tilde{u}(x)}{r(x, y_1)} = \frac{u(y_1) - H_x^\infty u}{r(x, y_1)} < \frac{v(y_1) - H_x^\infty v}{r(x, y_1)}.$$

Then

$$\nabla \tilde{v}(x, y_1) = \frac{\tilde{v}(y_1) - \tilde{v}(x)}{r(x, y_1)} = \nabla \tilde{v}(x, y_1) \leq M_{\tilde{v}}(x).$$
Similarly, using \( y_2 \in \partial x \) with \( \nabla \hat{v}(x, y) = -M_v(x) \), we obtain \( M_{\hat{v}}(x) < M_u(x) \), and a contradiction.

Lemma 3.4. Let \( D \subset V \). Let \( u \) be a function in \( \overline{D} \) and \( x \in D \). Let

\[
\hat{u}(y) = \begin{cases} 
H_{x}^{\infty} u & \text{if } y = x; \\
u(y) & \text{if } y \neq x.
\end{cases}
\]

(1) If \( u \) is an \( \infty \)-superharmonic function in \( D \), then \( \hat{u} \) is an \( \infty \)-superharmonic function in \( D \) such that \( \hat{u} \) is \( \infty \)-harmonic at \( x \) and that \( \hat{u} \leq u \).

(2) If \( u \) is an \( \infty \)-subharmonic function in \( D \), then \( \hat{u} \) is an \( \infty \)-subharmonic function in \( D \) such that \( \hat{u} \) is \( \infty \)-harmonic at \( x \) and that \( \hat{u} \geq u \).

Proof. We shall prove (1) only. It is obvious that \( \hat{u} \) is \( \infty \)-harmonic at \( x \). Since \( \hat{u}(x) = H_{x}^{\infty} u \leq u(x) \) and \( \hat{u}(z) = u(z) \) for \( z \neq x \), it follows that \( \hat{u} \leq u \). Lemma 3.3 shows that \( \hat{u}(z) = u(z) \geq H_{z}^{\infty} u \geq H_{z}^{\infty} \hat{u} \) for \( z \neq x \). This means that \( \hat{u} \) is \( \infty \)-superharmonic at \( z \).

Lemma 3.5. Let \( D \subset V \) and \( \{u_\lambda\}_{\lambda \in \Lambda} \) a family of functions in \( \overline{D} \).

(1) Suppose that \( u_\lambda \) is \( \infty \)-superharmonic in \( D \) for each \( \lambda \in \Lambda \) and that \( u := \inf_{\lambda \in \Lambda} u_\lambda \) is finite for each vertex in \( \overline{D} \). Then \( u \) is \( \infty \)-superharmonic in \( D \).

(2) Suppose that \( u_\lambda \) is \( \infty \)-subharmonic in \( D \) for each \( \lambda \in \Lambda \) and that \( u := \sup_{\lambda \in \Lambda} u_\lambda \) is finite for each vertex in \( \overline{D} \). Then \( u \) is \( \infty \)-subharmonic in \( D \).

Proof. We shall prove (1) only. Let \( x \in D \). Lemma 3.3 shows that \( H_{x}^{\infty} u \leq H_{x}^{\infty} u_\lambda \leq u_\lambda(x) \) for \( \lambda \in \Lambda \). Hence \( H_{x}^{\infty} u \leq u(x) \). This means that \( u \) is \( \infty \)-superharmonic at \( x \).

Lemma 3.6. Let \( \{u_\lambda\}_{\lambda \in \Lambda} \) be a family of functions in \( V \).

(1) Suppose that \( u := \inf_{\lambda \in \Lambda} u_\lambda \) is finite for each vertex in \( V \). Then \( D_\infty[u] \leq \sup_{\lambda \in \Lambda} D_\infty[u_\lambda] \).

(2) Suppose that \( u := \sup_{\lambda \in \Lambda} u_\lambda \) is finite for each vertex in \( V \). Then \( D_\infty[u] \leq \sup_{\lambda \in \Lambda} D_\infty[u_\lambda] \).

We remark that \( \sup_{\lambda \in \Lambda} D_\infty[u_\lambda] \leq \infty \).

Proof. We shall prove (1) only. Let \( (x, y) \in E \). We may assume \( u(y) \geq u(x) \). For \( \varepsilon > 0 \) there is \( \lambda \in \Lambda \) such that \( u_\lambda(x) \leq u(x) + \varepsilon \). Since \( u_\lambda(y) \geq u(y) \), it follows that

\[
0 \leq \nabla u(x, y) = \frac{u(y) - u(x)}{r(x, y)} \leq \frac{u_\lambda(y) - u_\lambda(x) + \varepsilon}{r(x, y)}.
\]

\[
= \nabla u_\lambda(x, y) + \frac{\varepsilon}{r(x, y)} \leq \sup_{\lambda \in \Lambda} D_\infty[u_\lambda] + \frac{\varepsilon}{r(x, y)}.
\]

Letting \( \varepsilon \to 0 \) we have that \( |\nabla u(x, y)| \leq \sup_{\lambda \in \Lambda} D_\infty[u_\lambda] \), so that \( D_\infty[u] \leq \sup_{\lambda \in \Lambda} D_\infty[u_\lambda] \).</p>
4. The ideal boundary of a network

For \(x, y \in V\) let \(R(x, y)\) be the geodesic distance between \(x\) and \(y\), i.e.,
\[
R(x, y) = \inf \{ \sum r(z_{i-1}, z_i); \{z_i\}_i \text{ is a path from } x \text{ to } y \} \quad \text{if } x \neq y,
\]
\[
R(x, x) = 0.
\]
Then \(R\) is a metric in \(V\). An infinite path is an infinite sequence \(\{x_i\}_{i=0}^{\infty}\) of distinct vertices such that \((x_{i-1}, x_i) \in E\) for \(i = 1, 2, \ldots\). Let \(P\) be the set of all infinite paths and let
\[
P_0 = \{ \{z_i\}_i \in P; \sum_{i=1}^{\infty} r(z_{i-1}, z_i) < \infty \}.
\]
For \(x \in V\) and for two infinite paths \(x = \{x_i\}_i, y = \{y_j\}_j \in P_0\) we let
\[
R(x, y) = R(y, x) = \lim_{n \to \infty} R(x, y_n), \quad R(x, y) = \lim_{n \to \infty} R(x_n, y_n).
\]
It is obvious that the right-hand side of each exists and that \(R\) satisfies the triangle inequality in \(V \cup P_0\). However it is not a metric in general; it happens that \(R(x, y) = 0\) for distinct \(x, y \in P_0\). We identify \(x, y \in P_0\) whenever \(R(x, y) = 0\).

We let \([x]\) be the equivalence class containing \(x \in P_0\) and let \(\Xi\) be the set of equivalence classes:
\[
[x] = \{ y \in P_0; R(x, y) = 0 \}, \quad \Xi = \{ [x]; x \in P_0 \}.
\]
For \(x, y \in V\) and \(\xi, \eta \in \Xi\) we let
\[
\rho(x, y) = R(x, y), \quad \rho(x, \eta) = \rho(\eta, x) = R(x, y), \quad \rho(\xi, \eta) = R(x, y),
\]
where \(x \in \xi\) and \(y \in \eta\). It is easy to see that \(\rho\) is well-defined. Also we have that, for \(\{x_m\}_m \in \xi\) and \(\{y_n\}_n \in \eta\),
\[
\rho(x, \eta) = \rho(\eta, x) = \lim_{n \to \infty} \rho(x, y_n), \quad \rho(\xi, \eta) = \lim_{m \to \infty} \rho(x_m, y_n),
\]
and that \(\rho\) is a metric in \(V \cup \Xi\). We call \(\Xi\) the ideal boundary of the network \((V, E, r)\).

**Lemma 4.1.** Let \(u \in D^{(\infty)}\) and \(\xi \in \Xi\). Then there exists a finite limit
\[
\lim_{n \to \infty} u(x_n) \text{ for } \{x_j\}_j \in \xi, \text{ which is independent of the choice of the representative.}
\]

**Proof.** Let \(\{x_j\}_j \in \xi\). It is easy to see that \(|u(x_m) - u(x_n)| \leq D_{\infty}[u] \rho(x_m, x_n)\), and that \(\{u(x_n)\}_n\) is a Cauchy sequence. There is a finite limit \(\lim_{n \to \infty} u(x_n)\).

Let \(x^{(i)}_n = \{x^{(i)}_n\}_n \in \xi\) for \(i = 1, 2\). Then \(|u(x^{(1)}_m) - u(x^{(2)}_m)| \leq D_{\infty}[u] \rho(x^{(1)}_m, x^{(2)}_m)\), and the right-hand side tends to 0 as \(m, n \to \infty\). Therefore \(\lim_{m \to \infty} u(x^{(1)}_m) = \lim_{n \to \infty} u(x^{(2)}_n)\). \(\Box\)

We simply write \(u(\xi) = \lim_{n \to \infty} u(x_n)\) for \(u \in D^{(\infty)}\) and \(\{x_j\}_j \in \xi \in \Xi\).

**Proposition 4.2.** Let \(u \in D^{(\infty)}\). Then \(|u(\xi) - u(\eta)| \leq D_{\infty}[u] \rho(\xi, \eta)\) for \(\xi, \eta \in \Xi\).
Proof. Let \( \{x_n\}_n \in \xi \) and \( \{y_m\}_m \in \eta \). Then \(|u(x_n) - u(y_m)| \leq D_{\infty}[u] \rho(x_n, y_m)\). It follows that \(|u(\xi) - u(\eta)| \leq D_{\infty}[u] \rho(\xi, \eta)\). \(\blacksquare\)

Next theorem implies the maximal principle.

**Theorem 4.3.** Let \( u \) be a function in \( V \) and \( x_0 \in V \) with \( M_u(x_0) > 0 \).

1. If \( u \) is \( \infty \)-superharmonic, then there is an infinite path \( x = \{x_i\}_{i=0}^{\infty} \in P \) such that

\[
u(x_n) \leq u(x_0) - M_u(x_0) \sum_{i=1}^{n} r(x_{i-1}, x_i) \quad \text{for each} \ n.
\]

Moreover, if \( u \) is bounded from below, then \( x \in P_0 \).

2. If \( u \) is \( \infty \)-subharmonic, then there is an infinite path \( x = \{x_i\}_{i=0}^{\infty} \in P \) such that

\[
u(x_n) \geq u(x_0) + M_u(x_0) \sum_{i=1}^{n} r(x_{i-1}, x_i) \quad \text{for each} \ n.
\]

Moreover, if \( u \) is bounded from above, then \( x \in P_0 \).

Proof. We shall prove (2) only. Theorem 3.1 shows that there is \( x_1 \in \partial x_0 \) such that \( \nabla u(x_0, x_1) = M_u(x_0) \). Note that \( u(x_1) = u(x_0) + \nabla u(x_0, x_1)r(x_0, x_1) > u(x_0) \). Again Theorem 3.1 shows that there is \( x_2 \in \partial x_1 \) such that

\[
\nabla u(x_1, x_2) = M_u(x_1) \geq \nabla u(x_0, x_1) = M_u(x_0).
\]

Note that \( u(x_2) = u(x_1) + \nabla u(x_1, x_2)r(x_1, x_2) > u(x_1) \), and that \( x_2 \neq x_0, x_1 \). Repeating this argument we obtain an infinite path \( x = \{x_i\}_i \) such that \( \nabla u(x_{i-1}, x_i) \geq M_u(x_0) \) and \( u(x_i) > u(x_{i-1}) \) for each \( i \). Therefore

\[
\nu(x_n) - u(x_0) = \sum_{i=1}^{n} \nabla u(x_{i-1}, x_i)r(x_{i-1}, x_i) \geq M_u(x_0) \sum_{i=1}^{n} r(x_{i-1}, x_i)
\]

for each \( n \).

If \( u \) is bounded from above, then \( \sum_{i=1}^{\infty} r(x_{i-1}, x_i) < \infty \), so that \( x \in P_0 \). \(\blacksquare\)

**Lemma 4.4.** Let \( u \in D^{(\infty)} \) and \( x_0 \in V \) with \( M_u(x_0) > 0 \).

1. If \( u \) is bounded from below and \( \infty \)-superharmonic, then there is \( \xi \in \Xi \) such that

\[
u(\xi) \leq u(x_0) - M_u(x_0) \rho(x_0, \xi).
\]

2. If \( u \) is bounded from above and \( \infty \)-subharmonic, then there is \( \xi \in \Xi \) such that

\[
u(\xi) \geq u(x_0) + M_u(x_0) \rho(x_0, \xi).
\]

Proof. We shall prove (2) only. Theorem 4.3 shows that there is \( \{x_n\}_n \in P_0 \) such that

\[
u(x_n) \geq u(x_0) + M_u(x_0) \sum_{i=1}^{n} r(x_{i-1}, x_i) \geq u(x_0) + M_u(x_0) \rho(x_0, x_n).
\]
Letting $\xi = \{x_n\}_n \in \Xi$ and tending $n \to \infty$ we have the assertion. \hfill \Box

**Corollary 4.5.** Let $u \in D^{(\infty)}$.

1. If $u$ is bounded from below and $\infty$-superharmonic, then $\inf_{V \cup \Xi} u = \inf_{\Xi} u$.
2. If $u$ is bounded from above and $\infty$-subharmonic, then $\sup_{V \cup \Xi} u = \sup_{\Xi} u$.

**Proof.** We shall prove (1) only. If $u$ is constant, then the assertion trivially holds. We assume that $u$ is not constant. It suffices to show that $\inf_V u \geq \inf_{\Xi} u$. Let $z_0 \in V$. We need to show that $u(z_0) \geq \inf_{\Xi} u$. Let $A = \{x \in V; u(x) = u(z_0)\}$. Since $u$ is not constant, it follows that there is $x_0 \in A$ with $M_u(x_0) > 0$. Lemma 4.4 implies that there is $\xi \in \Xi$ such that $u(\xi) \leq u(x_0) - M_u(x_0)\rho(x_0, \xi)$. Therefore $\inf_{\Xi} u \leq u(\xi) \leq u(x_0) = u(z_0)$.

**Lemma 4.6.** Let $\zeta \in \Xi$ and let $u(x) = \rho(\zeta, x)$. Then $u$ is $\infty$-superharmonic and $M_u \equiv 1$ in $V$. Especially $D_\infty[u] = 1$.

**Proof.** Let $x \in V$ and $y \in \partial x$. Since $|u(x) - u(y)| = |\rho(\zeta, x) - \rho(\zeta, y)| \leq \rho(x, y) \leq r(x, y)$, it follows that $|\nabla u(x, y)| \leq 1$, and that $M_u(x) \leq 1$.

Let $\{z_n\}_n \in \zeta$. Take a path $\{x_i\}_{i=0}^l$ from $x$ to $z_n$. Then

$$\sum_{i=1}^l r(x_{i-1}, x_i) = r(x, x_1) + \sum_{i=2}^l r(x_{i-1}, x_i) \geq r(x, x_1) + \rho(x_1, z_n) \geq \min_{y \in \partial x}(r(x, y) + \rho(y, z_n)).$$

It follows that $\rho(x, z_n) \geq \min_{y \in \partial x}(r(x, y) + \rho(y, z_n))$. Letting $n \to \infty$ we have $\rho(x, \zeta) \geq \min_{y \in \partial x}(r(x, y) + \rho(y, \zeta))$. This means that $u(x) \geq u(y) + r(x, y)$ for some $y \in \partial x$, or $\nabla u(x, y) \leq -1$. Therefore $\nabla u(x, y) = -1$, and $M_u(x) = 1$. Theorem 3.1 shows that $u$ is $\infty$-superharmonic at $x$. \hfill \Box

5. The Dirichlet Problem

A network is said to be $\infty$-hyperbolic if $P_0 \neq \emptyset$; otherwise a network is said to be $\infty$-parabolic.

First we shall show a Liouville type theorem for an $\infty$-parabolic network, namely Theorem 5.2, which immediately follows from the next proposition.

**Proposition 5.1.** Suppose that $(V, E, r)$ is an $\infty$-parabolic network.

1. Let $u$ be an $\infty$-superharmonic function such that $\liminf_{n \to \infty} u(x_n) > -\infty$ for each $\{x_n\}_n \in P$. Then $u$ must be constant.
2. Let $u$ be an $\infty$-subharmonic function such that $\limsup_{n \to \infty} u(x_n) < \infty$ for each $\{x_n\}_n \in P$. Then $u$ must be constant.

**Proof.** We shall prove (2) only. If $u$ is not constant, then there is $x_0 \in V$ such that $M_u(x_0) > 0$. Theorem 4.3 implies that there is $x = \{x_n\}_n \in P$ such that $u(x_n) \geq u(x_0) + M_u(x_0)\sum_{i=1}^n r(x_{i-1}, x_i)$.

Since $\limsup_{n \to \infty} u(x_n) < \infty$, it follows that $\sum_{i=1}^\infty r(x_{i-1}, x_i) < \infty$, and therefore $x \in P_0$, which is impossible because $P_0 = \emptyset$. \hfill \Box
Theorem 5.2. Suppose that \((V,E,r)\) is an \(\infty\)-parabolic network. Let \(u\) be an \(\infty\)-harmonic function which is either bounded from above or bounded from below. Then \(u\) must be constant.

There may be an unbounded \(\infty\)-harmonic function on an \(\infty\)-parabolic network.

Example 5.3. Let \(V = \{x_n\}_{n=-\infty}^{\infty}\), \(E = \{(x_{n-1}, x_n)\}_{n=-\infty}^{\infty}\) and \(r \equiv 1\). Then \((V,E,r)\) is an \(\infty\)-parabolic network. Let \(u(x_n) = n\). Then \(u\) is an \(\infty\)-harmonic function in \(V\).

From here to the end of this section we always assume that \((V,E,r)\) is an \(\infty\)-hyperbolic network. We formulate the Dirichlet problem for \(\infty\)-harmonic functions as follows:

For a bounded function \(f\) on \(\Xi\), find a bounded \(\infty\)-harmonic function \(h \in D^{(\infty)}\) such that \(h \equiv f\) on \(\Xi\).

We define the upper class \(U_f\), the lower class \(L_f\), the upper solution \(H_f\) and the lower solution \(H_f\) by

\[
U_f = \left\{ u \in D^{(\infty)}; \ u \text{ is a bounded from below and } \infty\text{-superharmonic function such that } u \geq f \text{ on } \Xi \right\},
\]

\[
L_f = \left\{ v \in D^{(\infty)}; \ v \text{ is a bounded from above and } \infty\text{-subharmonic function such that } v \leq f \text{ on } \Xi \right\},
\]

\[
H_f(x) = \inf\{u(x); u \in U_f\}, \quad H_f(x) = \sup\{v(x); v \in L_f\} \quad \text{for } x \in V.
\]

If \(U_f = \emptyset\), then we let \(H_f \equiv \infty\). If \(L_f = \emptyset\), then we let \(H_f \equiv -\infty\).

Proposition 5.4. Let \(f\) be a constant function on \(\Xi\). Then a solution to the Dirichlet problem for \(f\) must be constant.

Proof. Suppose that there is a nonconstant solution \(h\) to the Dirichlet problem. Let \(x_0 \in V\) with \(M_h(x_0) > 0\). Lemma 4.4 shows that there are \(\xi, \eta \in \Xi\) such that

\[
f(\xi) = h(\xi) \leq h(x_0) - M_h(x_0) \rho(x_0, \xi),
\]

\[
f(\eta) = h(\eta) \geq h(x_0) + M_h(x_0) \rho(x_0, \eta).
\]

Then

\[
0 = f(\eta) - f(\xi) \geq M_h(x_0)(\rho(x_0, \eta) + \rho(x_0, \xi)) > 0,
\]

which is a contradiction. \(\square\)

Proposition 4.2 shows that the boundary function must be a Lipschitz function on \(\Xi\) whenever a solution to the Dirichlet problem exists. By Proposition 5.4 we may assume that the boundary function is not constant. Therefore we restrict a boundary function to a nonconstant Lipschitz function on \(\Xi\). This also means that \(\Xi\) contains at least two points.
For a nonconstant Lipschitz function \( f \) on \( \Xi \) we let \( L_f \) be the Lipschitz constant:

\[
L_f = \sup_{\xi, \eta \in \Xi} \frac{|f(\xi) - f(\eta)|}{\rho(\xi, \eta)}.
\]

We define

\[
L_{f,\xi} = \sup_{\eta \in \Xi \setminus \{\xi\}} \frac{|f(\xi) - f(\eta)|}{\rho(\xi, \eta)},
\]

\[
\varphi_{f,\xi}(x) = f(\xi) + L_{f,\xi} \rho(\xi, x), \\
\psi_{f,\xi}(x) = f(\xi) - L_{f,\xi} \rho(\xi, x),
\]

\[
\varphi_f(x) = \inf_{\xi \in \Xi} \varphi_{f,\xi}(x), \\
\psi_f(x) = \sup_{\xi \in \Xi} \psi_{f,\xi}(x)
\]

for \( \xi \in \Xi \) and \( x \in V \).

**Lemma 5.5.** Let \( f \) be a nonconstant bounded Lipschitz function on \( \Xi \). Then \( \varphi_{f,\xi}, \psi_{f,\xi} \in \mathcal{U}_f \) and \( \psi_{f,\xi}, \psi_f \in \mathcal{L}_f \) for \( \xi \in \Xi \). Moreover

\[
M_{\varphi_{f,\xi}} = M_{\psi_{f,\xi}} = L_{f,\xi} \quad \text{in } V, \\
D_{\infty}[\varphi_{f,\xi}] = D_{\infty}[\psi_{f,\xi}] = L_{f,\xi}, \\
\varphi_{f,\xi}(\xi) = \psi_{f,\xi}(\xi) = f(\xi), \\
D_{\infty}[\varphi_f] = D_{\infty}[\psi_f] = L_f, \\
\varphi_f \equiv \psi_f \equiv f \quad \text{on } \Xi.
\]

**Proof.** Lemma 4.6 shows that \( \varphi_{f,\xi} \) is \( \infty \)-superharmonic, that \( M_{\varphi_{f,\xi}} = L_{f,\xi} \), and that \( D_{\infty}[\varphi_{f,\xi}] = L_{f,\xi} \). It is easy to see that \( \varphi_{f,\xi}(\xi) = f(\xi) \). Clearly \( \varphi_{f,\xi} \geq f(\xi) \) in \( V \), which means that \( \varphi_{f,\xi} \) is bounded from below. Let \( \eta \in \Xi \). Then \( \varphi_{f,\xi}(\eta) = f(\eta) + L_{f,\xi} \rho(\xi, \eta) \). Since \( f(\eta) - f(\xi) \leq L_{f,\xi} \rho(\xi, \eta) \), it follows that \( \varphi_{f,\xi}(\eta) \geq f(\eta) \).

Therefore \( \varphi_{f,\xi} \in \mathcal{U}_f \).

Since \( \varphi_{f,\xi} \geq f(\xi) \geq \inf_{\Xi} f \) in \( V \), it follows that \( \varphi_f \geq \inf_{\Xi} f \) in \( V \) and that \( \varphi_f \) is finite at each vertex in \( V \). Lemmas 3.5 and 3.6 show that \( \varphi_f \) is \( \infty \)-superharmonic and that \( D_{\infty}[\varphi_f] \leq \sup_{\xi \in \Xi} D_{\infty}[\varphi_{f,\xi}] = \sup_{\xi \in \Xi} L_{f,\xi} = L_f \). For \( \xi \in \Xi \) and \( \{y_n\}_n \in \eta \in \Xi \),

\[
\varphi_{f,\xi}(y_n) \geq \varphi_{f,\xi}(\eta) - D_{\infty}[\varphi_{f,\xi}] \rho(y_n, \eta) = \varphi_{f,\xi}(\eta) - L_{f,\xi} \rho(y_n, \eta) \\
\geq f(\eta) - L_f \rho(y_n, \eta).
\]

Taking the infimum with respect to \( \xi \) and tending \( n \to \infty \) we obtain \( \varphi_f(\eta) \geq f(\eta) \).

Since \( \varphi_f(\eta) \leq \varphi_{f,\xi}(\eta) = f(\eta) \), it follows that \( \varphi_f \equiv f \) on \( \Xi \) and that \( \varphi_f \in \mathcal{U}_f \). The fact \( |f(\xi) - f(\eta)| = |\varphi_f(\xi) - \varphi_f(\eta)| \leq D_{\infty}[\varphi_f] \rho(\xi, \eta) \) gives that \( D_{\infty}[\varphi_f] \geq L_f \), and that \( D_{\infty}[\varphi_f] = L_f \).

We can similarly prove the assertion for \( \psi_{f,\xi} \) and \( \psi_f \). \( \square \)

**Lemma 5.6.** Let \( f \) be a nonconstant bounded Lipschitz function on \( \Xi \) with Lipschitz constant \( L_f \). Let

\[
\mathcal{U}_f = \{u \in \mathcal{U}_f; u \leq \varphi_f \text{ in } V\}, \\
\mathcal{L}_f = \{v \in \mathcal{L}_f; v \geq \psi_f \text{ in } V\}.
\]

Then \( D_{\infty}[u] \leq L_f \) for \( u \in \mathcal{U}_f \cup \mathcal{L}_f \).
**Proposition 5.8.** Combining these we obtain

\[ M_u(x_0) \leq \frac{u(x_0) - u(\xi)}{\rho(x_0, \xi)}. \]

Lemma 5.5 shows that \( f(\xi) \leq u(\xi) \leq \varphi_f(\xi) = f(\xi) \), so that \( u(\xi) = f(\xi) \). Also

\[ u(x_0) \leq \varphi_f(x_0) \leq \varphi_{f, \xi}(x_0) = f(\xi) + L_f, \xi \rho(\xi, x_0) \leq f(\xi) + L_f \rho(\xi, x_0). \]

Combining these we have \( M_u(x_0) \leq L_f \). This means \( D_\infty[u] \leq L_f \).

We can similarly prove \( \overline{D} \leq L_f \) for \( u \in \bar{U}_f \).

**Theorem 5.7.** Let \( f \) be a nonconstant bounded Lipschitz function on \( \Xi \) with Lipschitz constant \( L_f \). Then both \( \overline{\mathcal{H}}_f \) and \( \mathcal{H}_f \) are bounded \( \infty \)-harmonic functions with

\[
D_\infty[\overline{\mathcal{H}}_f] \leq L_f, \quad D_\infty[\mathcal{H}_f] \leq L_f, \quad \overline{\mathcal{H}}_f \equiv \mathcal{H}_f \equiv f \quad \text{on} \; \Xi.
\]

In particular, both \( \overline{\mathcal{H}}_f \) and \( \mathcal{H}_f \) are solutions to the Dirichlet problem for \( f \).

**Proof.** First we shall show that \( \inf_{\Xi} f \leq \overline{\mathcal{H}}_f \leq \sup_{\Xi} f \) in \( V \). Since the constant function \( \sup_{\Xi} f \) is in \( U_f \), it follows that \( \overline{\mathcal{H}}_f \leq \sup_{\Xi} f \) in \( V \). Let \( u \in U_f \). Corollary 4.5 shows that \( \inf_{V \cap \Xi} u = \inf_{\Xi} u \geq \inf_{\Xi} f \), so that \( u \geq \inf_{\Xi} f \) in \( V \). Therefore \( \overline{\mathcal{H}}_f \geq \inf_{\Xi} f \) in \( V \).

Lemma 3.5 shows that \( \overline{\mathcal{H}}_f \) is \( \infty \)-superharmonic in \( V \). Let \( x \in V \). Let \( u(x) = \overline{\mathcal{H}}_f \) and \( u = \overline{\mathcal{H}}_f \) in \( V \setminus \{x\} \). Then Lemma 3.4 shows that \( u \leq \overline{\mathcal{H}}_f \), that \( u \in U_f \), and that \( u \) is \( \infty \)-harmonic at \( x \). Therefore \( \overline{\mathcal{H}}_f \equiv u \), and that \( \overline{\mathcal{H}}_f \) is \( \infty \)-harmonic at \( x \).

It is easy to see that \( \overline{\mathcal{H}}_f(x) = \inf\{u(x); u \in \overline{U}_f\} \) for \( x \in V \), where \( \overline{U}_f \) is defined as in Lemma 5.6. Lemmas 3.6 and 5.6 show that \( D_\infty[\overline{\mathcal{H}}_f] \leq \sup\{D_\infty[u]; u \in \overline{U}_f\} \leq L_f \).

Next we claim that \( \overline{\mathcal{H}}_f(\xi) = f(\xi) \) for \( \xi \in \Xi \). Lemma 5.5 shows that \( \overline{\mathcal{H}}_f(\xi) \leq \varphi_f(\xi) = f(\xi) \). For the converse, let \( \{x_n\}_n \in \xi \). Then

\[
\overline{\mathcal{H}}_f(\xi) \geq \overline{\mathcal{H}}_f(x_n) - D_\infty[\overline{\mathcal{H}}_f] \rho(x_n, \xi) \geq \overline{\mathcal{H}}_f(x_n) - L_f \rho(x_n, \xi).
\]

For \( n \in \mathbb{N} \) and \( \varepsilon > 0 \) there is \( u \in \overline{U}_f \) such that \( \overline{\mathcal{H}}_f(x_n) \geq u(x_n) - \varepsilon \). Lemma 5.6 implies that

\[
u(x_n) \geq u(\xi) - D_\infty[u] \rho(x_n, \xi) \geq f(\xi) - L_f \rho(x_n, \xi).
\]

Combining these we obtain \( \overline{\mathcal{H}}_f(\xi) \geq f(\xi) - 2L_f \rho(x_n, \xi) - \varepsilon \). Tending \( n \to \infty \) and \( \varepsilon \to 0 \) we have \( \overline{\mathcal{H}}_f(\xi) \geq f(\xi) \). Therefore \( \overline{\mathcal{H}}_f(\xi) = f(\xi) \).

Similarly we can prove the assertion for \( \mathcal{H}_f \).

**Proposition 5.8.** Let \( f \) be a nonconstant bounded Lipschitz function on \( \Xi \). Let \( h \) be a solution to the Dirichlet problem for \( f \). Then

\[ \psi_f \leq \overline{\mathcal{H}}_f \leq h \leq \mathcal{H}_f \leq \varphi_f \quad \text{in} \; V. \]
Proof. We note that the set of solutions to the Dirichlet problem coincides with $U_f \cap L_f$, especially $h \in U_f \cap L_f$. It follows that $\overline{H}_f = \inf_{u \in U_f} u \leq \inf_{u \in U_f \cap L_f} u \leq h$ in $V$.

We shall prove $H_{f, \xi} \leq \varphi_{f, \xi}$ in $V$ for $\xi \in \Xi$. On the contrary we assume that $A := \{y \in V; H_{f, \xi}(y) > \varphi_{f, \xi}(y)\} \neq \emptyset$ for a fixed $\xi \in \Xi$. Let $y_0 \in A$. Lemma 5.5 and Theorem 4.3 show that there is $\{y_n\}_n \in \eta \in \Xi$ such that

$$\varphi_{f, \xi}(y_n) \leq \varphi_{f, \xi}(y_0) - L_{f, \xi} \sum_{i=1}^n r(y_{i-1}, y_i).$$

Also

$$H_{f, \xi}(y_0) = H_f(y_0) + \sum_{i=1}^n \nabla H_f(y_i, y_{i-1}) r(y_{i-1}, y_i) \leq H_f(y_n) + \sum_{i=1}^n M_{H_f}(y_{i-1}) r(y_{i-1}, y_i).$$

Combining these and the fact that $y_0 \in A$ we have

$$\varphi_{f, \xi}(y_n) + L_{f, \xi} \sum_{i=1}^n r(y_{i-1}, y_i) < H_f(y_n) + \sum_{i=1}^n M_{H_f}(y_{i-1}) r(y_{i-1}, y_i). \tag{3}$$

Since $f(\eta) \leq \varphi_{f, \xi}(\eta)$ and $H_f(\eta) = f(\eta)$, it follows that $L_{f, \xi} \sum_{i=1}^\infty r(y_{i-1}, y_i) \leq \sum_{i=1}^\infty M_{H_f}(y_{i-1}) r(y_{i-1}, y_i)$. There is $n \geq 0$ with $L_{f, \xi} \leq M_{H_f}(y_n)$. We take the smallest such $n$. If $n \geq 1$, then, since $L_{f, \xi} > M_{H_f}(y_{i-1})$ for $i = 1, 2, \ldots, n$, the inequality (3) implies that

$$\varphi_{f, \xi}(y_n) + L_{f, \xi} \sum_{i=1}^n r(y_{i-1}, y_i) < H_f(y_n) + \sum_{i=1}^n L_{f, \xi} r(y_{i-1}, y_i),$$

so that $y_n \in A$. This also holds if $n = 0$.

Let $z_0 = y_n$. Then $M_{H_f}(z_0) \geq L_{f, \xi}$ and $z_0 \in A$. Lemma 4.4 shows that there is $\zeta \in \Xi$ such that

$$f(\zeta) = H_f(\zeta) \geq H_f(z_0) + M_{H_f}(z_0) \rho(z_0, \zeta) \geq H_f(z_0) + L_{f, \xi} \rho(z_0, \zeta).$$

Lemma 5.5 shows that

$$f(\zeta) \leq \varphi_{f, \xi}(\zeta) \leq \varphi_{f, \xi}(z_0) + D_\infty[\varphi_{f, \xi}] \rho(z_0, \zeta) = \varphi_{f, \xi}(z_0) + L_{f, \xi} \rho(z_0, \zeta).$$

These imply that $\varphi_{f, \xi}(z_0) \geq H_f(z_0)$, which contradicts $z_0 \in A$. This means that $H_f \leq \varphi_{f, \xi}$ in $V$ for $\xi \in \Xi$, and that $H_f \leq \varphi_f$ in $V$.

The other inequalities can be proved similarly. \hfill \Box

**Theorem 5.9.** Let $f$ be a nonconstant bounded Lipschitz function on $\Xi$ with Lipschitz constant $L_f$. Then a solution to the Dirichlet problem for $f$ is a solution to the variational problem:

Minimize $D_\infty[u]$ subject to $u \in D^{(\infty)}$ and $u \equiv f$ on $\Xi$. 

More precisely, if \( u \in \mathbf{D}(\infty) \) satisfies \( u \equiv f \) on \( \Xi \), then \( D_\infty[u] \geq L_f \), and the equality holds when \( u \) is bounded and \( \infty \)-harmonic.

We remark that the equality above can hold even when \( u \) is not \( \infty \)-harmonic. See Example 5.10.

**Proof.** Let \( u \in \mathbf{D}(\infty) \) with \( u \equiv f \) on \( \Xi \). Let \( \xi, \eta \in \Xi \). Then \( |f(\xi) - f(\eta)| \leq D_\infty[u] \rho(\xi, \eta) \). This means \( D_\infty[u] \geq L_f \).

Let \( h \) be a solution to the Dirichlet problem for \( f \). Proposition 5.8 and Lemma 5.6 show that \( h \in \mathcal{U}_f \) and that \( D_\infty[h] \leq L_f \). \( \square \)

A solution to the variational problem in Theorem 5.9 is not necessarily unique.

**Example 5.10.** We note that Lemma 5.5 shows that \( \varphi_f \) is a solution to the variational problem in Theorem 5.9. Let

\[
V = \{o\} \cup \{x_n, y_n, z_n\}_{n=1}^{\infty},
\]

\[
E = \{(x_{n-1}, x_n), (y_{n-1}, y_n), (z_{n-1}, z_n)\}_{n=1}^{\infty},
\]

\[
r(x_{n-1}, x_n) = r(y_{n-1}, y_n) = r(z_{n-1}, z_n) = 2^{-n},
\]

where \( x_0 = y_0 = z_0 = o \). Let \( \xi = \{x_n\}, \eta = \{y_n\}, \zeta = \{z_n\} \). Let \( f(\xi) = -1, f(\eta) = 1, f(\zeta) = 0 \). Then \( L_{f, \xi} = L_{f, \eta} = 1 \) and \( L_{f, \zeta} = 1/2 \). We have

\[
\overline{\mathcal{H}}_f(x) = \begin{cases} 
0 & \text{if } x = \sigma; \\
2^{-n} - 1 & \text{if } x = x_n; \\
1 - 2^{-n} & \text{if } x = y_n; \\
0 & \text{if } x = z_n,
\end{cases}
\]

\[
\varphi_f(x) = \begin{cases} 
0 & \text{if } x = \sigma; \\
2^{-n} - 1 & \text{if } x = x_n; \\
1 - 2^{-n} & \text{if } x = y_n; \\
2^{-n-1} & \text{if } x = z_n,
\end{cases}
\]

for \( n \geq 1 \). Note that \( \varphi_f \) is not \( \infty \)-harmonic at \( z_1 \). \( \square \)

Here we show a uniqueness result for a solution to the Dirichlet problem. Let

\[
\delta(x) = \inf_{\eta \in \Xi} \rho(x, \eta),
\]

\[
Q = \{x_n\} \in \mathcal{P}; \liminf_{n \to \infty} \delta(x_n) > 0 \text{ and } \limsup_{n \to \infty} \delta(x_n) < \infty\}.
\]

**Lemma 5.11.** Let \( v_1 \) and \( v_2 \) be bounded from above and \( \infty \)-subharmonic functions with \( v_1, v_2 \in \mathbf{D}(\infty) \). Suppose that \( \limsup_{n \to \infty} (v_1(x_n) + v_2(x_n)) \leq 0 \) for all \( \{x_n\} \in \mathcal{P}_0 \cup Q \). Then \( v_1 + v_2 \leq 0 \) in \( V \).

**Proof.** Let \( u = v_1 + v_2 \). It suffices to show that \( A := \{x \in V; u(x) > \alpha\} = \emptyset \) for all \( \alpha > 0 \). On the contrary we assume \( A \neq \emptyset \) for some \( \alpha > 0 \). Let \( M_i(x) = M_i(x) \) for \( i = 1, 2 \) and \( M(x) = \max\{M_1(x), M_2(x)\} \).

If \( M(x) = 0 \) for each \( x \in A \), then \( u \) is constant on \( \overline{\mathcal{A}} \), so that \( A = V \). For \( \{x_n\} \in \mathcal{P}_0 \), it follows that \( \alpha < \limsup_{n \to \infty} u(x_n) \leq 0 \), which contradicts \( \alpha > 0 \).

We may assume that \( M(x_0) > 0 \) for some \( x_0 \in A \).

We shall show that there exists \( x_1 \in \partial x_0 \) such that either

(1a) \( u(x_1) > u(x_0) \) and \( M(x_1) \geq M(x_0) \); or

(1b) \( u(x_1) = u(x_0), v_1(x_1) > v_1(x_0) \) and \( M(x_1) \geq M(x_0) \).
First we consider the case $M_1(x_0) > M_2(x_0)$. By Theorem 3.1, there exists $x_1 \in \partial x_0$ such that
\[ \nabla v_1(x_0, x_1) = M_1(x_0) > M_2(x_0) \geq -\nabla v_2(x_0, x_1), \]
and hence $\nabla u(x_0, x_1) > 0$, i.e., $u(x_1) > u(x_0)$. Since $M(x_1) \geq M_1(x_1) \geq \nabla v_1(x_0, x_1) = M_1(x_0) = M(x_0)$, it follows that $x_1$ satisfies (1a). In case $M_1(x_0) < M_2(x_0)$, similarly there is $x_1 \in \partial x_0$ with $\nabla v_2(x_0, x_1) = M_2(x_0)$, which satisfies (1a).

Next we consider the case $M_1(x_0) = M_2(x_0)$. By Theorem 3.1, there exists $x_1 \in \partial x_0$ such that
\[ \nabla v_1(x_0, x_1) = M_1(x_0) = M_2(x_0) \geq -\nabla v_2(x_0, x_1), \]
and hence $\nabla u(x_0, x_1) \geq 0$, i.e., $u(x_1) \geq u(x_0)$. If $u(x_1) > u(x_0)$, then an argument similar to the first case shows that $x_1$ satisfies (1a). If $u(x_1) = u(x_0)$, then, since $\nabla v_1(x_0, x_1) = M(x_0) > 0$, it follows that $v_1(x_1) > v_1(x_0)$. The fact $M(x_1) \geq \nabla v_1(x_0, x_1) = M(x_0)$ implies that $x_1$ satisfies (1b).

Since $x_1 \in A$ and $M(x_1) > 0$, there is $x_2 \in \partial x_1$ such that either
\begin{align*}
(2\text{a}) & \quad u(x_2) > u(x_1) \text{ and } M(x_2) \geq M(x_1); \text{ or} \\
(2\text{b}) & \quad u(x_2) = u(x_1), \quad v_1(x_2) > v_1(x_1) \text{ and } M(x_2) \geq M(x_1).
\end{align*}

Repeating this argument we obtain a sequence $\{x_n\}_n$ such that $x_n \in \partial x_{n-1}$ and that either
\begin{align*}
(n\text{a}) & \quad u(x_n) > u(x_{n-1}) \text{ and } M(x_n) \geq M(x_{n-1}); \text{ or} \\
(n\text{b}) & \quad u(x_n) = u(x_{n-1}), \quad v_1(x_n) > v_1(x_{n-1}) \text{ and } M(x_n) \geq M(x_{n-1}).
\end{align*}

Suppose that $x_k = x_i$ for some $k \leq l$. If (iia) holds for some $i$ with $k < i \leq l$, then
\[ u(x_l) \geq \cdots \geq u(x_{i}) > u(x_{i-1}) \geq \cdots \geq u(x_k) = u(x_i), \]
a contradiction; if (iib) holds for all $i$ with $k < i \leq l$, then $v_1(x_l) > \cdots > v_1(x_k) = v_1(x_i)$, a contradiction. Therefore $x := \{x_n\}_n$ is an infinite path.

If $M_1(x_n) \geq M_2(x_n)$, then, since $M_1(x_n) = M(x_n) \geq M(x_0) > 0$, Lemma 4.4 shows that there is $\eta_n \in \Xi$ such that $v_1(x_n) \geq v_1(x_n) + M_1(x_n)\rho(x_n, \eta_n) \geq v_1(x_n) + M(x_0)\rho(x_n, \eta_n)$. Let $s = \sup v_1 \vee \sup v_2$. Since $v_1(x_n) = u(x_n) - v_2(x_n) \geq u(x_0) - s$, it follows that
\[ s \geq u(x_0) - s + M(x_0)\rho(x_n, \eta_n), \]
and that
\[ \delta(x_n) \leq \rho(x_n, \eta_n) \leq \frac{2s - u(x_0)}{M(x_0)}. \]

The same inequality also holds when $M_1(x_n) < M_2(x_n)$. Therefore
\[ \limsup_{n \to \infty} \delta(x_n) \leq \frac{2s - u(x_0)}{M(x_0)} < \infty. \]

Let $\zeta \in \Xi$. Then $u(x_n) - u(\zeta) \leq D_\infty[u] \rho(x_n, \zeta)$. Since $u(\zeta) \leq 0$ and $u(x_n) > \alpha$, it follows that $D_\infty[u] \rho(x_n, \zeta) \geq \alpha$, so that
\[ \liminf_{n \to \infty} \delta(x_n) \geq \frac{\alpha}{D_\infty[u]} > 0. \]
Therefore $x \in Q$. The assumption shows that
\[ \alpha < \limsup_{n \to \infty} u(x_n) \leq 0, \]
which contradicts $\alpha > 0$. \hfill \qed

**Theorem 5.12.** Suppose that $Q = \emptyset$. Let $f$ be a nonconstant bounded Lipschitz function on $\mathbb{R}$. Then there exists a unique solution to the Dirichlet problem for $f$.

**Proof.** Let $u \in U_f$ and $v \in L_f$. We apply Lemma 5.11 to $v$ and $u$ and obtain $v - u \leq 0$ in $V$. This implies $\mathcal{H}_f \geq \mathcal{H}_f$. Proposition 5.8 shows that a solution $h$ to the Dirichlet problem satisfies $h \equiv \mathcal{H}_f \equiv \mathcal{H}_f$. \hfill \qed

Now we address the question:

Can we replace $P_0 \cup Q$ in the condition of Lemma 5.11 by $P_0$?

Let $w$ be a nonnegative function on $E$ with $w(x, y) = w(y, x)$ and $R \subset P$. Let
\[ t[w, R] = \inf \left\{ \sum_{j} r(x_{j-1}, x_j)w(x_{j-1}, x_j); \{x_j\} \in R \right\}, \]
\[ \mathcal{M}_\infty(R) = \inf \{ \sup_{E} w; t[w, R] \geq 1 \}. \]

We see that $\mathcal{M}_\infty$ is an outer measure on $P$ and we call it the $\infty$-modulus. It is easy to see that $\mathcal{M}_\infty(P \setminus P_0) = 0$ and $Q \subset P \setminus P_0$, so the above question seems to be affirmative. However the author has no idea to answer the question.

6. **AN $\infty$-HARMONIC FUNCTIONS AS A LIMIT OF $p$-HARMONIC FUNCTIONS**

Let $1 < p < \infty$ and let
\[ \varphi_p(t) = |t|^{p-1} \text{sgn } t = \begin{cases} \frac{t^{p-1}}{\|x\|} & \text{if } t > 0; \\ 0 & \text{if } t = 0; \\ \frac{(-t)^{p-1}}{\|x\|} & \text{if } t < 0. \end{cases} \]

For $x \in V$ and $u \in L(\partial x)$ we define
\[ \nu^p_{x,u}(t) = \sum_{y \in \partial x} \varphi_p \left( \frac{u(y) - t}{r(x, y)} \right). \]

Since $\nu^p_{x,u}$ is strictly decreasing and $\lim_{t \to \pm \infty} \nu^p_{x,u}(t) = \mp \infty$, there is a unique value $H^p_x u$ such that $\nu^p_{x,u}(H^p_x u) = 0$. Let $D \subset V$ and $u \in L(\partial D)$. If $u$ satisfies $u(x) \leq H^p_x u$ (or $u(x) = H^p_x u$, resp.) for each $x \in D$, then $u$ is said to be $p$-superharmonic ($p$-harmonic, resp.) in $D$..

Now we shall show that a limit of $p$-harmonic functions as $p \to \infty$ is an $\infty$-harmonic function.

**Lemma 6.1.** Let $x \in V$ and $u \in L(\partial x)$. Then
\[ \lim_{p \to \infty} H^p_x u = H^\infty_x u. \]
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Proof. Let $t_p = H_p^2 u$ and $t_{\infty} = H_{\infty}^2 u$. If $u$ is constant on $\partial x$, then $t_p = t_{\infty} = u$ and the assertion holds. We assume that $u$ is not constant on $\partial x$. Let $\tilde{u}$ be the function such that $\tilde{u}(x) = t_{\infty}$ and $\tilde{u} = u$ on $\partial x$. Then $M_\infty(x) > 0$.

$$J_+ = \{ y \in \partial x; u(y) > t_{\infty} \}, \quad J_- = \{ y \in \partial x; u(y) < t_{\infty} \}.$$ 

Theorem 3.1 shows that there is $y_1 \in \partial x$ such that

$$\frac{u(y_1) - t_{\infty}}{r(x, y_1)} = \frac{\tilde{u}(y_1) - \tilde{u}(x)}{r(x, y_1)} = \nabla \tilde{u}(x, y_1) = M_\infty(x) > 0.$$ 

This means $y_1 \in J_+$, especially $J_+ \neq \emptyset$. Since

$$\frac{u(y) - t_{\infty}}{r(x, y)} = \frac{\tilde{u}(y) - \tilde{u}(x)}{r(x, y)} = \nabla \tilde{u}(x, y) \leq M_\infty(x)$$

for $y \in \partial x$, it follows that

$$\max_{y \in J_+} \frac{u(y) - t_{\infty}}{r(x, y)} = M_\infty(x) = \mu_{x, \infty}^{\infty}(\tilde{u}(x)) = \mu_{x, \infty}^{\infty}(t_{\infty}).$$

Similarly $J_- \neq \emptyset$ and

$$\max_{y \in J_-} \frac{t_{\infty} - u(y)}{r(x, y)} = \mu_{x, \infty}^{\infty}(t_{\infty}).$$

Let $\varepsilon > 0$ with $\varepsilon < |u(y) - t_{\infty}|$ for every $y \in J_+ \cup J_-$. Let $J_0 = \{ y \in \partial x; u(y) = t_{\infty} \}$, which may be an empty set. We consider

$$\nu_{x, u}^p(t_{\infty} + \varepsilon) = \sum_{y \in J_+} \left( \frac{u(y) - t_{\infty} - \varepsilon}{r(x, y)} \right)^{p-1} - \sum_{y \in J_- \cup J_0} \left( \frac{t_{\infty} + \varepsilon - u(y)}{r(x, y)} \right)^{p-1}.$$ 

Let

$$\alpha_p = \sum_{y \in J_+} \left( \frac{u(y) - t_{\infty} - \varepsilon}{r(x, y)} \right)^{p-1}, \quad \beta_p = \sum_{y \in J_- \cup J_0} \left( \frac{t_{\infty} + \varepsilon - u(y)}{r(x, y)} \right)^{p-1}.$$ 

Let $q$ be a number with $(p - 1)(q - 1) = 1$. Then

$$\lim_{p \to \infty} \alpha_p^{q-1} = \max_{y \in J_+} \frac{u(y) - t_{\infty} - \varepsilon}{r(x, y)} < \mu_{x, \infty}^{\infty}(t_{\infty}),$$

$$\lim_{p \to \infty} \beta_p^{q-1} = \max_{y \in J_- \cup J_0} \frac{t_{\infty} + \varepsilon - u(y)}{r(x, y)} > \mu_{x, \infty}^{\infty}(t_{\infty}).$$

Therefore $\nu_{x, u}^p(t_{\infty} + \varepsilon) < 0$ for sufficiently large $p$. Similarly $\nu_{x, u}^p(t_{\infty} - \varepsilon) > 0$. Since $\nu_{x, u}^p$ is strictly decreasing, it follows that $t_{\infty} - \varepsilon < t_p < t_{\infty} + \varepsilon$, and hence $t_p \to t_{\infty}$. □

Theorem 6.2. Let $D \subset V$. Let $\{p_n\}_n$ be a sequence such that $1 < p_n < \infty$ and $\lim_{n \to \infty} p_n = \infty$. Let $\{u_n\}_n$ be a sequence of functions in $\overline{D}$ such that $u_n$ is $p_n$-harmonic in $D$ and converges pointwise to a function $u$ in $\overline{D}$. Then $u$ is \infty-harmonic in $D$. 

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Proof. Let $x \in D$ and $\varepsilon > 0$. By Lemma 6.1 there is $n$ such that $|H_p^n u - H_x^\infty u| < \varepsilon$. We may assume that $|u(y) - u_n(y)| < \varepsilon$ for all $y \in N x$. Then $|H_p^n u - H_x^2 u_n| < \varepsilon$. Since $H_p^n u_n = u_n(x)$, it follows that

$$|u(x) - H_x^\infty u| \leq |u(x) - u_n(x)| + |H_p^n u_n - H_x^p u| + |H_p^n u - H_x^n u|,$$

which means that $u$ is $\infty$-harmonic at $x$. □

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References


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