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CHARACTERIZATIONS OF ISOPARAMETRIC HYPERSURFACES IN A SPHERE

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Abstract. We characterize isoparametric hypersurfaces in a sphere by using covariant derivatives of their shape operators and by observing the extrinsic shape of their geodesics.

1. Introduction

It is well-known that a hypersurface M in an $(n+1)$-dimensional sphere $S^{n+1}(c)$ $(n \geqq 2)$ of constant sectional curvature c is isoparametric if and only if all of its principal curvatures in the ambient sphere $S^{n+1}(c)$ are constant. The study of isoparametric hypersurfaces is one of the most interesting objects in differential geometry. The classification problem of isoparametric hypersurfaces in a sphere is still open. However, we know that the number g of distinct principal curvatures of isoparametric hypersurfaces in a sphere is either $g = 1, 2, 3, 4$ or 6 (see [6, 7]). Note that this result is not obtained by classifying all isoparametric hypersurfaces.

We know that every isoparametric hypersurface with $g = 1$ (totally umbilic hypersurfaces) or $g = 2$ (Clifford hypersurfaces) has parallel shape operator but other isoparametric hypersurfaces do not have parallel shape operator. Moreover, a hypersurface M in $S^{n+1}(c)$ is totally umbilic if and only if every geodesic of M is a circle (i.e., either a great circle or a small circle of positive curvature) on $S^{n+1}(c)$.

The main purpose of this paper is to give a characterization of all isoparametric hypersurfaces in a sphere by weakening these geometric properties (see Theorem 1).

2. Statements of results

We first recall some basic terminology in the theory of hypersurfaces. Let M be a hypersurface in an $(n+1)$-dimensional sphere $S^{n+1}(c)$ $(n \geqq 2)$ of constant

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sectional curvature c and $\mathcal{N}$ a unit normal vector field on M in $S^{n+1}(c)$. Then the Riemannian connections $\widetilde{\nabla}$ of $S^{n+1}(c)$ and ∇ of M are related by Gauss formula

$$\widetilde{\nabla}_X Y = \nabla_X Y + \langle AX, Y\rangle \mathcal{N} \tag{2.1}$$

and Weingarten formula

$$\widetilde{\nabla}_X \mathcal{N} = -AX \tag{2.2}$$

for arbitrary vector fields X and Y on M, where $\langle\ ,\ \rangle$ denotes the Riemannian metric on M induced from the standard metric on $S^{n+1}(c)$ and A is the shape operator of M in $S^{n+1}(c)$. The Codazzi equation can be written as

$$\langle(\nabla_X A)Y, Z\rangle = \langle(\nabla_Y A)X, Z\rangle \tag{2.3}$$

for vector fields X, Y and Z tangent to M. Eigenvalues and eigenvectors of the shape operator A are called *principal curvatures* and *principal curvature vectors*, respectively.

Next we recall the definition of circles in Riemannian geometry. A smooth curve $\gamma = \gamma(s)$ in a Riemannian manifold $\widetilde{M}$ parametrized by its arclength s is called a *circle* of curvature k ($\geqq 0$) if it satisfies the following ordinary differential equation:

$$\widetilde{\nabla}_{\dot\gamma}\widetilde{\nabla}_{\dot\gamma}\dot\gamma = -k^2\dot\gamma,$$

where k is constant and $\widetilde{\nabla}_{\dot\gamma}$ denotes the covariant differentiation along γ with respect to the Riemannian connection $\widetilde{\nabla}$ of $\widetilde{M}$. Since $\|\widetilde{\nabla}_{\dot\gamma}\dot\gamma\| = k$, a circle of null curvature is nothing but a geodesic.

Theorem 1. *Let M be a connected hypersurface in an $(n+1)$-dimensional sphere $S^{n+1}(c)$ $(n \geqq 2)$ of constant sectional curvature c. Then the following three conditions are mutually equivalent.*

(1) *M is locally congruent to an isoparametric hypersurface in $S^{n+1}(c)$.*
(2) *The tangent bundle TM of M is decomposed as the direct sum of the principal distributions $V_{\lambda_i} = \{X \in TM | AX = \lambda_i X\}$ such that the covariant derivative of the shape operator A of M in $S^{n+1}(c)$ satisfies $(\nabla_X A)Y = 0$ for all $X, Y \in V_{\lambda_i}$ associated to every principal curvature λ_i, where ∇ denotes the Riemannian connection of M.*
(3) *For each point p of M, there exists an orthonormal basis $\{v_1, \ldots, v_{m_p}\}$ of the orthogonal complement of $\ker A_p$ in T_pM $(m_p = \mathrm{rank}\ A_p)$ such that every geodesic of M through p with initial vector v_i is a small circle of positive curvature in $S^{n+1}(c)$.*

In our previous paper [4], we proved that the above Conditions (1) and (3) are mutually equivalent. However, for readers we give a complete proof of Theorem 1.

Proof of Theorem 1. We first verify that Conditions (1) and (2) are mutually equivalent. Suppose Condition (1). Then the tangent bundle TM of M is decomposed as the direct sum of the principal distributions $V_{\lambda_i} = \{X \in TM | AX = \lambda_i X\}$ associated to constant principal curvature λ_i. For any $X, Y \in V_{\lambda_i}$ and any $Z \in TM$, the Codazzi equation (2.3) shows

$$\begin{aligned}\langle(\nabla_X A)Y, Z\rangle &= \langle(\nabla_Z A)Y, X\rangle = \langle\nabla_Z(AY) - A\nabla_Z Y, X\rangle \\ &= \langle(\lambda_i I - A)\nabla_Z Y, X\rangle = \langle\nabla_Z Y, (\lambda_i I - A)X\rangle = 0.\end{aligned}$$

Hence the covariant derivative of the shape operator A satisfies $(\nabla_X A)Y = 0$ for any $X, Y \in V_{\lambda_i}$.

We next suppose Condition (2). Taking a nonzero element X in V_{λ_i}, we have for all $Z \in TM$

$$\begin{aligned}\langle(\nabla_X A)X, Z\rangle &= \langle(\nabla_Z A)X, X\rangle = \langle\nabla_Z(AX) - A\nabla_Z X, X\rangle \\ &= \langle(Z\lambda_i)X + (\lambda_i I - A)\nabla_Z X, X\rangle \\ &= (Z\lambda_i)\|X\|^2 + \langle\nabla_Z X, (\lambda_i I - A)X\rangle = (Z\lambda_i)\|X\|^2.\end{aligned}$$

Since $(\nabla_X A)X = 0$ and X is not zero, we get $Z\lambda_i = 0$ for all $Z \in TM$. Thus we see that all principal curvatures of M in the ambient sphere $S^{n+1}(c)$ are constant, so that Conditions (1) and (2) are mutually equivalent.

In the following, we study the relation between Conditions (1) and (3). Suppose Condition (1). Let M be an isoparametric hypersurface of $S^{n+1}(c)$ with constant principal curvatures $\lambda_1, \dots, \lambda_g$. Then the tangent bundle TM is decomposed as: $TM = V_{\lambda_1} \oplus V_{\lambda_2} \oplus \cdots V_{\lambda_g}$. We here recall the fact that every V_{λ_i} $(1 \leqq i \leqq g)$ is integrable and moreover every leaf L_{λ_i} of V_{λ_i} is totally geodesic in the hypersurface M. (To do show that, we verify that $\nabla_X Y \in V_{\lambda_i}$ for all $X, Y \in V_{\lambda_i}$. For such vectors X, Y we have $(\nabla_X A)Y = 0$ by Condition (2), and hence

$$A(\nabla_X Y) = \nabla_X(AY) - (\nabla_X A)Y = \lambda_i(\nabla_X Y).)$$

The above fact, together with Gauss formula (2.1), implies that and every leaf L_{λ_i} is totally umbilic in the ambient sphere $S^{n+1}(c)$. Note that L_{λ_i} is nothing but a sphere $S^{m_i}(c_i)$ with $m_i = \dim V_{\lambda_i}$ and $c_i = c + \lambda_i^2$. So, when $\lambda_i \neq 0$ (resp. $\lambda_i = 0$), every geodesic $\gamma = \gamma(s)$ through $p = \gamma(0)$ on M with $\dot\gamma(0) \in V_{\lambda_i}$ is a small circle of positive curvature $|\lambda_i|$ (resp. a great circle) on $S^{n+1}(c)$. Therefore, choosing an orthonormal basis $\{v_1, \dots, v_{m_p}\}$ of the orthogonal complement of $\ker A_p$ in T_pM ($m_p = \text{rank } A_p$) as principal curvature vectors of M in $S^{n+1}(c)$, we obtain the desired Condition (3).

Conversely, suppose Condition (3). We consider the open dense subset $\mathcal{U} = \{p \in M |$the multiplicity of each principal curvature of M in $S^{n+1}(c)$ is constant on some neighborhood $\mathcal{V}_p(\subset \mathcal{U})$ of $p\}$ of M. Note that all principal curvature functions are differentiable on $\mathcal{U}$. In the following, we shall study on a fixed neighborhood $\mathcal{V}_p$. We remark that the shape operator A has constant rank on $\mathcal{V}_p$.

Let $\gamma_i = \gamma_i(s)$ $(1 \leqq i \leqq m_p)$ be geodesics of M with $\gamma_i(0) = p$ and $\dot\gamma(0) = v_i$, where $\{v_1, \dots, v_{m_p}\}$ is an orthonormal basis of the orthogonal complement of $\ker A_p$ in T_pM. We denote by $\widetilde\nabla$ and ∇ the Riemannian connections of $S^{n+1}(c)$ and M, respectively. Then they satisfy

$$\widetilde\nabla_{\dot\gamma_i}\widetilde\nabla_{\dot\gamma_i}\dot\gamma_i = -k_i^2\dot\gamma_i \tag{2.4}$$

for some positive constant k_i. Here, without loss of generality we can set $k_1 \leqq k_2 \leqq \cdots \leqq k_{m_p}$. It follows from Gauss formula (2.1) and Weingarten formula (2.2) that

$$\widetilde{\nabla}_{\dot{\gamma}_i}\widetilde{\nabla}_{\dot{\gamma}_i}\dot{\gamma}_i = -\langle A\dot{\gamma}_i, \dot{\gamma}_i\rangle A\dot{\gamma}_i + \langle(\nabla_{\dot{\gamma}_i}A)\dot{\gamma}_i, \dot{\gamma}_i\rangle\mathcal{N}. \tag{2.5}$$

Comparing the tangential components of (2.4) and (2.5), at $s = 0$ we obtain

$$\langle Av_i, v_i\rangle Av_i = k_i^2 v_i.$$

This, together with $k_i \neq 0$, implies

$$Av_i = k_i v_i \quad \text{or} \quad Av_i = -k_i v_i \quad (1 \leqq i \leqq m_p),$$

which means that the tangent space T_pM is decomposed as:

$$\begin{aligned} T_pM = \ker A_p &\oplus \{v \in T_pM | Av = -k_{i_1}v\} \oplus \{v \in T_pM | Av = k_{i_1}v\} \\ &\oplus \cdots \oplus \{v \in T_pM | Av = -k_{i_g}v\} \oplus \{v \in T_pM | Av = k_{i_g}v\}, \end{aligned}$$

where $0 < k_{i_1} < k_{i_2} < \cdots < k_{i_g}$ and g is the number of positive distinct k_j ($j = 1, \ldots, m_p$). Note that every k_{i_j} is differentiable on $\mathcal{V}_p$. We shall show the constancy of k_{i_j}. We first note that $v_{i_j}k_{i_j} = 0$ (see the normal component of Equation (2.5)). Let $\{v_{m_p+1}, \ldots, v_n\}$ be an orthonormal basis of $\ker A$. Then $\{v_1, \ldots, v_n\}$ forms an orthonormal basis of T_pM. For any v_ℓ ($1 \leqq \ell \neq i_j \leqq n$), since A is symmetric, we see

$$\langle(\nabla_{v_{i_j}}A)v_\ell, v_{i_j}\rangle = \langle v_\ell, (\nabla_{v_{i_j}}A)v_{i_j}\rangle. \tag{2.6}$$

In order to compute Equation (2.6) easily, we extend an orthonormal basis $\{v_1, \ldots, v_n\}$ to principal curvature unit vector fields on some neighborhood $\mathcal{W}_p(\subset \mathcal{V}_p)$, say $\{V_1, \ldots, V_n\}$. Moreover we can choose $\nabla_{V_{i_j}}V_{i_j} = 0$ at the point p, where $(V_{i_j})_p = v_{i_j}$. Such a principal curvature unit vector field V_{i_j} can be obtained as follows.

We first define a smooth vector field W_{i_j} on some sufficiently small neighborhood $\mathcal{W}_p(\subset \mathcal{V}_p)$ by using parallel displacement for the vector v_{i_j} along each geodesic with origin p. We remark that in general W_{i_j} is not principal on $\mathcal{W}_p$, but $AW_{i_j} = k_{i_j}W_{i_j}$ on the geodesic $\gamma = \gamma(s)$ with $\gamma(0) = p$ and $\dot{\gamma}(0) = v_{i_j}$. We here define the vector field U_{i_j} on $\mathcal{W}_p$ as $U_{i_j} = \Pi_{\alpha \neq k_{i_j}}(A - \alpha I)W_{i_j}$, where α runs over the set of all distinct principal curvatures of M except for the principal curvature k_{i_j}. We remark that $U_{i_j} \neq 0$ on the neighborhood $\mathcal{W}_p$ because $(U_{i_j})_p \neq 0$. Moreover, the vector field U_{i_j} satisfies $AU_{i_j} = k_{i_j}U_{i_j}$ on $\mathcal{W}_p$. We define V_{i_j} by normalizing U_{i_j} in some sense. That is, when $\Pi_{\alpha \neq k_{i_j}}(k_{i_j} - \alpha)(p) > 0$ (resp. $\Pi_{\alpha \neq k_{i_j}}(k_{i_j} - \alpha)(p) < 0$), we define $V_{i_j} = U_{i_j}/\|U_{i_j}\|$ (resp. $V_{i_j} = -U_{i_j}/\|U_{i_j}\|$). Then we know that $AV_{i_j} = k_{i_j}V_{i_j}$ on $\mathcal{W}_p$ and $(V_{i_j})_p = v_{i_j}$. Furthermore, our construction shows that the integral curve of V_{i_j} through the point p is a geodesic on M, so that in particular $\nabla_{V_{i_j}}V_{i_j} = 0$ at the point p.

Thanks to the Codazzi equation, at the point p we have

$$\begin{aligned}\text{(the left hand side of (2.6))} &= \langle(\nabla_{v_\ell}A)v_{i_j}, v_{i_j}\rangle = \langle(\nabla_{V_\ell}A)V_{i_j}, V_{i_j}\rangle_p \\ &= \langle\nabla_{V_\ell}(k_{i_j}V_{i_j}) - A\nabla_{V_\ell}V_{i_j}, V_{i_j}\rangle_p \\ &= \langle(V_\ell k_{i_j})V_{i_j} + (k_{i_j}I - A)\nabla_{V_\ell}V_{i_j}, V_{i_j}\rangle_p = v_\ell k_{i_j}\end{aligned}$$

and

$$\begin{aligned}\text{(the right hand side of (2.6))} &= \langle V_\ell, (\nabla_{V_{i_j}}A)V_{i_j}\rangle_p \\ &= \langle V_\ell, \nabla_{V_{i_j}}(k_{i_j}V_{i_j}) - A\nabla_{V_{i_j}}V_{i_j}\rangle_p \\ &= \langle v_\ell, (v_{i_j}k_{i_j})v_{i_j}\rangle = 0.\end{aligned}$$

Thus we can see that the differential dk_{i_j} of k_{i_j} vanishes at the point p, which shows that every $k_{i_j}(>0)$ is constant on $\mathcal{W}_p$, since we can take the point p as an arbitrarily fixed point of $\mathcal{W}_p$. So the principal curvature function k_{i_j} is constant locally on the open dense subset $\mathcal{U}$ of M. This, combined with the continuity of k_{i_j} and the connectivity of M, yields that k_{i_j} is constant on the hypersurface M. Hence all nonzero principal curvatures of M are constant, so that we obtain Condition (1). □

As an immediate consequence of Conditions (1) and (3) in Theorem 1 we have the following:

Theorem 2 ([4]). *Let M be a connected hypersurface in an $(n+1)$-dimensional sphere $S^{n+1}(c)$ $(n \geqq 2)$ of constant sectional curvature c. Then M is locally congruent to an isoparametric hypersurface with nozero principal curvatures in $S^{n+1}(c)$ if and only if for each point p of M there exists an orthonormal basis $\{v_1, \ldots, v_n\}$ of T_pM such that every geodesic of M through p with initial vector v_i is a small circle of positive curvature in $S^{n+1}(c)$.*

Isoparametric hypersurfaces in $S^{n+1}(c)$ with two distinct constant principal curvatures are called *Clifford hypersurfaces.* For a pair (c_1, c_2) of positive constants satisfying $1/c_1 + 1/c_2 = 1/c$ and a positive integer r with $1 \leqq r \leqq n-1$, we denote by $M_{r,n-r} = M_{r,n-r}(c_1, c_2)$ a naturally embedded hypersurface in $S^{n+1}(c)$ which is congruent to $S^r(c_1) \times S^{n-r}(c_2)$. It has two distinct constant constant principal curvatures $\lambda_1 = c_1/\sqrt{c_1+c_2}$ and $\lambda_2 = -c_2/\sqrt{c_1+c_2}$ with multiplicities r and $n-r$, respectively. Let $TM_{r,n-r} = V_{\lambda_1} \oplus V_{\lambda_2}$ be the decomposition into principal distributions coresponding to principal curvatures $\lambda_1,\ \lambda_2$.

Proposition 1 ([1]). *Let γ be a geodesic on a Clifford hypersurface $M_{r,n-r}(c_1, c_2)$ in $S^{n+1}(c)$. Then*

(1) *The curve γ is a geodesic in $S^{n+1}(c)$ if and only if the initial vector is of the form $\dot\gamma(0) = (\sqrt{c_2}\,w_1 + \sqrt{c_1}\,w_2)/\sqrt{c_1+c_2}$ with unit vectors $w_i \in V_{\lambda_i}$ $(i = 1, 2)$.*

(2) *If the initial vector $\dot\gamma(0)$ is neither principal nor of the form in (1), then γ is a circle (namely, a great circle or a small circle of positive curvature) on $S^{n+1}(c)$.*

Proof. Since $M_{r,n-r}$ has parallel shape operator, we find

$$\frac{d}{ds}\langle A\dot\gamma(s),\dot\gamma(s)\rangle = \langle(\nabla_{\dot\gamma}A)\dot\gamma(s),\dot\gamma(s)\rangle = 0.$$

Thus we may study geodesics at its initial point. We set $\dot\gamma(0) = a_1w_1 + a_2w_2$ with unit vectors $w_i \in V_{\lambda_i}$ $(i = 1, 2)$ and nonnegative constants a_1, a_2 satisfying $a_1^2 + a_2^2 = 1$. In this case we have $\langle A\dot\gamma(0),\dot\gamma(0)\rangle = a_1^2\lambda_1 + a_2^2\lambda_2$. Hence we can see that $\langle A\dot\gamma,\dot\gamma\rangle \equiv 0$ if and only if $a_1 = \sqrt{c_2}/\sqrt{c_1+c_2}$ and $a_2 = \sqrt{c_1}/\sqrt{c_1+c_2}$. Therefore, from Gauss formula (2.1), we get Statement (1).

Statement (2) is an immediate consequence of the proof of Theorem 1 and the above Statement (1). □

Paying attention to Proposition 1, we characterize all Clifford hypersurfaces $M_{r,n-r}(c_1, c_2)$ in $S^{n+1}(c)$.

Theorem 3 ([1])**.** *A connected hypersurface M in $S^{n+1}(c)$ is locally congruent to a Clifford hypersurface $M_{r,n-r}(c_1, c_2)$ with some r $(1 \leqq r \leqq n-1)$ if and only if there exist a function $d : M \to \{1, 2, \dots, n-1\}$, a constant α $(0 < \alpha < 1)$ and an orthonormal basis $\{v_1, \dots, v_n\}$ of T_pM at each point $p \in M$ satisfying the following two conditions*:

(1) *Every geodesic on M through p with initial vector v_i $(1 \leqq i \leqq n)$ is a small circle of positive curvature in $S^{n+1}(c)$;*

(2) *Every geodesic γ_{ij} on M through p with initial vector $\alpha v_i + \sqrt{1-\alpha^2}\, v_j$ $(1 \leqq i \leqq d(p) < j \leqq n)$ is a great circle in $S^{n+1}(c)$.*

In this case d is a constant function with $d \equiv r$ and

$$M = M_{n,n-r}(c/\alpha^2, c/(1-\alpha^2)).$$

Proof. The "only if" part follows from Theorem 2 and Proposition 1. So, we shall prove the "if" part. By Condition (1) we see that our real hypersurface M is isoparametric with nonzero principal curvatures in $S^{n+1}(c)$ (see Theorem 2). Consider a fixed point p_0. Setting $Av_i = \lambda_i v_i$ $(1 \leqq i \leqq n)$ at this point p_0, we can see that

$$\alpha^2\lambda_i + (1-\alpha^2)\lambda_j = 0 \quad \text{for } 1 \leqq i \leqq d(p_0) < j \leqq n. \tag{2.7}$$

Therefore M has just two distinct constant principal curvatures, so that M is locally congruent to some $M_{r,n-r}(c_1, c_2)$. Moreover, from the equalities $1/c_1 + 1/c_2 = 1/c$, $\lambda_i = c_1/\sqrt{c_1+c_2}$, $\lambda_j = -c_2/\sqrt{c_1+c_2}$ and (2.7) we can see that $c_1 = c/\alpha^2$ and $c_2 = c/(1-\alpha^2)$. □

Remark 1. In Theorem 3, setting $\alpha = \sqrt{r/n}$, we obtain a characterization of all *minimal* Clifford hypersurfaces $M = M_{r,n-r}(nc/r, nc/(n-r))$ in $S^{n+1}(c)$ (see [5]).

We finally study minimal isoparametric hypersurfaces with three distinct principal curvatures in $S^{n+1}(c)$ from the viewpoint of Theorem 1. Isoparametric hypersurfaces with three distinct principal curvatures are usually called *Cartan hypersurfaces.* If we denote by m_i the multiplicity of a principal curvature λ_i, then we find that these three principal curvatures have the same multiplicity (i.e.,

$m_1 = m_2 = m_3$) (see [6, 7]). When a Cartan hypersurface is minimal, it is congruent to one of the following hypersurfaces:

$$\begin{aligned} M^3 &= \mathrm{SO}(3)/(\mathbb{Z}_2 + \mathbb{Z}_2) \to S^4(c), \\ M^6 &= \mathrm{SU}(3)/T^2 \to S^7(c), \\ M^{12} &= \mathrm{Sp}(3)/\mathrm{Sp}(1) \times \mathrm{Sp}(1) \times \mathrm{Sp}(1) \to S^{13}(c), \\ M^{24} &= F_4/\mathrm{Spin}(8) \to S^{25}(c). \end{aligned}$$

Principal curvatures of a Cartan minimal hypersurface are $\sqrt{3c}$, 0, $-\sqrt{3c}$ (see [2, 3]).

Theorem 4 ([1])**.** *Let M be a connected hypersurface of $S^{n+1}(c)$. Suppose that at each point $p \in M$ there exists an orthogonal basis $\{v_1, \ldots, v_{m_p}\}$ of the orthogonal complement of* $\ker A_p$ *in T_pM ($m_p = \mathrm{rank}\ A_p$) such that*

(1) *all geodesics through p with initial vector v_i ($1 \leqq i \leqq m_p$) are small circles of positive curvature in $S^{n+1}(c)$,*
(2) *they have the same curvature k_p.*

Then $k_p = k$ (constant) on M and M is locally congruent either a totally umbilc hypersurface $S^n(c_1)$ with $k = \sqrt{c_1 - c}$, a Clifford hypersurface $M_{r,n-r}(2c, 2c)$ ($1 \leqq r \leqq n-1$) with $k = \sqrt{c}$, or a Cartan minimal hypersurface with $k = \sqrt{3c}$.

Proof. A totally umbilic hypersurface satisfies the hypothesis of Theorem 4 trivially. By the discussion in the proof of Theorem 1 we see that a hypersurface satisfying the hypothesis of Theorem 4 is isoparametric with at most three principal curvatures k, $-k$, 0 in $S^{n+1}(c)$. Thus we get the desired result. □

Remark 2. (1) In Theorems 3 and 4, we only need Condition (2) at some point $p_0 \in M$.
(2) If we add a condition that M is complete to assumptions of Theorems 1, 2, 3 and 4, then these theorems are global results. So, we can delete "locally" in these statements.
(3) In the assumptions of Theorems 1, 2, 3 and 4, we do not need to take the vectors $\{v_i\}$ as a local smooth field of orthonormal frames on M.

At the end of this paper, we pose the following open problem related to Theorem 1:

Problem. *Let M be a connected hypersurface in an $(n+1)$-dimensional sphere $S^{n+1}(c)$ ($n \geqq 2$) of constant sectional curvature c. If for each point p of M, there exists an orthonormal basis $\{v_1, \ldots, v_n\}$ of T_pM such that every geodesic of M through p with initial vector v_i is a circle (i.e., either a small circle of positive curvature or a great circle) in $S^{n+1}(c)$, then is M locally congruent to an isoparametric hypersurface in this ambient sphere?*

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