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QUASICONVEX DUALITY THEOREMS WITH QUASICONJUGATES AND GENERATOR

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Abstract. This paper is based on the author's thesis, "On duality theorems for quasiconvex programming". In this paper, we investigate duality theorems for quasiconvex programming as generalizations of results in convex programming, and consists of three topics. The first topic is about quasiconjugates and polar sets. The second is about three types of set containment characterizations. The third is about constraint qualifications for Lagrange-type duality theorem in quasiconvex programming.

1. Introduction

Mathematical programming is the use of mathematical models in order to assist in taking decisions, and is one of the most powerful techniques for making optimal decisions. In the research of mathematical programming, duality theorems have been investigated by many researchers. Especially, in convex programming, duality theorems are very useful and powerful tool to find a solution. The Fenchel conjugate is one of the well known conjugate function for convex functions, and by using this conjugate, some types of set containment characterizations have been investigated. Recently, these set containment characterizations imply the weakest constraint qualification for Lagrange-type duality theorems which play important roles in convex programming problem.

Quasiconvex functions is well known as a generalized notion of convex functions. Since the class of quasiconvex functions is wide, and include many functions which arise in mathematical programming problem in practice, quasiconvex programming can be applied to a lot of problems. In quasiconvex programming, various important theorems and notions were investigated, for example, quasiconjugates [9, 16, 17, 30, 31], duality theorems [9, 30, 31], and optimality conditions [15, 31]. However, these results are not generalizations of results in convex

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programming although quasiconvex programming is a generalization of convex programming.

In this paper, we investigate duality theorems for quasiconvex programming as generalizations of results in convex programming, by using quasiconjugates in [9, 16, 17, 30, 31] and Penot and Volle's result in [15].

This paper consists of three topics. Section 2 deals with quasiconjugates and polar sets. We introduce quasiconjugates in [9, 16, 17, 30, 31], and investigate mainly H-quasiconjugate. Also, we introduce various types of polar sets which play important roles in set containment characterizations.

Section 3 deals with three types of set containment characterizations. We investigate the characterization by using H and R-quasiconjugates, λ-quasi and λ-semiconjugates, and the generator of quasiconvex functions, respectively. Also, we introduce the notion of generator for quasiconvex functions, and show that set containment characterizations by using the generator is generalized results of set containment characterizations in convex programming. In the last of Section 3, we compare characterizations in this paper with the previous ones.

Section 4 deals with newly constraint qualifications for Lagrange-type duality theorem in quasiconvex programming. These constraint qualifications is concerned with the set containment characterization by using the notion of generator. In Section 4.1, we investigated the closed cone constraint qualification for quasiconvex programming. We show that this constraint qualification is a generalized notion of constraint qualification in convex programming and the weakest constraint qualification of Lagrange-type (strong) duality theorem for quasiconvex programming. In Section 4.2, we introduce a new subdifferential for quasiconvex functions by using generator. Also, we investigate optimality conditions as generalizations of convex ones, and establish the weakest constraint qualification for these optimality conditions.

2. Quasiconjugates and polar sets

It is well known that the Fenchel conjugate provides dual problems of convex programming problems. In a similar way, different notions of conjugate for quasiconvex functions have been introduced in order to obtain dual problems of quasiconvex programming problems. For example, the λ-quasiconjugate ($\lambda \in \mathbb{R}$), defined by Greenberg and Pierskalla [9], plays an important role in quasiconvex programming and in the theory of surrogate duality, corresponding to that of the Fenchel conjugate in convex programming and Lagrangian duality. But λ-quasiconjugate involves an extra parameter that many authors have tried to eliminate. Thach [30, 31] established two dualities without the extra parameter for a general quasiconvex minimization (maximization) problem, by using the concepts of H-quasiconjugate and R-quasiconjugate, which are similar to 1 and -1-quasiconjugate.

In this section, we investigate some quasiconjugates and polar sets. All results of this section is based on [20, 29].

2.1. **H-quasiconjugate and R-quasiconjugate.** First of all, we introduce the H-quasiconjugate of functions. In [30], Thach defined the H-quasiconjugate in $\mathbb{R}^n$, and in this section, we investigate the H-quasiconjugate also in $\mathbb{R}^n$.

Definition 2.1. [30] Let f be a function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$. The H-quasiconjugate of f is the function $f^H : \mathbb{R}^n \to \overline{\mathbb{R}}$ such that

$$f^H(u) = \begin{cases} -\inf\{f(x) \mid \langle u, x\rangle \geq 1\} & \text{if} \quad u \neq 0 \\ -\sup\{f(x) \mid x \in \mathbb{R}^n\} & \text{if} \quad u = 0. \end{cases}$$

The H-quasiconjugate of f^H, denoted by f^{HH}, is called the H-biquasiconjugate of f.

Clearly, $f^H(0) \leq f^H(u)$ for all $u \in \mathbb{R}^n \setminus \{0\}$, $f^{HH} \leq f$ on $\mathbb{R}^n \setminus \{0\}$, $f^H \leq g^H$ on $\mathbb{R}^n \setminus \{0\}$ when $f \geq g$ on $\mathbb{R}^n \setminus \{0\}$, and $f^H = g^H$ on $\mathbb{R}^n \setminus \{0\}$ when $f = g$ on $\mathbb{R}^n \setminus \{0\}$. Also we have the following inequalities.

Proposition 2.2. *The following inequalities hold:*

(i) $\sup_{u\in\mathbb{R}^n} f^H(u) \leq -\inf_{x\in\mathbb{R}^n} f(x)$,

(ii) $-\sup_{x\in\mathbb{R}^n} f(x) \leq \inf_{u\in\mathbb{R}^n\setminus\{0\}} f^H(u)$.

We can check that f^H is H-evenly quasiconvex in a similar way of [31]. We can also see that the equality $f^{HH}(0) = \inf\{f(x) \mid x \in \mathbb{R}^n \setminus \{0\}\}$ holds in Lemma 4.1 of [30]. From this equality, we characterize the identity $f = f^{HH}$ in the next theorem.

Theorem 2.3. *The following properties are satisfied:*

(i) $f = f^{HH}$ *on* $\mathbb{R}^n \setminus \{0\}$ *if* f *is* H*-evenly quasiconvex,*

(ii) $f = f^{HH}$ *if and only if* f *is* H*-evenly quasiconvex and*

$$f(0) = \inf\{f(x) \mid x \in \mathbb{R}^n \setminus \{0\}\}.$$

Proof. At first, we show that (i) holds. It is clear that $f(x) \geq f^{HH}(x)$ for all $x \in \mathbb{R}^n \setminus \{0\}$. Assume that there exists $x_0 \in \mathbb{R}^n \setminus \{0\}$ such that $f(x_0) > f^{HH}(x_0)$. We can choose $\alpha \in \mathbb{R}$ satisfying

$$f(x_0) > \alpha > f^{HH}(x_0),$$

and then $x_0 \notin L(f, \leq, \alpha)$. Since $L(f, \leq, \alpha)$ is H-evenly convex, there exists $v \in \mathbb{R}^n \setminus \{0\}$ such that $\langle v, x_0\rangle \geq 1 > \langle v, y\rangle$ for all $y \in L(f, \leq, \alpha)$ by using separation theorem. This shows that $f^H(v) = -\inf\{f(x) \mid \langle v, x\rangle \geq 1\} \leq -\alpha$. Hence,

$$f^{HH}(x_0) = -\inf\{f^H(u) \mid \langle u, x_0\rangle \geq 1\} \geq -f^H(v) \geq \alpha,$$

and this is a contradiction.

Next, we show that (ii) holds. By Lemma 4.1 of [30], $f^{HH}(0) = \inf\{f(x) \mid x \in \mathbb{R}^n \setminus \{0\}\}$. Hence, by using (i), we can prove that if f is H-evenly quasiconvex and $f(0) = \inf\{f(x) \mid x \in \mathbb{R}^n \setminus \{0\}\}$, then $f = f^{HH}$. The converse is clear since f^{HH} is H-evenly quasiconvex and $f^{HH}(0) = \inf\{f(x) \mid x \in \mathbb{R}^n \setminus \{0\}\}$. □

A function f is said to achieve the maximum value at infinity if $f(x_k) \to \sup\{f(x) \mid x \in \mathbb{R}^n\}$ for any sequence $\{x_k\}$ with $\|x_k\| \to +\infty$, and f is said to achieve the minimum value at the origin if $f(x_k) \to \inf\{f(x) \mid x \in \mathbb{R}^n \setminus \{0\}\}$ for any sequence $\{x_k\} \subset \mathbb{R}^n \setminus \{0\}$ with $x_k \to 0$. Let Γ^∞ and γ^0 be the set of all functions that achieve the maximum value at infinity, and the set of all functions that achieve the minimum value at the origin, respectively, that is,

$$\begin{aligned} \Gamma^\infty &= \{g : \mathbb{R}^n \to \overline{\mathbb{R}} \mid g \text{ achieves the maximum value at infinity}\}, \\ \gamma^0 &= \{g : \mathbb{R}^n \to \overline{\mathbb{R}} \mid g \text{ achieves the minimum value at the origin}\}. \end{aligned}$$

We denote by S^c the complement of $S \subset \mathbb{R}^n$ and by $B(z, r)$ the open ball centered at $z \in \mathbb{R}^n$ with radius $r > 0$.

Proposition 2.4. *The following properties are satisfied,*

(i) $f \in \Gamma^\infty$ *if and only if for any* $M < \sup\{f(x) \mid x \in \mathbb{R}^n\}$ *there exists* $\delta > 0$ *such that*

$$B(0, \delta)^c \subset L(f, \geq, M),$$

(ii) $f \in \gamma^0$ *if and only if for any* $m > \inf\{f(x) \mid x \in \mathbb{R}^n \setminus \{0\}\}$ *there exists* $\delta > 0$ *such that*

$$B(0, \delta) \setminus \{0\} \subset L(f, <, m).$$

Proof. We only show (ii), and we can show (i) in the similar way. Assume that f achieves the minimum value at the origin and there exists $m_0 > \inf\{f(x) \mid x \in \mathbb{R}^n \setminus \{0\}\}$ such that for any $\delta > 0$, there exists $x \in B(0, \delta) \setminus \{0\}$ such that $f(x) \geq m_0$. Then we can choose a sequence $\{x_k\} \subset \mathbb{R}^n \setminus \{0\}$ which converges to 0 and $f(x_k) \geq m_0$ for each $k \in \mathbb{N}$. This contradicts that f achieves the minimum value at the origin. Conversely, assume that for any $m > \inf\{f(x) \mid x \in \mathbb{R}^n \setminus \{0\}\}$, there exists $\delta > 0$ such that $B(0, \delta) \setminus \{0\} \subset L(f, <, m)$. If $\{x_k\} \subset \mathbb{R}^n \setminus \{0\}$ converges to 0, then there exists $K \in \mathbb{N}$ such that $\|x_k\| < \delta$ for any $k \geq K$. This shows that $\inf\{f(x) \mid x \in \mathbb{R}^n \setminus \{0\}\} \leq f(x_k) < m$ for any $k \geq K$. This shows that $\{f(x_k)\}$ converges to $\inf\{f(x) \mid x \in \mathbb{R}^n \setminus \{0\}\}$. □

According to [30], f is usc then f^H is lsc, and if a function $f \in \Gamma^\infty$ is lsc, then f^H is usc. Also, we investigate the following theorem.

Theorem 2.5. *The following properties are satisfied:*

(i) *If* $f \in \gamma^0$ *then* $f^H \in \Gamma^\infty$,

(ii) *if* $f \in \Gamma^\infty$ *then* $f^H \in \gamma^0$.

Proof. (i) Let $f \in \gamma^0$ and $\{u_k\} \subset \mathbb{R}^n$ be a sequence satisfying $\|u_k\|$ tends to $+\infty$. By using (ii) of Proposition 2.4, for any $m > \inf\{f(x) \mid x \in \mathbb{R}^n \setminus \{0\}\}$, there exists $\delta > 0$ such that

$$B(0, \delta) \setminus \{0\} \subset L(f, <, m).$$

Since $\|u_k\|$ tends to $+\infty$, we can find an integer K such that for any $k \geq K$, $\frac{u_k}{\|u_k\|^2} \in B(0, \delta) \setminus \{0\}$. By using (i) of Proposition 2.2, we can show that

$$\inf\{f(x) \mid x \in \mathbb{R}^n \setminus \{0\}\} \leq -\sup_{u \in \mathbb{R}^n} f^H(u) \leq -f^H(u_k) \leq f\left(\frac{u_k}{\|u_k\|^2}\right) < m$$

because $\left\langle u_k, \frac{u_k}{\|u_k\|^2} \right\rangle = 1$. This shows that $\{f^H(u_k)\}$ converges to $\sup\{f^H(u) \mid u \in \mathbb{R}^n\}$, and then $f^H \in \Gamma^\infty$.
(ii) Let $f \in \Gamma^\infty$ and $\{u_k\} \subset \mathbb{R}^n \setminus \{0\}$ be a sequence satisfying $\{u_k\}$ converges to 0. For any $M < \sup\{f(x) \mid x \in \mathbb{R}^n\}$, there exists $\delta > 0$ such that

$$B(0,\delta)^c \subset L(f, \geq, M),$$

by Proposition 2.4 (i). Since $\{u_k\}$ converges to 0, we can find an integer K such that for any $k \geq K$,

$$\{x \mid \langle u_k, x\rangle \geq 1\} \subset B(0,\delta)^c,$$

that is,

$$\langle u_k, x\rangle \geq 1 \Longrightarrow f(x) \geq M.$$

From this implication and by using (ii) of Proposition 2.2, we have

$$\sup\{f(x) \mid x \in \mathbb{R}^n\} \geq -\inf_{u \in \mathbb{R}^n \setminus \{0\}} f^H(u) \geq -f^H(u_k) \geq M,$$

for any $k \geq K$. This shows that $\{f^H(u_k)\}$ converges to $\inf\{f^H(u) \mid u \in \mathbb{R}^n \setminus \{0\}\}$, and then $f^H \in \gamma^0$. □

Next we show properties of level sets of H-biquasiconjugate. For this purpose, we prove the following proposition.

Proposition 2.6. *Let α, $\beta \in \mathbb{R}$, and $v \in \mathbb{R}^n \setminus \{0\}$. If $f \in \Gamma^\infty$ is lsc, then the following two conditions are equivalent:*

(i) $L(f, \leq, \beta) \subset \{x \mid \langle v, x\rangle < \alpha\}$,
(ii) $\exists \varepsilon > 0$ *s.t.* $L(f, <, \beta + \varepsilon) \subset \{x \mid \langle v, x\rangle < \alpha\}$.

Proof. We show that condition (i) implies condition (ii). Assume that $L(f, \leq, \beta) \subset \{x \mid \langle v, x\rangle < \alpha\}$ and for all $\varepsilon > 0$ there exists $x_\varepsilon \in L(f, <, \beta + \varepsilon)$ such that $\langle v, x_\varepsilon\rangle \geq \alpha$, then we can choose a sequence $\{x_k\} \subset \mathbb{R}^n$ such that for all $k \in \mathbb{N}$, $\beta < f(x_k) < \beta + \frac{1}{k}$ and $\langle v, x_k\rangle \geq \alpha$, and we have

$$f(x_k) \to \beta < f(x_1) \leq \sup\{f(x) \mid x \in \mathbb{R}^n\}.$$

If $\|x_k\| \to +\infty$, then $f(x_k) \to \sup\{f(x) \mid x \in \mathbb{R}^n\}$ since $f \in \Gamma^\infty$ and this is a contradiction. If $\{x_k\}$ is bounded, then we can choose a subsequence $\{x_{k_i}\}$ and $x_0 \in \mathbb{R}^n$ such that $x_{k_i} \to x_0$. Clearly $\langle v, x_0\rangle \geq \alpha$, but $x_0 \in L(f, \leq, \beta)$ since $f(x_0) \leq \liminf_{i\to\infty} f(x_{k_i}) = \beta$. This is a contradiction. The converse implication is obvious. □

Now we can give results on the level sets of the H-biquasiconjugate.

Theorem 2.7. *The following properties are satisfied:*

(i) $L(f, \leq, \alpha) \setminus \{0\} \subset L(f^{HH}, \leq, \alpha)$,
(ii) $L(f, <, \alpha) \setminus \{0\} \subset L(f^{HH}, <, \alpha)$,
(iii) *if* $\alpha \geq \inf\{f(x) \mid x \in \mathbb{R}^n \setminus \{0\}\}$, *then* $\mathrm{Hec}L(f, \leq, \alpha) \subset L(f^{HH}, \leq, \alpha)$,
(iv) $L(f^{HH}, <, \alpha) \subset \mathrm{Hec}L(f, <, \alpha)$,

(v) *if* $f \in \Gamma^\infty$ *is lsc, and* $\alpha \geq \inf\{f(x) \mid x \in \mathbb{R}^n \setminus \{0\}\}$, *then*

$$\mathrm{Hec}L(f, \leq, \alpha) = L(f^{HH}, \leq, \alpha) = \bigcap_{\varepsilon>0} \mathrm{Hec}L(f, <, \alpha + \varepsilon).$$

Proof. (i), (ii) and (iii) are obvious. At first, we show (iv). Assume that $x(\neq 0) \notin \mathrm{Hec}L(f, <, \alpha)$. By using separation theorem, there exists $a \in \mathbb{R}^n \setminus \{0\}$ such that $\langle a, x\rangle \geq 1 > \langle a, y\rangle$ for all $y \in L(f, <, \alpha)$. Then

$$f^{HH}(x) = -\inf\{f^H(v) \mid \langle v, x\rangle \geq 1\} \geq -f^H(a) = \inf\{f(y) \mid \langle a, y\rangle \geq 1\} \geq \alpha.$$

Therefore $x \notin L(f^{HH}, <, \alpha)$. If $L(f^{HH}, <, \alpha)$ contains 0, then $L(f, <, \alpha)$ is not empty since $\alpha > f^{HH}(0) = \inf\{f(x) \mid x \in \mathbb{R}^n \setminus \{0\}\}$. Hence $\mathrm{Hec}L(f, <, \alpha)$ contains 0.

Next we show (v). By using (iii) and (iv), we can check easily that

$$\mathrm{Hec}L(f, \leq, \alpha) \subset L(f^{HH}, \leq, \alpha) \subset \bigcap_{\varepsilon>0} \mathrm{Hec}L(f, <, \alpha + \varepsilon).$$

We assume that $x \notin \mathrm{Hec}L(f, \leq, \alpha)$. By using separation theorem, there exists $a \in \mathbb{R}^n \setminus \{0\}$ such that $\langle a, x\rangle \geq 1 > \langle a, y\rangle$ for all $y \in L(f, \leq, \alpha)$, that is, $L(f, \leq, \alpha) \subset \{y \mid \langle a, y\rangle < 1\}$. By using Proposition 2.6, there exists $\varepsilon_0 > 0$ such that $\langle a, x\rangle \geq 1 > \langle a, y\rangle$ for all $y \in L(f, <, \alpha + \varepsilon_0)$. By using separation theorem again, we have $x \notin \bigcap_{\varepsilon>0} \mathrm{Hec}L(f, <, \alpha + \varepsilon)$, and consequently

$$\bigcap_{\varepsilon>0} \mathrm{Hec}L(f, <, \alpha + \varepsilon) \subset \mathrm{Hec}L(f, \leq, \alpha).$$

This completes the proof. □

By the definition of H-quasiconjugate and Theorem 2.3, we can see

(i) $(\inf_{i\in I} f_i)^H = (\sup_{i\in I} f_i^H)$ on $\mathbb{R}^n \setminus \{0\}$, and

(ii) If f is H-evenly quasiconvex, then $f^{HH} = f$ on $\mathbb{R}^n \setminus \{0\}$.

When every f_i is H-evenly quasiconvex, by substituting f_i^H into (i) we have

$$(\inf_{i\in I} f_i^H)^{HH} = (\sup_{i\in I} f_i^{HH})^H = (\sup_{i\in I} f_i)^H$$

on $\mathbb{R}^n \setminus \{0\}$. However, the H-evenly quasiconvexity assumption is too strong because it assures $f_i(0) \leq f_i(x)$ for all $x \in \mathbb{R}^n$ and $i \in I$. The assumption of the next theorem is weaker than the previous one and guarantees that $(\inf_{i\in I} f_i^H)^{HH} = (\sup_{i\in I} f_i)^H$ on $\mathbb{R}^n \setminus \{0\}$.

Theorem 2.8. *Let I be an arbitrary index set, and f_i be an evenly quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$ for each $i \in I$. If the condition*

(A1) $\sup_{i\in I} f_i(x) > \sup_{i\in I} f_i(0)$ *for all* $x \in \mathbb{R}^n \setminus \{0\}$,

is satisfied, then

$$(\sup_{i\in I} f_i)^H(v) = (\inf_{i\in I} f_i^H)^{HH}(v) \text{ for all } v \in \mathbb{R}^n \setminus \{0\}.$$

Proof. In general, the following equality about H-quasiconjugate of inf-function is satisfied: for all $v \in \mathbb{R}^n \setminus \{0\}$,

$$(\inf_{i\in I} f_i)^H(v) = \sup_{i\in I} f_i^H(v),$$

see for example [30]. Then, for all $x \in \mathbb{R}^n \setminus \{0\}$, we have

$$(\inf_{i\in I} f_i^H)^H(x) = \sup_{i\in I} f_i^{HH}(x) \le \sup_{i\in I} f_i(x),$$

hence, for all $v \in \mathbb{R}^n \setminus \{0\}$, we have

$$(\sup_{i\in I} f_i)^H(v) \le (\inf_{i\in I} f_i^H)^{HH}(v).$$

If the equality does not hold in the above inequality, then there exists $v \in \mathbb{R}^n \setminus \{0\}$ such that $(\sup_{i\in I} f_i)^H(v) < (\inf_{i\in I} f_i^H)^{HH}(v)$, and hence, there exists $\alpha \in \mathbb{R}$ and $x' \in \mathbb{R}^n$ such that $\langle v, x'\rangle \ge 1$ and

$$(\sup_{i\in I} f_i)^H(v) < \alpha < \inf_{\langle w,x'\rangle \ge 1} \inf_{i\in I} f_i^H(w).$$

From $x' \neq 0$ and the assumption, we have $\sup_{i\in I} f_i(x') > \sup_{i\in I} f_i(0)$, and put $\varepsilon' = (\sup_{i\in I} f_i(x') - \sup_{i\in I} f_i(0))/2 > 0$. For all $\varepsilon \in (0, \varepsilon')$, there exists $i_0 \in I$ such that

$$f_{i_0}(x') > \sup_{i\in I} f_i(x') - \varepsilon > \sup_{i\in I} f_i(0) \ge f_{i_0}(0).$$

Since $L(f_{i_0}, \le, \sup_{i\in I} f_i(x') - \varepsilon)$ does not contain x', contains 0, and it is evenly convex, there exists $a \in \mathbb{R}^n \setminus \{0\}$ such that for all $x \in L(f_{i_0}, \le, \sup_{i\in I} f_i(x') - \varepsilon)$,

$$\langle a, x'\rangle \ge 1 > \langle a, x\rangle .$$

Therefore,

$$\alpha < \inf_{\langle w,x'\rangle \ge 1} \inf_{i\in I} f_i^H(w) \le f_{i_0}^H(a) \le -\sup_{i\in I} f_i(x') + \varepsilon.$$

Since ε is arbitrary, we have

$$\alpha \le -\sup_{i\in I} f_i(x') \le -\inf_{\langle v,x\rangle \ge 1} \sup_{i\in I} f_i(x) = (\sup_{i\in I} f_i)^H(v).$$

This is a contradiction. □

Next, we introduce the R-quasiconjugate.

Definition 2.9. [31] The R-quasiconjugate of f is the function $f^R : X \to \overline{\mathbb{R}}$ such that

$$f^R(u) = -\inf\{f(x) \mid \langle u, x\rangle \ge -1\}, \forall u \in X.$$

Thach [31] investigated the R-quasiconjugate and introduced the following duality theorem.

Let A be a nonempty convex subset of $\mathbb{R}^n$, and f is a quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$. Consider the following problem (P) and (D),

$$(P) \begin{cases} \text{minimize } f(x), \\ \text{subject to } x \in A, \end{cases}$$

$$(D) \begin{cases} \text{minimize } f^R(u), \\ \text{subject to } u \in \{v \in \mathbb{R}^n \mid \forall x \in A, \langle v, x \rangle \leq 1\}. \end{cases}$$

Let $v(P) = \inf\{f(x) \mid x \in A\}$ and $v(D) = \inf\{f^R(u) \mid \forall x \in A, \langle u, x \rangle \leq 1\}$, then $v(P) \geq -v(D)$. Thach investigated necessary and sufficient conditions for $v(P) = -v(D)$.

Also, Thach introduced some optimality conditions by using R-quasiconjugate.

2.2. **λ-quasiconjugate and λ-semiconjugate.** Next, we introduce two concepts of quasiconjugate for quasiconvex functions. In [9], the following λ-quasiconjugate was investigated in $\mathbb{R}^n$. In this thesis, we define the λ-quasiconjugates in a locally convex topological vector space X.

Definition 2.10. [9] The λ-quasiconjugate of f is the function $f^\nu_\lambda : X^* \to \overline{\mathbb{R}}$ such that

$$f^\nu_\lambda(u) = \lambda - \inf\{f(x) \mid \langle u, x \rangle \geq \lambda\}, \forall u \in X^*.$$

Definition 2.11. [16] The λ-semiconjugate of f is the function $f^\theta_\lambda : X^* \to \overline{\mathbb{R}}$ such that

$$f^\theta_\lambda(u) = \lambda - \inf\{f(x) \mid \langle u, x \rangle > \lambda\}, \forall u \in X^*.$$

Singer [16] defined the λ-semiconjugate in the following form,

$$f^\theta_\lambda(u) = \lambda - 1 - \inf\{f(x) \mid \langle u, x \rangle > \lambda - 1\}, \forall u \in X^*.$$

But we redefine the λ-semiconjugate in this paper. Clearly, $f^\nu_1 = f^H + 1$ and $f^\nu_{-1} = f^R - 1$. Also, we can check that f^ν_λ is H-evenly quasiconvex and f^θ_λ is lsc H-evenly quasiconvex if $\lambda > 0$ in the similar way of [16, 31]. Also, in [9, 16, 17, 31], researchers investigated the condition which guarantees that a function is equal to its biconjugate function and duality theorems by these quasiconjugates.

2.3. **Polar sets.** In this section, we introduce some types of polar sets in locally convex topological vector spaces.

Definition 2.12. Let A be a nonempty subset of X and $\alpha \in \mathbb{R}$. We define polar sets as follows.

$$A^{*(<,\alpha)} = \{v \in X^* \mid \forall x \in A, \langle v, x \rangle < \alpha\},$$
$$A^{*(\leq,\alpha)} = \{v \in X^* \mid \forall x \in A, \langle v, x \rangle \leq \alpha\}.$$

Also, S be a nonempty subset of X^* and $\alpha \in \mathbb{R}$, we define polar sets as follows. $S^{*(<,\alpha)} = \{x \in X \mid \forall v \in S, \langle v, x \rangle < \alpha\}$, and $S^{*(\leq,\alpha)} = \{x \in X \mid \forall v \in S, \langle v, x \rangle \leq \alpha\}$. Clearly, if $\alpha > 0$ then $A^{*(<,\alpha)}$ is H-evenly convex, $(A^{*(<,\alpha)})^{*(<,\alpha)}$ is $\mathrm{Hec}A$, $A^{*(\leq,\alpha)}$ is closed H-evenly convex, and $(A^{*(\leq,\alpha)})^{*(\leq,\alpha)}$ is $\mathrm{cl}\,\mathrm{Hec}A$. Moreover, for all $\alpha \in \mathbb{R}$,

$$\left(\bigcup_{i \in I} A_i\right)^{*(\leq,\alpha)} = \bigcap_{i \in I} \left(A_i^{*(\leq,\alpha)}\right) \text{ and } \left(\bigcup_{i \in I} A_i\right)^{*(<,\alpha)} = \bigcap_{i \in I} \left(A_i^{*(<,\alpha)}\right).$$

Also, we introduce following two propositions.

Proposition 2.13. *Let I be an arbitrary set, A_i be a nonempty H-evenly convex subset of X for each $i \in I$, and $\alpha > 0$.*

Then,

$$\left(\bigcap_{i\in I} A_i\right)^{*(<,\alpha)} = \operatorname{Hec}\bigcup_{i\in I}\left(A_i^{*(<,\alpha)}\right).$$

Furthermore, if A_i is closed for each $i \in I$, then

$$\left(\bigcap_{i\in I} A_i\right)^{*(\leq,\alpha)} = \operatorname{cl}\operatorname{Hec}\bigcup_{i\in I}\left(A_i^{*(\leq,\alpha)}\right).$$

Proof. Let $S_i \subset X^*$ for each $i \in I$, then it is clear that $\left(\bigcup_{i\in I} S_i\right)^{*(<,\alpha)} = \bigcap_{i\in I}\left(S_i^{*(<,\alpha)}\right)$. Hence, for $\{A_i^{*(<,\alpha)} \mid i \in I\}$,

$$\left(\bigcup_{i\in I} A_i^{*(<,\alpha)}\right)^{*(<,\alpha)} = \bigcap_{i\in I}\left((A_i^{*(<,\alpha)})^{*(<,\alpha)}\right) = \bigcap_{i\in I} A_i$$

by the assumption of A_i. Therefore,

$$\left(\bigcap_{i\in I} A_i\right)^{*(<,\alpha)} = \left(\left(\bigcup_{i\in I} A_i^{*(<,\alpha)}\right)^{*(<,\alpha)}\right)^{*(<,\alpha)} = \operatorname{Hec}\bigcup_{i\in I}\left(A_i^{*(<,\alpha)}\right).$$

The proof of second equation is similar. □

Proposition 2.14. *Let A be a nonempty subset of X and $\alpha \in \mathbb{R}$. Then, following statements hold.*

(i) $(\operatorname{cl}\operatorname{co}A)^{*(\leq,\alpha)} = A^{*(\leq,\alpha)}$ *and* $(\operatorname{ec}A)^{*(<,\alpha)} = A^{*(<,\alpha)}$,

(ii) *if* $\alpha > 0$ *then* $(\operatorname{cl}\operatorname{Hec}A)^{*(\leq,\alpha)} = A^{*(\leq,\alpha)}$ *and* $(\operatorname{Hec}A)^{*(<,\alpha)} = A^{*(<,\alpha)}$.

Proof. At first, we show the statement (i). By separation theorem, for all $v \in X^*$ and $\alpha \in \mathbb{R}$, $A \subset \{x \in X \mid \langle v, x\rangle \leq \alpha\}$ if and only if $\operatorname{cl}\operatorname{co}A \subset \{x \in X \mid \langle v, x\rangle \leq \alpha\}$. Also, by separation theorem, $A \subset \{x \in X \mid \langle v, x\rangle < \alpha\}$ if and only if $\operatorname{ec}A \subset \{x \in X \mid \langle v, x\rangle < \alpha\}$, this completes the proof of (i). The proof of (ii) is similar by separation theorem. □

3. Set containment characterization

Classification is one of the basic problems in data mining which addresses the question of how best to use historical data to improve the process of making decisions and to discover regularities. Motivated by general nonpolyhedral knowledge-based data classification, the containment problem which consists of characterizing the inclusion $A \subset B$, where

$$A = \{x \in X \mid \forall i \in I, f_i(x) \leq 0,\ \forall j \in J, g_j(x) < 0\},$$

and

$$B = \{x \in X \mid \forall s \in S, k_s(x) < 0,\ \forall w \in W, h_w(x) \leq 0\},$$

was studied by many researchers [6, 8, 10, 14]. The first characterizations were given by Mangasarian [14] for linear systems and for systems involving differentiable convex functions, and keys to this approach were Farkas' Lemma and the duality theorems of convex programming, respectively. In [10], Jeyakumar proved the following set containment characterizations.

Theorem 3.1. [10] *Let I be an arbitrary set, and for each $i \in I$, let f_i be a convex function from $\mathbb{R}^n$ to $\mathbb{R}$. In addition, let $\{x \in \mathbb{R}^n \mid \forall i \in I, f_i(x) \leq 0\}$ be a non-empty set, $u \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$. Then, (i) and (ii) given below are equivalent:*

(i) $\{x \in \mathbb{R}^n \mid \forall i \in I, f_i(x) \leq 0\} \subset \{x \in \mathbb{R}^n \mid \langle u, x\rangle \leq \alpha\}$,
(ii) $(u, \alpha) \in \mathrm{cl}\,\mathrm{cone}\,\mathrm{co} \bigcup_{i \in I} \mathrm{epi} f_i^*$.

Theorem 3.2. [10] *Let I be an arbitrary set, and for each $i \in I$, let f_i be a convex function from $\mathbb{R}^n$ to $\mathbb{R}$, $\{x \in \mathbb{R}^n \mid \forall i \in I, f_i(x) \leq 0\}$ be a non-empty set. In addition, let h be a convex function from $\mathbb{R}^n$ to $\mathbb{R}$. Then, (i) and (ii) given below are equivalent:*

(i) $\{x \in \mathbb{R}^n \mid \forall i \in I, f_i(x) \leq 0\} \subset \{x \in \mathbb{R}^n \mid h(x) \geq 0\}$,
(ii) $(0, 0) \in \mathrm{epi} h^* + \mathrm{cl}\,\mathrm{cone}\,\mathrm{co} \bigcup_{i \in I} \mathrm{epi} f_i^*$.

Jeyakumar investigated Theorem 3.1 and 3.2 in $\mathbb{R}^n$. However, these results are also valid in locally convex topological vector space because these results were proved by separation theorem. Also, these characterizations are generalization of Mangasarian's characterization, and play important roles in convex programming problems.

In this section, we investigate three types of set containment characterizations for quasiconvex programming, by using H and R-quasiconjugates, λ-quasi and λ-semiconjugates, and the generator of quasiconvex functions, respectively. This section is based on [20, 29, 24].

3.1. **By H-quasiconjugate and R-quasiconjugate.** In this section, we establish dual characterizations of the set containment in $\mathbb{R}^n$, assuming the quasiconvexity of g_j, $j \in J$, the linearity or quasiconcavity of k_s, $s \in S$, that A is defined by strict inequalities and B by both types of inequalities, so that A is convex whereas B is either convex or reverse convex. The dual characterizations are provided in terms of level sets of H-quasiconjugate and R-quasiconjugate of quasiconvex functions.

We present a characterization of the containment of a convex set, defined by quasiconvex constraints, in an open halfspace. We start with a result on the containment for the case $|J| = 1$.

Theorem 3.3. *Let $v \in \mathbb{R}^n \setminus \{0\}$, $\alpha \in (0, \infty)$ and $\beta \in \mathbb{R}$. Then the following conditions are equivalent:*

(i) $L(g, <, \beta) \subset \{x \mid \langle v, x\rangle < \alpha\}$,
(ii) $\frac{v}{\alpha} \in L(g^H, \leq, -\beta)$.

Proof. Assume that $L(g,<,\beta) \subset \{x \mid \langle v,x\rangle < \alpha\}$, that is, $g(x) < \beta$ implies $\langle v,x\rangle < \alpha$ or, equivalently, $\left\langle \frac{v}{\alpha},x\right\rangle \geq 1$ implies $g(x) \geq \beta$. This shows that

$$g^H\left(\frac{v}{\alpha}\right) = -\inf\left\{g(x) \;\middle|\; \left\langle \frac{v}{\alpha},x\right\rangle \geq 1\right\} \leq -\beta.$$

Conversely, if $g^H(\frac{v}{\alpha}) \leq -\beta$, then $\inf\{g(x) \mid \left\langle \frac{v}{\alpha},x\right\rangle \geq 1\} \geq \beta$. Therefore the inequality $\left\langle \frac{v}{\alpha},x\right\rangle \geq 1$ implies $g(x) \geq \beta$, that is, $g(x) < \beta$ implies $\langle v,x\rangle < \alpha$. Thus $L(g,<,\beta) \subset \{x \mid \langle v,x\rangle < \alpha\}$. □

The theorem is valid when the constraint function is unique. Substituting $\sup_{j\in J} g_j$ into g, we have

$$L(\sup_{j\in J} g_j,<,\beta) \subset \{x \mid \langle v,x\rangle < \alpha\} \Longleftrightarrow \frac{v}{\alpha} \in L((\sup_{j\in J} g_j)^H,\leq,-\beta),$$

for $v \in \mathbb{R}^n \setminus \{0\}$, $\alpha \in (0,\infty)$ and $\beta \in \mathbb{R}$. Combining Theorems 2.8 and 3.3, we get the first characterization theorem.

Theorem 3.4. *Let J be an arbitrary index set, and g_j be an evenly quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$ for each $j \in J$, and assume the condition (A1):*

(A1) $\quad \forall x \in \mathbb{R}^n \setminus \{0\} \quad \sup_{j\in J} g_j(x) > \sup_{j\in J} g_j(0).$

Then, for $v \in \mathbb{R}^n \setminus \{0\}$, $\alpha \in (0,\infty)$ and $\beta \in \mathbb{R}$, the following conditions (i) and (ii) are equivalent:

(i) $\{x \in \mathbb{R}^n \mid \sup_{j\in J} g_j(x) < \beta\} \subset \{x \in \mathbb{R}^n \mid \langle v,x\rangle < \alpha\}$,

(ii) $\dfrac{v}{\alpha} \in L((\inf_{j\in J} g_j^H)^{HH},\leq,-\beta)$.

From Theorem 2.7 (iii) and Theorem 2.8,

$$\mathrm{Hec}\bigcup_{j\in J} L(g_j^H,\leq,-\beta) \subset L((\inf_{j\in J}(g_j^H))^{HH},\leq,-\beta) = L((\sup_{j\in J} g_j)^H,\leq,-\beta),$$

which yields the following result.

Corollary 3.5. *Let $v \in \mathbb{R}^n \setminus \{0\}$, $\alpha \in (0,\infty)$ and $\beta \in \mathbb{R}$. If there exists $m \in \mathbb{N}$, $v_1,\dots,v_m \in \mathbb{R}^n$, and $\lambda_1,\dots,\lambda_m \in [0,\infty)$ with $\sum_{k=1}^m \lambda_k \leq 1$ such that*

$$\frac{v}{\alpha} = \sum_{k=1}^m \lambda_k v_k \text{ and for all } k \in \{1,\cdots,m\}, g_j^H(v_k) \leq -\beta \text{ for some } i \in I$$

then,

$$\{x \in \mathbb{R}^n \mid \sup_{j\in J} g_j(x) < \beta\} \subset \{x \in \mathbb{R}^n \mid \langle v,x\rangle < \alpha\}.$$

Next, we show a result on the containment for the case J is arbitrary.

Theorem 3.6. *Let J be an arbitrary index set, and g_j be an evenly quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$ for each $j \in J$. Assume that the following conditions (A1) and (A2) are satisfied:*

(A1) $\forall x \in \mathbb{R}^n \setminus \{0\} \quad \sup_{j\in J} g_j(x) > \sup_{j\in J} g_j(0),$

(A2) $\inf_{j\in J}(g_j^H)$ *is l.s.c and included in* Γ^∞.

Then for $v \in \mathbb{R}^n \setminus \{0\}$, $\alpha \in (0,\infty)$ *and* $\beta \in \mathbb{R}$, *the following statements are equivalent:*

(i) $L(\sup_{j\in J} g_j, <, \beta) \subset \{x \in \mathbb{R}^n \mid \langle v, x\rangle < \alpha\}$,

(ii) $\dfrac{v}{\alpha} \in \mathrm{Hec}L(\inf_{j\in J} g_j^H, \leq, -\beta)$.

Proof. Firstly, we show that (ii) implies (i). Assume that (ii) holds. Then, from Theorem 2.7 (iii) and Theorem 2.8,

$$\mathrm{Hec}L(\inf_{j\in J} g_j^H, \leq, -\beta) \subset L((\inf_{j\in J}(g_j^H))^{HH}, \leq, -\beta) = L((\sup_{j\in J} g_j)^H, \leq, -\beta).$$

Then we have $\frac{v}{\alpha} \in L((\sup_{j\in J} g_j)^H, \leq, -\beta)$ and, by Theorem 3.3, (i) is derived. Next, we show (i) implies (ii). By using Theorem 2.8 and Theorem 3.3, (i) implies

$$\frac{v}{\alpha} \in L((\inf_{j\in J}(g_j^H))^{HH}, \leq, -\beta).$$

From assumption (A2) and Theorem 2.7, we get

$$L((\inf_{j\in J}(g_j^H))^{HH}, \leq, -\beta) = \mathrm{Hec}L(\inf_{j\in J} g_j^H, \leq, -\beta).$$

Then (ii) is satisfied. □

In the following Corollary 3.7, we show a set containment characterization when all g_j are usc quasiconvex, J is a finite set, all k_s are affine, S is an arbitrary set.

Corollary 3.7. *Let* J *be a finite set,* S *be an arbitrary set,* g_j *be an usc quasiconvex function from* $\mathbb{R}^n$ *to* $\overline{\mathbb{R}}$ *and included in* γ^0 *for each* $j \in J$, $v_s \in \mathbb{R}^n \setminus \{0\}$ *and* $\alpha_s \in (0,\infty)$ *for each* $s \in S$. *If condition (A1) holds, then the following conditions (i) and (ii) are equivalent:*

(i) $\{x \in \mathbb{R}^n \mid \forall j \in J, g_j(x) < \beta\} \subset \{x \in \mathbb{R}^n \mid \forall s \in S, \langle v_s, x\rangle < \alpha_s\}$,

(ii) $\forall s \in S$, $\dfrac{v_s}{\alpha_s} \in \mathrm{Hec} \bigcup_{j\in J} L(g_j^H, \leq, -\beta)$.

Proof. We can check that $\bigcup_{j\in J} L(g_j^H, \leq, -\beta) = L(\inf_{j\in J} g_j^H, \leq, -\beta)$, $L(\sup_{j\in J} g_j, <, \beta) = \{x \in \mathbb{R}^n \mid \forall j \in J, g_j(x) < \beta\}$ and (A2) by using assumptions. □

We present a characterization of the containment of a convex set, defined by finitely many constraints, in a reverse convex set, defined by a quasiconvex constraint.

Theorem 3.8. *Let* $v \in \mathbb{R}^n \setminus \{0\}$, $\beta \in \mathbb{R}$. *Then the following conditions are equivalent:*

(i) $L(g, <, \beta) \subset \{x \mid \langle v, x\rangle < -1\}$,

(ii) $v \in L(g^R, \leq, -\beta)$.

The proof is similar to the one of Theorem 3.3, hence it is omitted.

Theorem 3.9. *Let g and h be usc quasiconvex functions from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$. Assume that $L(h,<,\alpha) \neq \emptyset$ and $0 \in L(g,<,\beta)$ for some α, $\beta \in \mathbb{R}$. Then the following conditions are equivalent:*

(i) $L(g,<,\beta) \subset L(h,\geq,\alpha)$,
(ii) $0 \in L(g^H,\leq,-\beta) \setminus \{0\} + L(h^R,\leq,-\alpha) \setminus \{0\}$,
(iii) *there exists $v \in \mathbb{R}^n \setminus \{0\}$ such that $g^H(v) \leq -\beta$ and $h^R(-v) \leq -\alpha$.*

Proof. It is clear that (ii) and (iii) are equivalent. Since $L(g,<,\beta)$ and $L(h,<,\alpha)$ are nonempty open convex subsets and $0 \in L(g,<,\beta)$, we have

$$\begin{aligned}
\text{(i)} \quad &\iff \quad L(g,<,\beta) \cap L(h,<,\alpha) = \emptyset \\
&\iff \quad \exists v \in \mathbb{R}^n \setminus \{0\},\ \exists \gamma \in \mathbb{R} \ \text{ s.t.} \\
&\qquad\quad \langle v, x\rangle > \gamma > \langle v, x'\rangle, \forall x \in L(h,<,\alpha), \forall x' \in L(g,<,\beta) \\
&\iff \quad \exists v \in \mathbb{R}^n \setminus \{0\} \ \text{ s.t.} \\
&\qquad\quad \langle v, x\rangle > 1 > \langle v, x'\rangle, \forall x \in L(h,<,\alpha), \forall x' \in L(g,<,\beta) \\
&\iff \quad \exists v \in \mathbb{R}^n \setminus \{0\} \ \text{ s.t. } \ g^H(v) \leq -\beta \text{ and } h^R(-v) \leq -\alpha,
\end{aligned}$$

by separation theorem. □

Substituting $\sup_{j\in J} g_j$ into g, we obtain the following theorem.

Theorem 3.10. *Let J be a finite set, g_j be an usc quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$ for each $j \in J$, and h be an usc quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$. Assume that $L(h,<,\alpha) \neq \emptyset$ and $\sup_{j\in J} g_j(0) < \beta$ for some α, $\beta \in \mathbb{R}$. Then the following conditions are equivalent:*

(i) $\{x \in \mathbb{R}^n \mid \forall j \in J, g_j(x) < \beta\} \subset L(h,\geq,\alpha)$,
(ii) $0 \in L((\sup_{j\in J} g_j)^H,\leq,-\beta) \setminus \{0\} + L(h^R,\leq,-\alpha) \setminus \{0\}$,
(iii) *there exists $v \in \mathbb{R}^n \setminus \{0\}$ such that $(\sup_{j\in J} g_j)^H(v) \leq -\beta$ and $h^R(-v) \leq -\alpha$.*

Proof. Since J is a finite set, we have $\sup_{j\in J} g_j$ is usc and $L(\sup_{j\in J} g_j,\leq,\beta) = \{x \in \mathbb{R}^n \mid \forall j \in J, g_j(x) < \beta\}$. By using Theorem 3.9, we conclude the proof. □

The next corollary characterizes the set containment in the case that all g_j are usc quasiconvex, J is a finite set, all h_w are usc quasiconvex, W is an arbitrary set.

Corollary 3.11. *Let J be a finite set, W be an arbitrary set, g_j be an usc quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$ included in γ^0 for each $j \in J$, h_w be an usc quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$ and $\alpha_w \in (0,\infty)$ for each $w \in W$. Assume that*

(A1) $\forall x \in \mathbb{R}^n \setminus \{0\} \quad \sup_{j\in J} g_j(x) > \sup_{j\in J} g_j(0)$,

and $L(h_w,<,\alpha_w) \neq \emptyset$ for each $j \in J$ and $\sup_{i\in I} f_i(0) < \beta$ for some $\beta \in \mathbb{R}$. Then the following conditions are equivalent:

(i) $\{x \in \mathbb{R}^n \mid \forall j \in J, g_j(x) < \beta\} \subset \{x \in \mathbb{R}^n \mid \forall w \in W, h_w(x) \geq \alpha_w\}$,
(ii) *for each $w \in W$,* $0 \in \operatorname{Hec} \bigcup_{j\in J} L(g_j^H,\leq,-\beta) \setminus \{0\} + L(h_w^R,\leq,-\alpha_w) \setminus \{0\}$,

(iii) *for each $w \in W$, there exists $v \in \mathbb{R}^n \setminus \{0\}$ such that*

$$v \in \mathrm{Hec} \bigcup_{j \in J} L(g_j^H, \leq, -\beta) \text{ and } h_w^R(-v) \leq -\alpha_w.$$

3.2. **By λ-quasiconjugate and λ-semiconjugate.** In this section, we show set containment characterizations, by λ-quasiconjugate and λ-semiconjugate. These dual characterizations are provided in terms of level sets of λ-quasiconjugate and λ-semiconjugate, especially 1, -1-quasiconjugate, 1-semiconjugate. In [29], we investigated characterizations in $\mathbb{R}^n$. However, in this section we investigate characterizations in a locally convex topological vector space.

At first, we introduce the following proposition in a locally convex topological vector space X.

Proposition 3.12. *Let A be a convex subset of X, $x \in \mathrm{cl}A$ and $y \in \mathrm{int}A$. Then, for all $\alpha \in (0, 1)$, $(1-\alpha)x + \alpha y \in \mathrm{int}A$.*

Proof. Since $y \in \mathrm{int}A$, there exists a convex and symmetric neighborhood U of 0 such that $y + U \subset A$. Then, $(1-\alpha)x + \alpha y + \alpha^2 U \subset A$. Actually, since $x \in \mathrm{cl}A$ and U is symmetric, there exists $z \in \alpha U$ such that $z + x \in A$ and $-z \in \alpha U$. For all $a \in \alpha^2 U$,

$$\frac{a + (1-\alpha)(-z)}{\alpha} \in \alpha U + (1-\alpha)U \subset U,$$

that is $y + \frac{a+(1-\alpha)(-z)}{\alpha} \in A$. Since A is convex,

$$(1-\alpha)x + \alpha y + a = (1-\alpha)(z + x) + \alpha \left(y + \frac{a + (1-\alpha)(-z)}{\alpha} \right) \in A.$$

This completes the proof. □

The following proposition, which concerns the closure of the intersection of a family of convex sets, plays an important role in set containment characterizations.

Proposition 3.13. *Let I be an arbitrary set, and A_i be a convex subset of X for each $i \in I$. If $\mathrm{int} \cap_{i \in I} A_i$ is nonempty, then $\mathrm{cl} \cap_{i \in I} A_i = \cap_{i \in I} \mathrm{cl}A_i$.*

Proof. Let $x \in \cap_{i \in I} \mathrm{cl}A_i$. Since $\mathrm{int} \cap_{i \in I} A_i \neq \emptyset$, there exists $z \in \mathrm{int} \cap_{i \in I} A_i$. Then for each $i \in I$, $\{(1-\alpha)x + \alpha z \mid \alpha \in (0, 1]\} \subset \mathrm{int}A_i$, because of Proposition 3.12. Therefore $\{(1-\alpha)x + \alpha z \mid \alpha \in (0, 1]\} \subset \cap_{i \in I} A_i$, that is, $x \in \mathrm{cl} \bigcap_{i \in I} A_i$. The converse is clear. □

We present characterizations of the containment of a convex set, defined by infinite quasiconvex constraints, in an evenly convex set.

In the beginning, we show a result of the containment when $|J| = |S| = 1$.

Theorem 3.14. *Let g be a function from X to $\overline{\mathbb{R}}$, $v \in X^* \setminus \{0\}$, $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$. Then, following conditions (i), (ii) and (iii) are equivalent:*

(i) $L(g, <, \beta) \subset \{x \mid \langle v, x \rangle < \alpha\}$,

(ii) $v \in (L(g, <, \beta))^{*(<, \alpha)}$,

(iii) $v \in L(g_\alpha^\nu, \leq, \alpha - \beta)$.

Proof. It is clear that (i) and (ii) are equivalent. Assume that $L(g,<,\beta) \subset \{x \mid \langle v,x\rangle < \alpha\}$, then the implication $g(x) < \beta$ implies $\langle v,x\rangle < \alpha$, or equivalently, $\langle v,x\rangle \geq \alpha$ implies $g(x) \geq \beta$ holds. This shows

$$g_\alpha^\nu(v) = \alpha - \inf\{g(x) \mid \langle v,x\rangle \geq \alpha\} \leq \alpha - \beta.$$

Conversely, if $g_\alpha^\nu(v) \leq \alpha - \beta$, then $\inf\{g(x) \mid \langle v,x\rangle \geq \alpha\} \geq \beta$. Hence, the implication $\langle v,x\rangle \geq \alpha$ implies $g(x) \geq \beta$, or $g(x) < \beta$ implies $\langle v,x\rangle < \alpha$ holds. This derives $L(g,<,\beta) \subset \{x \mid \langle v,x\rangle < \alpha\}$. □

Next, we show the set containment characterization, assuming that all g_j $(j \in J)$ are strictly evenly quasiconvex, J and S are possibly infinite.

Theorem 3.15. *Let J, S be arbitrary sets, $\beta \in \mathbb{R}$, g_j be a strictly evenly quasiconvex function from X to $\overline{\mathbb{R}}$ for each $j \in J$, and $v_s \in X^*$ and $\alpha_s \in (0,\infty)$ for each $s \in S$. Assume that $g_j(0) < \beta$ for each $j \in J$. Then, following conditions (i) and (ii) are equivalent:*

(i) $\{x \in X \mid \forall j \in J, g_j(x) < \beta\} \subset \{x \in X \mid \forall s \in S, \langle v_s,x\rangle < \alpha_s\}$,

(ii) $\forall s \in S,\ \dfrac{v_s}{\alpha_s} \in \mathrm{Hec}\bigcup_{j\in J} L((g_j)_1^\nu, \leq, 1-\beta)$.

Proof. It is clear that (i) and

$$\forall s \in S,\ \frac{v_s}{\alpha_s} \in \left(\bigcap_{j\in J} L(g_j,<,\beta)\right)^{*(<,1)}$$

are equivalent. By using the assumption, $L(g_j,<,\beta)$ is a H-evenly convex set for each $j \in J$. Hence, by using Proposition 2.13, $\frac{v_s}{\alpha_s} \in \mathrm{Hec}\bigcup_{j\in J}(L(g_j,<,\beta))^{*(<,1)}$ for each $s \in S$. Furthermore, by using Theorem 3.14, $\frac{v_s}{\alpha_s} \in \mathrm{Hec}\bigcup_{j\in J} L((g_j)_1^\nu, \leq, 1-\beta)$ for each $s \in S$. □

In the following theorem, we show the set containment characterization, assuming that all f_i $(i \in I)$ are evenly quasiconvex, I and S are possibly infinite.

Theorem 3.16. *Let I and S be arbitrary sets, $\beta \in \mathbb{R}$, f_i be an evenly quasiconvex function from X to $\overline{\mathbb{R}}$ for each $i \in I$, and $v_s \in X^*$ and $\alpha_s \in (0,\infty)$ for each $s \in S$. Assume that $f_i(0) \leq \beta$ for each $i \in I$. Then, following conditions (i) and (ii) are equivalent:*

(i) $\{x \in X \mid \forall i \in I, f_i(x) \leq \beta\} \subset \{x \in X \mid \forall s \in S, \langle v_s,x\rangle < \alpha_s\}$,

(ii) $\forall s \in S,\ \dfrac{v_s}{\alpha_s} \in \mathrm{Hec}\bigcup_{i\in I} L((f_i)_1^\nu, <, 1-\beta)$.

Proof. It is clear that (i) and

$$\forall s \in S,\ \frac{v_s}{\alpha_s} \in \left(\bigcap_{i\in I} L(f_i,\leq,\beta)\right)^{*(<,1)}$$

are equivalent. By using the assumption, $L(f_i,\leq,\beta)$ is a H-evenly convex set for each $i \in I$. Therefore, by using Proposition 2.13, for each $s \in S$, $\frac{v_s}{\alpha_s} \in \mathrm{Hec}\bigcup_{i\in I}(L(f_i,\leq,\beta))^{*(<,1)}$. Because $L(f_i,\leq,\beta) = \bigcap_{\varepsilon>0} L(f_i,\leq,\beta+\varepsilon)$, by using

Proposition 2.13 again, for each $s \in S$, $\frac{v_s}{\alpha_s} \in \mathrm{Hec} \bigcup_{i \in I} \mathrm{Hec} \bigcup_{\varepsilon > 0} (L(f_i, \leq, \beta + \varepsilon))^{*(<,1)}$. Furthermore, $\bigcup_{\varepsilon > 0} (L(f_i, \leq, \beta + \varepsilon))^{*(<,1)} = \bigcup_{\varepsilon > 0} (L(f_i, <, \beta + \varepsilon))^{*(<,1)}$. Hence we can prove that $\frac{v_s}{\alpha_s} \in \mathrm{Hec} \bigcup_{i \in I} \bigcup_{\varepsilon > 0} L((f_i)_1^{\nu}, \leq, 1 - \beta - \varepsilon)$ for each $s \in S$ by using Theorem 3.14. Therefore, for each $s \in S$, $\frac{v_s}{\alpha_s} \in \mathrm{Hec} \bigcup_{i \in I} L((f_i)_1^{\nu}, <, 1 - \beta)$. The converse is similar. □

Next, we show the set containment characterization, assuming that all f_i $(i \in I)$ are evenly quasiconvex, all g_j $(j \in J)$ are strictly evenly quasiconvex, I, J and S are arbitrary sets.

Theorem 3.17. *Let I, J and S be arbitrary sets, $\beta \in \mathbb{R}$, f_i be an evenly quasiconvex function from X to $\overline{\mathbb{R}}$ for each $i \in I$, g_j be a strictly evenly quasiconvex function from X to $\overline{\mathbb{R}}$ for each $j \in J$, and $v_s \in X^*$ and $\alpha_s \in (0, \infty)$ for each $s \in S$. Assume that $f_i(0) \leq \beta$ and $g_j(0) < \beta$ for each $i \in I$ and $j \in J$. Then, following conditions (i) and (ii) are equivalent:*

(i) $\{x \mid \forall i \in I, f_i(x) \leq \beta, \forall j \in J, g_j(x) < \beta\} \subset \{x \mid \forall s \in S, \langle v_s, x \rangle < \alpha_s\}$,
(ii) $\forall s \in S$,
$$\frac{v_s}{\alpha_s} \in \mathrm{Hec} \left[\left(\bigcup_{i \in I} L((f_i)_1^{\nu}, <, 1 - \beta) \right) \bigcup \left(\bigcup_{j \in J} L((g_j)_1^{\nu}, \leq, 1 - \beta) \right) \right].$$

Proof. The proof is similar to Theorem 3.15 and 3.16. □

In the following theorem, we show the result of the characterizing set containment when $|J| = |W| = 1$ by using λ-semiconjugate.

Theorem 3.18. *Let g be a function from X to $\overline{\mathbb{R}}$, $u \in X^*$, $\gamma \in \mathbb{R}$ and $\beta \in \mathbb{R}$. Then, following conditions (i), (ii) and (iii) are equivalent:*

(i) $L(g, <, \beta) \subset \{x \in X \mid \langle u, x \rangle \leq \gamma\}$,
(ii) $u \in (L(g, <, \beta))^{*(\leq, \gamma)}$,
(iii) $u \in L(g_{\gamma}^{\theta}, \leq, \gamma - \beta)$.

Proof. It is clear that (i) and (ii) are equivalent. We may assume that $L(g, <, \beta) \subset \{x \in X \mid \langle u, x \rangle \leq \gamma\}$, then the implication $g(x) < \beta$ implies $\langle u, x \rangle \leq \gamma$, or equivalently, $\langle u, x \rangle > \gamma$ implies $g(x) \geq \beta$ holds. This shows
$$g_{\gamma}^{\theta}(u) = \gamma - \inf\{g(x) \mid \langle u, x \rangle > \gamma\} \leq \gamma - \beta.$$
Conversely, if $g_{\gamma}^{\theta}(u) \leq \gamma - \beta$, then $\inf\{g(x) \mid \langle u, x \rangle > \gamma\} \geq \beta$. Therefore the implication $\langle u, x \rangle > \gamma$ implies $g(x) \geq \beta$, or $g(x) < \beta$ implies $\langle u, x \rangle \leq \gamma$ holds. This derives $L(g, <, \beta) \subset \{x \mid \langle u, x \rangle \leq \gamma\}$. □

Next, we show the set containment characterization, assuming that g_j $(j \in J)$ are quasiconvex, J and W are arbitrary sets.

Theorem 3.19. *Let J and W be arbitrary sets, $\beta \in \mathbb{R}$, g_j be a quasiconvex function from X to $\overline{\mathbb{R}}$ for each $j \in J$, and $u_w \in X^*$ and $\gamma_w \in (0, \infty)$ for each $w \in W$. Assume that $g_j(0) < \beta$ for each $j \in J$ and* $\mathrm{int}\{x \in X \mid \forall j \in J, g_j(x) < \beta\}$ *is nonempty. Then, following conditions (i) and (ii) are equivalent:*

(i) $\{x \in X \mid \forall j \in J, g_j(x) < \beta\} \subset \{x \in X \mid \forall w \in W, \langle u_w, x\rangle \le \gamma_w\}$,
(ii) $\forall w \in W$, $\dfrac{u_w}{\gamma_w} \in \operatorname{cl} \operatorname{Hec} \bigcup_{j \in J} L((g_j)^\theta_1, \le, 1-\beta)$.

Proof. It is easy to show that (i) is equivalent to

$$\forall w \in W,\ \frac{u_w}{\gamma_w} \in \left(\bigcap_{j\in J} L(g_j, <, \beta)\right)^{*(\le,1)}.$$

Since $\operatorname{int}\{x \in X \mid \forall j \in J, g_j(x) < \beta\}$ is nonempty, we can prove that

$$(\bigcap_{j\in J} L(g_j, <, \beta))^{*(\le,1)} = (\operatorname{cl}\bigcap_{j\in J} L(g_j, <, \beta))^{*(\le,1)} = (\bigcap_{j\in J} \operatorname{cl}L(g_j, <, \beta))^{*(\le,1)}$$

by using Proposition 3.13. From the assumption, $\operatorname{cl}L(g_j, <, \beta)$ is closed H-evenly convex for each $j \in J$. Hence, by using Proposition 2.13, for each $w \in W$, $\frac{u_w}{\gamma_w} \in \operatorname{cl}\operatorname{Hec}\bigcup_{j\in J}(\operatorname{cl}L(g_j, <, \beta))^{*(\le,1)}$. Also, by using Theorem 3.18, for each $w \in W$, $\frac{u_w}{\gamma_w} \in \operatorname{cl}\operatorname{Hec}\bigcup_{j\in J} L((g_j)^\theta_1, \le, 1-\beta)$. The converse is similar. □

In the following theorem, we show the set containment characterization, assuming that all f_i $(i \in I)$ are quasiconvex, I and W are arbitrary sets.

Theorem 3.20. *Let I and W be arbitrary sets, $\beta \in \mathbb{R}$, f_i be a quasiconvex function from X to $\overline{\mathbb{R}}$ for each $i \in I$, and $u_w \in X^*$ and $\gamma_w \in (0,\infty)$ for each $w \in W$. Assume that $f_i(0) \le \beta$ for each $i \in I$ and $\operatorname{int}\{x \in X \mid \forall i \in I, f_i(x) \le \beta\}$ is nonempty. Then, following conditions (i) and (ii) are equivalent:*

(i) $\{x \in X \mid \forall i \in I, f_i(x) \le \beta\} \subset \{x \in X \mid \forall w \in W, \langle u_w, x\rangle \le \gamma_w\}$,
(ii) $\forall w \in W$, $\dfrac{u_w}{\gamma_w} \in \operatorname{cl}\operatorname{Hec}\bigcup_{i\in I} L((f_i)^\theta_1, <, 1-\beta)$.

Proof. It is clear that (i) is equivalent to for each $w \in W$, $\frac{u_w}{\gamma_w} \in (\bigcap_{i\in I} L(f_i, \le, \beta))^{*(\le,1)}$. By the similar way in Theorem 3.19, we can prove that for each $w \in W$, $\frac{u_w}{\gamma_w} \in (\bigcap_{i\in I}\operatorname{cl}L(f_i, \le, \beta))^{*(\le,1)}$. From the assumption, $\operatorname{cl}L(f_i, \le, \beta)$ is a closed H-evenly convex set for each $j \in J$. Therefore, by using Proposition 2.13, $\frac{u_w}{\gamma_w} \in \operatorname{cl}\operatorname{Hec}\bigcup_{i\in I}(\operatorname{cl}L(f_i, \le, \beta))^{*(\le,1)}$ for each $w \in W$. Also, by using Proposition 3.13 again,

$$\operatorname{cl}\operatorname{Hec}\bigcup_{i\in I}\left(\operatorname{cl}\bigcap_{\varepsilon>0} L(f_i, <, \beta+\varepsilon)\right)^{*(\le,1)} = \operatorname{cl}\operatorname{Hec}\bigcup_{i\in I}\left(\bigcap_{\varepsilon>0}\operatorname{cl}L(f_i, <, \beta+\varepsilon)\right)^{*(\le,1)}.$$

By using the assumption, for each $\varepsilon > 0$, $\operatorname{cl}L(f_i, <, \beta+\varepsilon)$ is a closed H-evenly convex set. Therefore, by using Proposition 2.13, for each $w \in W$,

$$\frac{u_w}{\gamma_w} \in \operatorname{cl}\operatorname{Hec}\bigcup_{i\in I}\bigcup_{\varepsilon>0}(\operatorname{cl}L(f_i, <, \beta+\varepsilon))^{*(\le,1)}.$$

Furthermore,

$$\operatorname{cl}\operatorname{Hec}\bigcup_{i\in I}\bigcup_{\varepsilon>0}(\operatorname{cl}L(f_i, <, \beta+\varepsilon))^{*(\le,1)} = \operatorname{cl}\operatorname{Hec}\bigcup_{i\in I}\bigcup_{\varepsilon>0}(L(f_i, <, \beta+\varepsilon))^{*(\le,1)},$$

and by using Theorem 3.18, we can prove that for each $w \in W$,

$$\frac{u_w}{\gamma_w} \in \mathrm{cl}\,\mathrm{Hec} \bigcup_{i \in I} \bigcup_{\varepsilon > 0} L((f_i)_1^\theta, \leq, 1 - \varepsilon - \beta),$$

that is, $\frac{u_w}{\gamma_w} \in \mathrm{cl}\,\mathrm{Hec} \bigcup_{i \in I} L((f_i)_1^\theta, <, 1 - \beta)$. □

Next, we show the set containment characterization, assuming that f_i and g_j are quasiconvex for each $i \in I$ and $j \in J$, I, J and W are arbitrary sets.

Theorem 3.21. *Let I, J and W be arbitrary sets, $\beta \in \mathbb{R}$, f_i and g_j be quasiconvex functions from X to $\overline{\mathbb{R}}$ for each $i \in I$ and $j \in J$, and $u_w \in X^*$ and $\gamma_w \in (0, \infty)$ for each $w \in W$. Assume that $f_i(0) \leq \beta$ for each $i \in I$, $g_j(0) < \beta$ for each $j \in J$, and $\mathrm{int}\{x \in X \mid \forall i \in I, f_i(x) \leq \beta, \forall j \in J, g_j(x) < \beta\}$ is nonempty. Then, following conditions (i) and (ii) are equivalent:*

(i) $\{x \mid \forall i \in I, f_i(x) \leq \beta, \forall j \in J, g_j(x) < \beta\} \subset \{x \mid \forall w \in W, \langle u_w, x\rangle \leq \gamma_w\}$,

(ii) $\forall w \in W$,

$$\frac{u_w}{\gamma_w} \in \mathrm{cl}\,\mathrm{Hec}\left[\left(\bigcup_{i \in I} L((f_i)_1^\theta, <, 1 - \beta)\right) \bigcup \left(\bigcup_{j \in J} L((g_j)_1^\theta, \leq, 1 - \beta)\right)\right].$$

Proof. The proof is similar to Theorem 3.19 and 3.20. □

In the following theorem, we show the set containment characterization, assuming that all f_i $(i \in I)$ are evenly quasiconvex, all g_j $(j \in J)$ are strictly evenly quasiconvex, and I, J, S and W are arbitrary sets.

Theorem 3.22. *Let I, J, S and W be arbitrary sets, $\beta \in \mathbb{R}$, f_i be an evenly quasiconvex function from X to $\overline{\mathbb{R}}$ for each $i \in I$, g_j be a strictly evenly quasiconvex function from X to $\overline{\mathbb{R}}$ for each $j \in J$, $v_s \in X^*$ and $\alpha_s \in (0, \infty)$ for each $s \in S$, and $u_w \in X^*$ and $\gamma_w \in (0, \infty)$ for each $w \in W$. Assume that $f_i(0) \leq \beta$ for each $i \in I$, $g_j(0) < \beta$ for each $j \in J$ and $\mathrm{int}\{x \in X \mid f_i(x) \leq \beta, i \in I, g_j(x) < \beta, j \in J\}$ is nonempty. Then, following conditions (i) and (ii) are equivalent:*

(i) $A \subset B$,

(ii) $\forall s \in S$,

$$\frac{v_s}{\alpha_s} \in \mathrm{Hec}\left[\left(\bigcup_{i \in I} L((f_i)_1^\nu, <, 1 - \beta)\right) \bigcup \left(\bigcup_{j \in J} L((g_j)_1^\nu, \leq, 1 - \beta)\right)\right],$$

$\forall w \in W$,

$$\frac{u_w}{\gamma_w} \in \mathrm{cl}\,\mathrm{Hec}\left[\left(\bigcup_{i \in I} L((f_i)_1^\theta, <, 1 - \beta)\right) \bigcup \left(\bigcup_{j \in J} L((g_j)_1^\theta, \leq, 1 - \beta)\right)\right],$$

where

$$\begin{aligned} A &= \{x \in X \mid \forall i \in I, f_i(x) \leq \beta, \forall j \in J, g_j(x) < \beta\}, \\ B &= \{x \in X \mid \forall s \in S, \langle v_s, x\rangle < \alpha_s, \forall w \in W, \langle u_w, x\rangle \leq \gamma_w\}. \end{aligned}$$

Proof. The proof is similar to Theorem 3.15, 3.16, 3.19 and 3.20. □

We present characterizations of the containment of a convex set, defined by infinite quasiconvex constraints, in a reverse convex set, defined by infinite quasiconvex constraints.

In the beginning, we show the result of the containment when $|J| = |W| = 1$.

Theorem 3.23. *Let g be a quasiconvex function from X to $\overline{\mathbb{R}}$, h be an usc quasiconvex function from X to $\overline{\mathbb{R}}$, $\gamma \in \mathbb{R}$ and $\beta \in \mathbb{R}$. Assume that $L(g, <, \beta)$ and $L(h, <, \gamma)$ are nonempty. Then, following conditions (i), (ii) and (iii) are equivalent:*

(i) $L(g, <, \beta) \subset L(h, \geq, \gamma)$,
(ii) $L(g, <, \beta) \bigcap L(h, <, \gamma) = \emptyset$,
(iii) *there exists $\alpha \in \mathbb{R}$ such that*
$0 \in L(g^{\theta}_{\alpha}, \leq, \alpha - \beta) \setminus \{0\} + L(h^{\nu}_{-\alpha}, \leq, -\alpha - \gamma) \setminus \{0\}$.

Proof. It is clear that (i) and (ii) are equivalent. We may assume that the condition (ii) holds. Then, there exists $v \in X^* \setminus \{0\}$ and $\alpha \in \mathbb{R}$ such that for all $x \in L(h, <, \gamma)$ and $y \in L(g, <, \beta)$,

$$\langle v, x \rangle > \alpha \geq \langle v, y \rangle \,,$$

since g is quasiconvex and k is usc quasiconvex. Clearly, $v \in (L(g, <, \beta))^{*(\leq, \alpha)}$ and $-v \in (L(h, <, \gamma))^{*(<, -\alpha)}$. By using Theorem 3.14 and Theorem 3.18, $v \in (L(g^{\theta}_{\alpha}, \leq, \alpha - \beta)$ and $-v \in L(h^{\nu}_{-\alpha}, \leq, -\alpha - \gamma)$. Hence, $0 \in L(g^{\theta}_{\alpha}, \leq, \alpha - \beta) \setminus \{0\} + L(h^{\nu}_{-\alpha}, \leq, -\alpha - \gamma) \setminus \{0\}$. The converse is similar. □

Next, we show the set containment characterization, assuming that all g_j $(j \in J)$ are quasiconvex, all h_w $(w \in W)$ are usc quasiconvex, J and W are arbitrary sets.

Theorem 3.24. *Let J and W be arbitrary sets, g_j be a quasiconvex function from X to $\overline{\mathbb{R}}$ for each $j \in J$, h_w be an usc quasiconvex function from X to $\overline{\mathbb{R}}$ and $\gamma_w \in \mathbb{R}$ for each $w \in W$, and $\beta \in \mathbb{R}$. Assume that $0 \in \operatorname{int} \cap_{j \in J} L(g_j, <, \beta)$ and $\cap_{w \in W} L(h_w, <, \gamma_w)$ is nonempty. Then, following conditions (i) and (ii) are equivalent:*

(i) $\displaystyle\bigcap_{j \in J} L(g_j, <, \beta) \subset \bigcap_{w \in W} L(h_w, \geq, \gamma_w)$,
(ii) $\forall w \in W$,
$$0 \in \left(\operatorname{cl} \operatorname{Hec} \bigcup_{j \in J} L((g_j)^{\theta}_{1}, \leq, 1 - \beta) \setminus \{0\}\right) + L((h_w)^{\nu}_{-1}, \leq, -1 - \gamma_w) \setminus \{0\}.$$

Proof. Assume that the condition (i) is hold. Then, for each $w \in W$, $\cap_{j \in J} L(g_j, <, \beta) \bigcap L(h_w, <, \gamma_w) = \emptyset$. Since all g_j are quasiconvex, k_w are usc quasiconvex and $0 \in \operatorname{int} \cap_{j \in J} L(g_j, <, \beta)$, there exists $v \in X^* \setminus \{0\}$ such that for all $x \in L(h_w, <, \gamma_w)$ and $y \in \cap_{j \in J} L(g_j, <, \beta)$, $\langle v, x \rangle > 1 \geq \langle v, y \rangle$. By using Theorem 3.14 and Theorem 3.19, we can prove that $v \in \operatorname{cl} \operatorname{Hec} \cup_{j \in J} L((g_j)^{\theta}_{1}, \leq, 1 - \beta)$ and $-v \in L((h_w)^{\nu}_{-1}, \leq, -1 - \gamma_w)$, that is, $0 \in (\operatorname{cl} \operatorname{Hec} \cup_{j \in J} L((g_j)^{\theta}_{1}, \leq, 1 - \beta) \setminus \{0\}) + L((h_w)^{\nu}_{-1}, \leq, -1 - \gamma_w) \setminus \{0\}$. The converse implication is similar. □

In the following theorem, we show the set containment characterization, assuming that f_i $(i \in I)$ are quasiconvex, h_w $(w \in W)$ are usc quasiconvex, I and W are arbitrary sets.

Theorem 3.25. *Let I and W be arbitrary sets, f_i be a quasiconvex function from X to $\overline{\mathbb{R}}$ for each $i \in I$, h_w be an usc quasiconvex function from X to $\overline{\mathbb{R}}$ and $\gamma_w \in \mathbb{R}$ for each $w \in W$, and $\beta \in \mathbb{R}$. Assume that $0 \in \operatorname{int} \cap_{i\in I} L(f_i, \leq, \beta)$ and $\cap_{w\in W} L(h_w, <, \gamma_w)$ is nonempty. Then, following conditions (i) and (ii) are equivalent:*

(i) $\displaystyle\bigcap_{i\in I} L(f_i, \leq, \beta) \subset \bigcap_{w\in W} L(h_w, \geq, \gamma_w)$,

(ii) $\forall w \in W$,
$$0 \in \left(\operatorname{cl}\operatorname{Hec}\bigcup_{i\in I} L((f_i)^\theta_1, <, 1-\beta) \setminus \{0\}\right) + L((h_w)^\nu_{-1}, \leq, -1-\gamma_w) \setminus \{0\}.$$

Proof. Assume that the condition (i) is hold. Then, $\cap_{i\in I} L(f_i, \leq, \beta) \bigcap L(h_w, <, \gamma_w) = \emptyset$ for each $w \in W$. Since all f_i are quasiconvex, h_w are usc quasiconvex and $0 \in \operatorname{int} \cap_{i\in I} L(f_i, \leq, \beta)$, there exists $v \in X^* \setminus \{0\}$ such that for all $x \in L(h_w, <, \gamma_w)$ and $y \in \cap_{i\in I} L(f_i, \leq, \beta)$, $\langle v, x\rangle > 1 \geq \langle v, y\rangle$. By using Theorem 3.14 and Theorem 3.20, we can prove that $v \in \operatorname{cl}\operatorname{Hec} \cup_{j\in J} L((f_i)^\theta_1, <, 1-\beta)$ and $-v \in L((h_w)^\nu_{-1}, \leq, -1-\gamma_w)$. The converse is similar. □

We show the set containment characterization, assuming that f_i and g_j are quasiconvex for each $i \in I$ and $j \in J$, h_w are usc quasiconvex for each $w \in W$, and I, J and W are arbitrary sets.

Theorem 3.26. *Let I, J and W be arbitrary sets, f_i and g_j be quasiconvex functions from X to $\overline{\mathbb{R}}$ for each $i \in I$ and $j \in J$, h_w be an usc quasiconvex function from X to $\overline{\mathbb{R}}$ and $\gamma_w \in \mathbb{R}$ for each $w \in W$, and $\beta \in \mathbb{R}$. Assume that $0 \in \operatorname{int}[(\cap_{i\in I} L(f_i, \leq, \beta)) \bigcap (\cap_{j\in J} L(g_j, <, \beta))]$ and $\cap_{w\in W} L(h_w, <, \gamma_w)$ is nonempty. Then, following conditions (i) and (ii) are equivalent:*

(i) $\displaystyle(\bigcap_{i\in I} L(f_i, \leq, \beta)) \bigcap (\bigcap_{j\in J} L(g_j, <, \beta)) \subset \bigcap_{w\in W} L(h_w, \geq, \gamma_w)$,

(ii) $\forall w \in W$,
$$0 \in \left(\operatorname{cl}\operatorname{Hec}\left\{(\bigcup_{i\in I} L((f_i)^\theta_1, <, 1-\beta)) \bigcup (\bigcup_{j\in J} L((g_j)^\theta_1, \leq, 1-\beta))\right\} \setminus \{0\}\right)$$
$$+ L((h_w)^\nu_{-1}, \leq, -1-\gamma_w) \setminus \{0\}.$$

Proof. The proof is similar to Theorem 3.24 and 3.25.

□

In this section, we show set containment characterizations in an evenly convex set, assuming that the inequalities in A and B can be either weak or strict. However, on set containment characterizations in a reverse convex set, we show only the case where inequalities in B are weak. Hereinafter, we show that it is difficult to characterize the set containment characterization in a reverse convex set, assuming that the inequalities in B are strict.

We consider the characterization of $A \subset B$, where I, J and S are arbitrary sets, f_i and g_j are quasiconvex functions from X to $\overline{\mathbb{R}}$ for each $i \in I$ and $j \in J$, k_s is a quasiconvex function from X to $\overline{\mathbb{R}}$ and $\alpha_s \in \mathbb{R}$ for each $s \in S$, $\beta \in \mathbb{R}$, and

$$\begin{aligned} A &= \{x \in X \mid f_i(x) \leq \beta, i \in I, g_j(x) < \beta, j \in J\}, \\ B &= \{x \in X \mid k_s(x) > \alpha_s, s \in S\}. \end{aligned}$$

Assume that J is empty, f_i is lsc quasiconvex for each $i \in I$, k_s is lsc quasiconvex and $L(k_s, \leq, \alpha_s)$ is compact for each $s \in S$, and $0 \in \operatorname{int} \cap_{i \in I} L(f_i, \leq, \beta)$. Then, following conditions (i) and (ii) are equivalent:

$$\begin{array}{ll} \text{(i)} & \cap_{i\in I} L(f_i, \leq, \beta) \subset \cap_{s\in S} L(k_s, >, \alpha_s), \\ \text{(ii)} & \forall s \in S, \exists v \in X^* \setminus \{0\} \text{ s.t.} \\ & \forall x \in L(k_s, \leq, \alpha_s), \forall y \in \cap_{i\in I} L(f_i, \leq, \beta), \langle v, x\rangle > 1 \geq \langle v, y\rangle \,. \end{array}$$

Of course, we can rewrite the condition (ii) by using level sets of quasiconjugate functions. Assume that I is empty, $|J| < \infty$, g_j is usc quasiconvex for each $j \in J$, k_s is quasiconvex for each $s \in S$, and $0 \in \operatorname{int} \cap_{j\in J} L(g_j, <, \beta)$, then, following conditions (i) and (ii) are equivalent:

$$\begin{array}{ll} \text{(i)} & \cap_{j\in J} L(g_j, <, \beta) \subset \cap_{s\in S} L(k_s, >, \alpha_s), \\ \text{(ii)} & \forall s \in S, \exists v \in X^* \setminus \{0\} \text{ s.t.} \\ & \forall x \in L(k_s, \leq, \alpha_s), \forall y \in \cap_{j\in J} L(g_j, <, \beta), \langle v, x\rangle \geq 1 > \langle v, y\rangle \,. \end{array}$$

Hence, we can show the set containment characterization by using 1-quasiconjugate and -1-semiconjugate.

However, if J is an arbitrary set, then $\cap_{j\in J} L(g_j, <, \beta)$ is not always open even if g_j is usc quasiconvex for each $j \in J$. Hence, if g_j is usc quasiconvex for each $j \in J$, k_s is lsc quasiconvex and $L(k_s, \leq, \alpha_s)$ is compact for each $s \in S$, $0 \in \operatorname{int} \cap_{j\in J} L(g_j, <, \beta)$ and $\cap_{j\in J} L(g_j, <, \beta) \subset \cap_{s\in S} L(k_s, >, \alpha_s)$, then, for each $s \in S$, there exists $v \in X^* \setminus \{0\}$ such that for all $x \in \cap_{j\in J} L(g_j, <, \beta)$ and $y \in L(k_s, \leq, \alpha_s)$,

$$\langle v, x\rangle \geq 1 \geq \langle v, y\rangle \,.$$

Also, the above inequality does not imply that $\cap_{j\in J} L(g_j, <, \beta) \subset \cap_{s\in S} L(k_s, >, \alpha_s)$, and these assumptions of functions are the strongest one in this problem. Therefore, it is hard to characterize set containments by using quasiconjugate function.

3.3. **Application.** We show that set containment characterizations in this section is useful to consider quasiconvex programming problem. Let I be an arbitrary set, f_i be a lsc quasiconvex function from X to $\overline{\mathbb{R}}$ for each $i \in I$, $A = \{x \in X \mid \forall i \in I, f_i(x) \leq 0\}$, and h be an usc quasiconvex function. Assume that $0 \in \operatorname{int} A$, and consider the following problem (P),

$$(P) \begin{cases} \text{minimize } h(x), \\ \text{subject to } x \in A. \end{cases}$$

Remember that all lsc quasiconvex functions are the supremum of some family of lsc quasi-affine functions. Hence, without loss of generality, we can assume that f_i is a lsc quasi-affine function for each $i \in I$, that is, there exist $\{(l_i, v_i) \mid i \in I\} \subset Q \times X^*$

such that $f_i = l_i \circ v_i$ for each $i \in I$. By using Theorem 3.25, for each $\gamma \in \mathbb{R}$, following conditions (i), (ii) and (iii) are equivalent:

(i) $\cap_{i\in I} L(f_i, \leq, 0) \subset L(h, \geq, \gamma)$,
(ii) $0 \in \mathrm{cl}\,\mathrm{Hec} \cup_{i\in I} L((f_i)^\theta_1, <, 1) + L(h^\nu_{-1}, \leq, -1-\gamma) \setminus \{0\}$,
(iii) $0 \in \mathrm{cl}\,\mathrm{co}(\{\frac{1}{(l_i)^{-1}(0)} v_i \mid i \in I\} \cup \{0\}) + L(h^\nu_{-1}, \leq, -1-\gamma) \setminus \{0\}$.

Actually, for each $i \in I$,

$$\begin{aligned} L((l_i \circ v_i)^\theta_1, <, 1) &= \{z \in X \mid (l_i \circ v_i)^\theta_1(z) < 1\} \\ &= \{z \in X \mid 1 - \inf\{l_i \circ v_i(x) \mid \langle z, x\rangle > 1\} < 1\} \\ &= \{z \in X \mid \inf\{l_i \circ v_i(x) \mid \langle z, x\rangle > 1\} > 0\}. \end{aligned}$$

If $z \notin \mathbb{R}_+\{v_i\}$, it is clear that $\inf\{l_i \circ v_i(x) \mid \langle z, x\rangle > 1\} = \inf_{t\in\mathbb{R}} l_i(t) \leq 0$ because S is nonempty. And if $z \in \mathbb{R}_+\{v_i\} \setminus \{0\}$, there exists $\lambda > 0$ such that $z = \lambda v_i$, hence $\inf\{l_i \circ v_i(x) \mid \langle z, x\rangle > 1\} = l_i(\frac{1}{\lambda})$ because l_i is non-decreasing. Also, it is clear that $\inf\{l_i \circ v_i(x) \mid \langle 0, x\rangle > 1\} = \infty$, hence we can prove that $L((l_i \circ v_i)^\theta_1, <, 1) = [0, \frac{1}{(l_i)^{-1}(0)})\{v_i\}$. Furthermore, $\mathrm{cl}\,\mathrm{Hec} \cup_{i\in I} L((l_i \circ v_i)^\theta_1, <, 1) = \mathrm{cl}\,\mathrm{Hec} \cup_{i\in I} [0, \frac{1}{(l_i)^{-1}(0)})\{v_i\} = \mathrm{cl}\,\mathrm{ec}(\cup_{i\in I}[0, \frac{1}{(l_i)^{-1}(0)})\{v_i\} \cup \{0\})$ because $\cup_{i\in I}[0, \frac{1}{(l_i)^{-1}(0)})\{v_i\}$ is nonempty. Also $\mathrm{cl}\,\mathrm{ec}(\cup_{i\in I}[0, \frac{1}{(l_i)^{-1}(0)})\{v_i\} \cup \{0\}) = \mathrm{cl}\,\mathrm{co}(\cup_{i\in I}[0, \frac{1}{(l_i)^{-1}(0)})\{v_i\} \cup \{0\}) = \mathrm{cl}\,\mathrm{co}(\{\frac{1}{(l_i)^{-1}(0)} v_i \mid i \in I\} \cup \{0\})$. Hence, the above conditions (i), (ii) and (iii) are equivalent.

Clearly, $\inf_{x\in A} h(x) = \sup\{\gamma \in \mathbb{R} \mid \cap_{i\in I} L(f_i, \leq, 0) \subset L(h, \geq, \gamma)\}$. Hence, we can prove that

$$\inf_{x\in A} h(x) = \sup\left\{\gamma \,\middle|\, 0 \in \mathrm{cl}\,\mathrm{co}(\{\frac{v_i}{(l_i)^{-1}(0)} \mid i \in I\} \cup \{0\}) + L(h^\nu_{-1}, \leq, -1-\gamma) \setminus \{0\}\right\},$$

that is, we get the following new duality problem of (P),

$$(D)\begin{cases} \text{maximize } \gamma, \\ \text{subject to } 0 \in \mathrm{cl}\,\mathrm{co}(\{\frac{1}{(l_i)^{-1}(0)} v_i \mid i \in I\} \cup \{0\}) + L(h^\nu_{-1}, \leq, -1-\gamma) \setminus \{0\}. \end{cases}$$

The value of the dual problem (D) is equal to $-\inf_{z\in T}(h^\nu_{-1}(z) + 1)$, where $T = -\mathrm{cl}\,\mathrm{co}(\{\frac{1}{(l_i)^{-1}(0)} v_i \mid i \in I\} \cup \{0\})$. Furthermore, $A^{*(\leq,1)} = \mathrm{cl}\,\mathrm{co}(\{\frac{1}{(l_i)^{-1}(0)} v_i \mid i \in I\} \cup \{0\}) = -T$ and $h^\nu_{-1} + 1 = h^R$. Hence,

$$\inf_{x\in A} h(x) = -\inf_{z\in -A^{*(\leq,1)}} h^R(z),$$

and we can get another duality problem of (P),

$$(D')\begin{cases} \text{minimize } h^R(z), \\ \text{subject to } z \in -A^{*(\leq,1)}, \end{cases}$$

This duality problem (D') is a same problem of the duality problem by Thach [31].

3.4. **By the generator of quasiconvex functions.** In this section, we show two set containment characterizations for quasiconvex constraints. First, we present a characterization of a containment of a convex set, defined by a quasiconvex constraint, in a closed halfspace.

First, we introduce a notion of a quasiaffine function which is a generalized notion of an affine function. A function f is said to be quasiaffine if quasiconvex and quasiconcave. In [15], Penot and Volle proved the following theorem.

Theorem 3.27. [15] *The following conditions (i) and (ii) are equivalent:*

(i) *f is lsc quasiaffine,*

(ii) *there exists $k \in Q$ and $w \in X^*$ such that $f = k \circ w$,*

where $Q = \{h : \mathbb{R} \to \overline{\mathbb{R}} \mid h$ is lsc and non-decreasing$\}$. Also, the following conditions (iii) and (iv) are equivalent:

(iii) *f is lsc quasiconvex,*

(iv) *there exists $\{(k_i, w_i) \mid i \in I\} \subset Q \times X^*$ such that $f = \sup_{i\in I} k_i \circ w_i$.*

Theorem 3.27 indicates that a lsc quasiconvex function f consists of a supremum of some family of lsc quasiaffine functions. Based on this result, we define a notion of generator for quasiconvex functions, that is, $G = \{(k_i, w_i) \mid i \in I\} \subset Q \times X^*$ is said to be a generator of f if $f = \sup_{i\in I} k_i \circ w_i$, and we can see that all lsc quasiconvex functions have at least one generator by Theorem 3.27. Also, when f is a proper lsc convex function, $B_f = \{(k_v, v) \mid v \in \mathrm{dom} f^*, k_v(t) = t - f^*(v), \forall t \in \mathbb{R}\} \subset Q \times X^*$ is a generator of f. Actually, for all $x \in X$,

$$f(x) = f^{**}(x) = \sup\{\langle v, x\rangle - f^*(v) \mid v \in \mathrm{dom} f^*\} = \sup_{v\in \mathrm{dom} f^*} k_v(\langle v, x\rangle).$$

We call the generator B_f "the basic generator" of convex function f. The basic generator is very important with respect to the comparison of convex and quasiconvex programming.

Moreover, we introduce a generalized notion of inverse function of $h \in Q$. The following function h^{-1} is said to be the hypo-epi-inverse of h :

$$h^{-1}(a) = \inf\{b \in \mathbb{R} \mid a < h(b)\} = \sup\{b \in \mathbb{R} \mid h(b) \leq a\}.$$

If h has the inverse function, then the inverse function of $h$is equal to the hypo-epi-inverse of h. in detail see [15]. In this thesis, we denote the hypo-epi-inverse of h by h^{-1}. Also, we denote the lower left-hand Dini derivative of $h \in Q$ at t by $D_-h(t)$, that is $D_-h(t) = \liminf_{\varepsilon\to 0-} \frac{h(t+\varepsilon)-h(t)}{\varepsilon}$. A function h is said to be lower left-hand Dini differentiable if $D_-h(t)$ is finite for all $t \in \mathbb{R}$.

By using the notion of generator, we investigate the following set containment characterization.

Theorem 3.28. *Let f be a lsc quasiconvex function from X to $\overline{\mathbb{R}}$ with generator $\{(k_i, w_i) \mid i \in I\} \subset Q \times X^*$, $u \in X^*$, and α, $\beta \in \mathbb{R}$. Assume that $\{x \in X \mid f(x) \leq \beta\} \neq \emptyset$ and there exists $i_0 \in I$ such that $k_{i_0}^{-1}(\beta) \in \mathbb{R}$. Then (i) and (ii) given below are equivalent:*

(i) $\{x \in X \mid f(x) \leq \beta\} \subset \{x \in X \mid \langle u, x\rangle \leq \alpha\}$,

(ii) $(u, \alpha) \in \mathrm{cl\,cone\,co} \bigcup_{i\in I} \{(w_i, \delta) \in X^* \times \mathbb{R} \mid k_i^{-1}(\beta) \leq \delta\}$.

Proof. Let $(u, \alpha) \in \mathrm{cl\,cone\,co} \bigcup_{i\in I} \{(w_i, \delta) \in X^* \times \mathbb{R} \mid k_i^{-1}(\beta) \leq \delta\}$, then there exists $\{(u_k, \alpha_k)\} \subset \mathrm{cone\,co} \bigcup_{i\in I} \{(w_i, \delta) \in X^* \times \mathbb{R} \mid k_i^{-1}(\beta) \leq \delta\}$ such that $\{(u_k, \alpha_k)\}$

converges to (u,α). For each $k \in \mathbb{N}$, there exist $\lambda_k \geq 0$, $m_k \in \mathbb{N}$, $i_1, \cdots, i_{m_k} \in I$, $\delta_1, \cdots, \delta_{m_k} \in \mathbb{R}$, and $\beta_1, \cdots, \beta_{m_k} \geq 0$ such that $\Sigma_{j=1}^{m_k} \beta_j = 1$, $\delta_j \geq k_{i_j}^{-1}(\beta)$, and $(u_k, \alpha_k) = \lambda_k(\Sigma_{j=1}^{m_k} \beta_j w_{i_j}, \Sigma_{j=1}^{m_k} \beta_j \delta_j)$. Then, for all $x \in X$ with $f(x) \leq \beta$ and $i \in I$, $k_i(\langle w_i, x\rangle) \leq \beta$. Hence, for all $k \in \mathbb{N}$,

$$\langle u_k, x\rangle = \lambda_k \sum_{j=1}^{m_k} \beta_j \left\langle w_{i_j}, x\right\rangle \leq \lambda_k \sum_{j=1}^{m_k} \beta_j \delta_j = \alpha_k.$$

Therefore, $\langle u, x\rangle \leq \alpha$. Conversely, let $(u,\alpha) \notin \operatorname{cl}\operatorname{cone}\operatorname{co}\bigcup_{i\in I} \{(w_i,\delta) \in X^* \times \mathbb{R} \mid k_i^{-1}(\beta) \leq \delta\}$. By using separation theorem, there exists $(v_0, \gamma_0) \in (X \times \mathbb{R})\backslash\{0\}$ such that for all $(w,\delta) \in \bigcup_{i\in I} \{(w_i,\delta) \in X^* \times \mathbb{R} \mid k_i^{-1}(\beta) \leq \delta\}$,

$$\langle (u,\alpha),(v_0,\gamma_0)\rangle > 0 \geq \langle (w,\delta),(v_0,\gamma_0)\rangle .$$

(a) If $\gamma_0 < 0$, then we may assume that $\gamma_0 = -1$. Therefore, for each $i \in I$ with $k_i^{-1}(\beta) \in \mathbb{R}$, $\langle u, v_0\rangle - \alpha > 0 \geq \langle w_i, v_0\rangle - k_i^{-1}(\beta)$, and hence, $\langle u, v_0\rangle > \alpha$ and for each $i \in I$, $k_i^{-1}(\beta) \geq \langle w_i, v_0\rangle$. Then, $k_i(\langle w_i, v_0\rangle) \leq \beta$, that is, $f(v_0) \leq \beta$, which contradicts (i).

(b) If $\gamma_0 = 0$, then for each $i \in I$ with $k_i^{-1}(\beta) \in \mathbb{R}$, $\langle u, v_0\rangle > 0 \geq \langle w_i, v_0\rangle$, and since $L(f,\leq,\beta) \neq \emptyset$, there exists $x_0 \in L(f,\leq,\beta)$. Then, for all $t > 0$, for each $i \in I$ with $k_i^{-1}(\beta) \in \mathbb{R}$, $\langle w_i, x_0 + tv_0\rangle \leq \langle w_i, x_0\rangle$, which implies $k_i(\langle w_i, x_0 + tv_0\rangle) \leq k_i(\langle w_i, x_0\rangle) \leq \beta$. Therefore, $x_0 + tv_0 \in L(f,\leq,\beta)$. However, since $\langle u, v_0\rangle > 0$, there exists $t_0 > 0$ such that $\langle u, x_0 + t_0 v_0\rangle > \alpha$. This is contradiction.

(c) If $\gamma_0 > 0$, we can assume that $\gamma_0 = 1$, then for each $i \in I$ with $k_i^{-1}(\beta) \in \mathbb{R}$ and $\delta \geq k_i^{-1}(\beta)$, $\langle u, v_0\rangle + \alpha > 0 \geq \langle w_i, v_0\rangle + \delta$. However, this is a contradiction for sufficiently large δ. □

A generator of the quasiconvex function obtained in Theorem 3.27 is not unique, and any lsc quasiconvex function has infinite generators. However, the set in condition (ii) of Theorem 3.28 does not depend on the generator of the function f. When $\{(k_i, w_i) \mid i \in I\} \subset Q \times X^*$ and $\{(l_j, u_j) \mid j \in J\} \subset Q \times X^*$ are generators of f, the following three conditions can be proven to be equivalent by using Theorem 3.28:

(i) $\{x \in X \mid f(x) \leq \beta\} \subset \{x \in X \mid \langle u, x\rangle \leq \alpha\}$,

(ii) $(u,\alpha) \in \operatorname{cl}\operatorname{cone}\operatorname{co}\displaystyle\bigcup_{i\in I} \{(w_i,\delta) \in X^* \times \mathbb{R} \mid k_i^{-1}(\beta) \leq \delta\}$,

(iii) $(u,\alpha) \in \operatorname{cl}\operatorname{cone}\operatorname{co}\displaystyle\bigcup_{j\in J} \{(u_j,\delta) \in X^* \times \mathbb{R} \mid l_j^{-1}(\beta) \leq \delta\}$.

Moreover, the set in the above condition (ii) is equal to $\operatorname{epi}\delta_A^*$, that is

$$\operatorname{epi}\delta_A^* = \operatorname{cl}\operatorname{cone}\operatorname{co}\bigcup_{i\in I} \{(w_i,\delta) \in X^* \times \mathbb{R} \mid k_i^{-1}(\beta) \leq \delta\},$$

where $A = L(f,\leq,\beta)$. This equation is very important to define the newly proposed CCCQ in the way described in [7].

Next, we present a characterization of the containment of a convex set, defined by a quasiconvex constraint, in a reverse convex set, as defined by a convex constraint.

Theorem 3.29. *Let f be a lsc quasiconvex function from X to $\overline{\mathbb{R}}$ with generator $\{(k_i, w_i) \mid i \in I\} \subset Q \times X^*$, and let h be a proper lsc convex function from X to $\mathbb{R} \cup \{+\infty\}$. In addition, let $\beta \in \mathbb{R}$. Assume that $A = \{x \in X \mid f(x) \leq \beta\} \neq \emptyset$, $\mathrm{dom} h \cap A \neq \emptyset$, $\mathrm{epi} h^* + \mathrm{epi} \delta_A^*$ is w^*-closed and that there exists $i_0 \in I$ such that $k_{i_0}^{-1}(\beta) \in \mathbb{R}$. Then, (i) and (ii) given below are equivalent:*

(i) $\{x \in X \mid f(x) \leq \beta\} \subset \{x \in X \mid h(x) \geq 0\}$,

(ii) $(0,0) \in \mathrm{epi} h^* + \mathrm{cl\,cone\,co} \bigcup_{i \in I} \{(w_i, \delta) \in X^* \times \mathbb{R} \mid k_i^{-1}(\beta) \leq \delta\}$.

Proof. Assume the condition (ii) holds. Based on the above assumption, subdifferential sum rule, and Theorem 3.28, we can show that

$$(0,0) \in \mathrm{epi} h^* + \mathrm{epi} \delta_A^* = \mathrm{epi}(h + \delta_A)^*.$$

This shows $\inf_{x \in A} h(x) \geq 0$, that is, condition (i) holds. The inverse implication is similar. □

3.5. **Discussion.** We compare the main results in this section with previous ones in Refs. [6, 8, 10, 14]. Consider the sets

$$A = \{x \in X \mid f_i(x) \leq 0, \forall i \in I; g_j(x) < 0, \forall j \in J; l_e(x) = 0, \forall e \in E\},$$

and

$$B = \{x \in X \mid k_s(x) < 0, \forall s \in S; h_w(x) \leq 0, \forall w \in W\},$$

where I, J, E, S and W are arbitrary sets, $I \cup J \cup E \neq \emptyset$, $S \cup W \neq \emptyset$, and $\{f_i, i \in I\}$, $\{g_j, j \in J\}$, $\{l_e, e \in E\}$, $\{k_s, s \in S\}$ and $\{h_w, w \in W\}$ are functions from X to $\overline{\mathbb{R}}$. We summarize in Table 1 the results on set containments in the similar way in [8]. No. from 1 to 7 are previous results, No. 8 to 13 are our results in this section. The column 2 inform about reference or section. The columns 3, 4, 5, 6 and 7 inform on the cardinality of the index sets, which can be empty, finite or arbitrary (abbreviated as "$\emptyset$", "Fin" and "Arb", respectively), and the columns 8, 9, 10, 11 and 12 inform about assumptions of functions, which can be affine, quadratic concave, differentiable convex, differentiable concave, convex, concave, quasiconvex and quasiconcave (abbreviated as "Aff", "Quad", "Dconv", "Dconc", "Conv", "Conc", "Qconv" and "Qconc", respectively). "***" means that $J \cup W \neq \emptyset$.

No.		I	J	E	S	W	$\{f_i\}$	$\{g_j\}$	$\{l_e\}$	$\{k_s\}$	$\{h_w\}$
1	[14]	Fin	$\emptyset$	$\emptyset$	$\emptyset$	Fin	Aff	–	–	–	Aff
2	[14]	Fin	$\emptyset$	$\emptyset$	$\emptyset$	Fin	Aff	–	–	–	Quad
3	[14]	Fin	$\emptyset$	$\emptyset$	$\emptyset$	Fin	Dconv	–	–	–	Dconc
4	[10]	Arb	$\emptyset$	$\emptyset$	$\emptyset$	Fin	Conv	–	–	–	Aff
5	[10]	Arb	$\emptyset$	$\emptyset$	$\emptyset$	Fin	Conv	–	–	–	Conc
6	[8]	Arb	***	Arb	Arb	***	Aff	Aff	Aff	Aff	Aff
7	[6]	Arb	Arb	$\emptyset$	$\emptyset$	Fin	Conv	Conv	–	–	Conc
8	3.1	$\emptyset$	Fin	$\emptyset$	Fin	$\emptyset$	–	Qconv	–	Aff	–
9	3.1	$\emptyset$	Fin	$\emptyset$	$\emptyset$	Fin	–	Qconv	–	–	Qconc
10	3.2	Arb	Arb	$\emptyset$	Arb	Arb	Qconv	Qconv	–	Aff	Aff
11	3.2	Arb	Arb	$\emptyset$	$\emptyset$	Arb	Qconv	Qconv	–	–	Qconc
12	3.4	Arb	$\emptyset$	$\emptyset$	$\emptyset$	Arb	Qconv	–	–	–	Aff
13	3.4	Arb	$\emptyset$	$\emptyset$	$\emptyset$	Arb	Qconv	–	–	–	Conc

Table 1. Literature on set containments

In the rest of the section, we compare No. 4 with No. 8 especially. Section 3.1 characterizes the containment in the form

$$L(f,<,\beta) \subset \{x \mid \langle v,x\rangle < \alpha\},$$

whereas, in Theorem 3.1, Jeyakumar considered inclusions of the form

$$L(f,\leq,\beta) \subset \{x \mid \langle v,x\rangle \leq \alpha\}.$$

We discuss conditions guaranteeing the equivalence of both inclusions. It is easy to show that for any $v \in \mathbb{R}^n \setminus \{0\}$,

$$\mathrm{int}\{x \mid \langle v,x\rangle \leq \alpha\} = \{x \mid \langle v,x\rangle < \alpha\}, \quad \mathrm{cl}\{x \mid \langle v,x\rangle < \alpha\} = \{x \mid \langle v,x\rangle \leq \alpha\}.$$

Moreover, if f is continuous, we have

$$L(f,<,\beta) \subset \mathrm{int}L(f,\leq,\beta), \quad \mathrm{cl}L(f,<,\beta) \subset L(f,\leq,\beta)$$

are satisfied, but the converse inclusions are not true in general. When the equalities are fulfilled in these inclusions, we can show easily that our form and Jeyakumar's form are equivalent. For our purpose, we show the following lemmas:

Lemma 3.30. *Let A, $B \subset \mathbb{R}^n$. If $\mathrm{int}A = \emptyset$ and $\mathrm{int}(\mathrm{cl}B) = \mathrm{int}B$, then we have $\mathrm{int}(A \cup B) = \mathrm{int}B$.*

Proof. Inclusion $\mathrm{int}(A \cup B) \supset \mathrm{int}B$ is obvious. Conversely, for any $x \in \mathrm{int}(A \cup B)$, there exists $r > 0$ satisfying $B(x,r) \subset A \cup B$. If $\mathrm{int}(B(x,r) \cap B^c) \neq \emptyset$, then we have a contradiction since $\mathrm{int}A = \emptyset$ and $B(x,r) \cap B^c \subset (A \cup B) \cap B^c \subset A$ hold. Therefore

$$\emptyset = \mathrm{int}(B(x,r) \cap B^c) = B(x,r) \cap \mathrm{int}(B^c) = B(x,r) \cap (\mathrm{cl}B)^c,$$

and then $B(x,r) \subset \mathrm{cl}B$. By using assumption $\mathrm{int}(\mathrm{cl}B) = \mathrm{int}B$, we obtain $B(x,r) \subset \mathrm{int}(\mathrm{cl}B) = \mathrm{int}B \subset B$. This shows that $x \in \mathrm{int}B$. □

Lemma 3.31. *Let A, $B \subset \mathbb{R}^n$. If $\mathrm{int}A = \emptyset$ and B is a convex set with $\mathrm{int}B \neq \emptyset$, then we have*

(i) $\mathrm{int}(A \cup B) = \mathrm{int}B$,
(ii) $\mathrm{int}(A \cup B^c) = \mathrm{int}B^c$.

Proof. Since B is convex and $\mathrm{int}B \neq \emptyset$, we have

$$\mathrm{int}(\mathrm{cl}B) = \mathrm{int}B \quad \text{and} \quad \mathrm{cl}(\mathrm{int}B) = \mathrm{cl}B.$$

The second equation yields $\mathrm{int}(\mathrm{cl}(B^c)) = \mathrm{int}(B^c)$. Therefore (i) and (ii) are proved by using Lemma 3.30. □

Theorem 3.32. *Let f be a continuous quasiconvex function from $\mathbb{R}^n$ to $\mathbb{R}$, $v \in \mathbb{R}^n \setminus \{0\}$ and $\alpha \in \mathbb{R}$. If $\mathrm{int}L(f, =, \beta) = \emptyset$ and $\mathrm{int}L(f, <, \beta) \neq \emptyset$ for some $\beta \in \mathbb{R}$, then we have*

(i) $L(f, <, \beta) = \mathrm{int}L(f, \leq, \beta)$,
(ii) $\mathrm{cl}L(f, <, \beta) = L(f, \leq, \beta)$.

Moreover

$$L(f, <, \beta) \subset \{x \mid \langle v, x\rangle < \alpha\} \iff L(f, \leq, \beta) \subset \{x \mid \langle v, x\rangle \leq \alpha\}.$$

Proof. (i). Put $A = L(f, =, \beta)$ and $B = L(f, <, \beta)$. By using Lemma 3.31 (i), $\mathrm{int}L(f, \leq, \beta) = \mathrm{int}L(f, <, \beta) = L(f, <, \beta)$ because f is usc. Next we show (ii). Put $A = L(f, =, \beta)$ and $B = L(f, \leq, \beta)$. By using the Lemma 3.31 (ii), we have $\mathrm{int}L(f, \geq, \beta) = \mathrm{int}L(f, >, \beta)$, and equivalently $\mathrm{cl}L(f, <, \beta) = \mathrm{cl}L(f, \leq, \beta)$. Since f is lsc, $\mathrm{cl}L(f, \leq, \beta) = L(f, \leq, \beta)$. The equivalence is straightforward consequence of statements (i) and (ii). □

Remark 3.33. If every f_i is convex, $\mathrm{dom}(\sup_{i\in I} f_i) = \mathbb{R}^n$ and condition [A1] in Theorem 2.8 holds, then the assumptions of Theorem 3.4 is satisfied. Also if $\inf_{x\in\mathbb{R}^n} \sup_{i\in I} f_i(x) < 0$, we can check that $\mathrm{int}L(\sup_{i\in I} f_i, =, 0) = \emptyset$ and $\mathrm{int}L(\sup_{i\in I} f_i, <, 0) \neq \emptyset$. For any $\alpha \in (0, \infty)$ and any $v \in \mathbb{R}^n \setminus \{0\}$, we have the following characterization concerned with the Fenchel conjugate and H-quasiconjugate:

$$(v, \alpha) \in \mathrm{cl}\left(\mathrm{cone}\,\mathrm{co}\bigcup_{i\in I} \mathrm{epi} f_i^*\right) \iff \frac{v}{\alpha} \in L((\inf_{i\in I} f_i^H)^{HH}, \leq, 0).$$

4. Constraint qualifications

We consider the following mathematical programming problem:

$$\begin{cases} \text{minimize } f(x), \\ \text{subject to } g_i(x) \leq 0, \forall i \in I, \end{cases}$$

where I is an arbitrary set, f and g_i are extended real-valued functions from locally convex Hausdorff topological vector space X.

In convex programming, a constraint qualification is an essential ingredient of the elegant and powerful duality theory. The best-known constraint qualifications are the Slater-type constraint qualifications. Often, however, such constraint qualifications are not satisfied for problems that arise in applications. The lack of a constraint qualification can cause theoretical and numerical difficulties in applications. For convex programming, Jeyakumar, Dinh, and Lee [11] developed the closed cone constraint qualification involving epigraphs and extending the Slater-type conditions. Constraint qualifications involving epigraphs have been used extensively in various studies, see [1, 7, 13]. Such constraint qualifications concern Jeyakumar's set containment characterization (Theorem 3.1).

Recall $\Gamma_0(X)$, the set of all proper lsc convex functions from X to $\overline{\mathbb{R}}$. In [7], the condition of Farkas-Minkowski (FM) was investigated as the weakest constraint qualification for Lagrange (strong) duality. Let I be an index set, for each $i \in I$, let $g_i \in \Gamma_0(X)$. The convex system $\{g_i(x) \le 0 \mid i \in I\}$ is said to be FM if the characteristic cone

$$\mathrm{cone}\,\mathrm{co} \bigcup_{i \in I} \mathrm{epi} g_i^*$$

is w^*-closed.

The following theorem indicates that FM is the weakest constraint qualification for Lagrange duality.

Theorem 4.1. [7] *Let I be an index set, for each $i \in I$, let $g_i \in \Gamma_0(X)$. Assume that $A = \{x \in X \mid \forall i \in I, g_i(x) \le 0\} \neq \emptyset$. Then, the following statements are equivalent:*

(i) $\{g_i(x) \le 0 \mid i \in I\}$ *is FM,*

(ii) *for all $v \in X^*$,*

$$\inf_{x \in A} \langle v, x \rangle = \max_{\lambda \in \mathbb{R}^{(I)}_+} \inf_{x \in X} \{\langle v, x \rangle + \sum_{i \in I} \lambda_i g_i(x)\},$$

(iii) *for all $f \in \Gamma_0(X)$ with $\mathrm{dom} f \cap A \neq \emptyset$, where $\mathrm{epi} f^* + \mathrm{epi} \delta_A^*$ is w^*-closed,*

$$\inf_{x \in A} f(x) = \max_{\lambda \in \mathbb{R}^{(I)}_+} \inf_{x \in X} \Big\{ f(x) + \sum_{i \in I} \lambda_i g_i(x) \Big\},$$

where $\mathbb{R}^{(I)}_+ = \{\lambda \in \mathbb{R}^I \mid \forall i \in I, \lambda_i \ge 0, \{i \in I \mid \lambda_i \neq 0\}$ is finite$\}$.

FM and Theorem 4.1 is closely related to Jeyakumar's set containment characterization (Theorem 3.1).

In [13], Li, Ng and Pong investigated the basic constraint qualification for convex programming. Let $\{g_i \mid i \in I\} \subset \Gamma_0(X)$, then the family $\{g_i \mid i \in I\}$ is said to satisfy the basic constraint qualification (the BCQ) at $x \in A$ if

$$N_A(x) = \mathrm{cone}\,\mathrm{co} \bigcup_{i \in I(x)} \partial g_i(x),$$

where $A = \{x \in X \mid \forall i \in I, g_i(x) \le 0\}$ and $I(x) = \{i \in I \mid g_i(x) = 0\}$. Also, they investigated the following theorem which indicates that the BCQ is the weakest constraint qualification for a certain optimality condition.

Theorem 4.2. [13] $\{g_i \mid i \in I\} \subset \Gamma_0(X)$. *Assume that $A = \{x \in X \mid \forall i \in I, g_i(x) \le 0\} \neq \emptyset$ and $x_0 \in A$. Then, the following statements are equivalent:*

(i) $\{g_i(x) \le 0 \mid i \in I\}$ *satisfies the BCQ at x_0,*

(ii) *for all* $v \in X^*$*,* x_0 *is a minimizer of* v *in* A *if and only if there exists* $\lambda \in \mathbb{R}_+^{(I(x_0))}$ *such that*

$$-v \in \sum_{i \in I(x_0)} \lambda_i \partial g_i(x_0),$$

(iii) *for all* $f \in \Gamma_0(X)$ *with* $\mathrm{dom} f \cap A \neq \emptyset$ *and* $\mathrm{epi} f^* + \mathrm{epi}\delta_A^*$ *is* w^**-closed,* x_0 *is a minimizer of* f *in* A *if and only if there exists* $\lambda \in \mathbb{R}_+^{(I(x_0))}$ *such that*

$$0 \in \partial f(x_0) + \sum_{i \in I(x_0)} \lambda_i \partial g_i(x_0).$$

In this section, we investigate two constraint qualifications for a Lagrange-type duality and a newly optimality condition on quasiconvex programming. These constraint qualifications is similar to FM and the BCQ. This section is based on [24, 23].

4.1. **Closed cone constraint qualification.** In this section, we show a duality theorem for quasiconvex programming with the new constraint qualification which deals with the set containment characterization by the generator (Theorem 3.28). At first, we introduce a new closed cone constraint qualification for quasiconvex programming.

Definition 4.3. Let g be a lsc quasiconvex function from X to $\overline{\mathbb{R}}$ with a generator $\{(k_i, w_i) \mid i \in I\} \subset Q \times X^*$, $A = \{x \in X \mid g(x) \leq 0\} \neq \emptyset$ and there exists $i_0 \in I$ such that $k_{i_0}^{-1}(0) \in \mathbb{R}$. Then, the quasiconvex system $\{g(x) \leq 0\}$ satisfies the closed cone constraint qualification for quasiconvex programming (the Q-CCCQ) w.r.t. $\{(k_i, w_i) \mid i \in I\}$ if

$$\mathrm{cone}\,\mathrm{co} \bigcup_{i \in I} \{(w_i, \delta) \in X^* \times \mathbb{R} \mid k_i^{-1}(0) \leq \delta\}$$

is w^*-closed.

As a consequence of Theorem 3.28, $\{g(x) \leq 0\}$ satisfies the Q-CCCQ w.r.t. $\{(k_i, w_i) \mid i \in I\}$ if and only if the alternative form of the Q-CCCQ,

$$\mathrm{epi}\delta_A^* \subset \mathrm{cone}\,\mathrm{co} \bigcup_{i \in I} \{(w_i, \delta) \in X^* \times \mathbb{R} \mid k_i^{-1}(0) \leq \delta\}$$

holds.

In the remainder of the present paper, we fix a generator $\{(k_i, w_i) \mid i \in I\} \subset Q \times X^*$ of g and assume that there exists $i_0 \in I$ such that $w_{i_0} = 0$ and

$$k_{i_0}(t) = \begin{cases} -\infty & \text{if } t \leq 0, \\ 1 & \text{if } t > 0. \end{cases}$$

This assumption is trivial because $\{(k_i, w_i) \mid i \in I \setminus \{i_0\}\}$ is also generator of f. However, the assumption is important because it ensures the following inclusion, which is critical in the proof of Theorem 4.4:

$$\{0\} \times [0, \infty) \subset \mathrm{cone}\,\mathrm{co} \bigcup_{i \in I} \{(w_i, \delta) \in X^* \times \mathbb{R} \mid k_i^{-1}(0) \leq \delta\}. \tag{1}$$

Theorem 4.4. *Let g be a lsc quasiconvex function from X to $\overline{\mathbb{R}}$ with a generator $\{(k_i, w_i) \mid i \in I\} \subset Q \times X^*$. Assume that $A = \{x \in X \mid g(x) \leq 0\} \neq \emptyset$. Then, the following statements are equivalent:*

(i) *$\{g(x) \leq 0\}$ satisfies the Q-CCCQ w.r.t. $\{(k_i, w_i) \mid i \in I\}$,*

(ii) *for all $v \in X^*$,*

$$\inf_{x\in A} \langle v, x\rangle = \max_{\lambda\in\mathbb{R}^{(I)}_+} \inf_{x\in X} \left\{ \langle v, x\rangle + \sum_{i\in I} \lambda_i(\langle w_i, x\rangle - k_i^{-1}(0)) \right\},$$

(iii) *for all $f \in \Gamma_0(X)$ with $\mathrm{dom} f \cap A \neq \emptyset$ and $\mathrm{epi} f^* + \mathrm{epi}\delta^*_A$ is w^*-closed,*

$$\inf_{x\in A} f(x) = \max_{\lambda\in\mathbb{R}^{(I)}_+} \inf_{x\in X} \left\{ f(x) + \sum_{i\in I} \lambda_i(\langle w_i, x\rangle - k_i^{-1}(0)) \right\}.$$

Proof. First, we prove that (i) implies (iii). Let f be a lsc convex function with $\inf_{x\in A} f(x) \in \mathbb{R}$, where $\mathrm{dom} f \cap A \neq \emptyset$ and $\mathrm{epi} f^* + \mathrm{epi}\delta^*_A$ is w^*-closed. Then, as a result of Fenchel duality,

$$\inf_{x\in A} f(x) = \inf_{x\in X}\{f(x) + \delta_A(x)\} = \max_{v\in X^*}\{-f^*(v) - \delta^*_A(-v)\}.$$

Therefore, there exists $v_0 \in X^*$ such that $\inf_{x\in A} f(x) = -f^*(v_0) - \delta^*_A(-v_0)$. Furthermore, there exist $\lambda \in \mathbb{R}^{(I)}_+$ and $\delta \in \mathbb{R}^{(I)}$ such that $-v_0 = \Sigma_{i\in I}\lambda_i w_i$, for each $i \in I$ with $\lambda_i \neq 0$, $k_i^{-1}(0) \leq \delta_i$, for each $i \in I$ with $\lambda_i = 0$, $\delta_i = 0$, and $\delta^*_A(-v_0) = \Sigma_{i\in I}\lambda_i\delta_i$. If there exists i_0 such that $\lambda_{i_0} \neq 0$ and $\delta_{i_0} > k_{i_0}^{-1}(0)$, then there exists $\gamma \in \mathbb{R}$ such that $\gamma < \delta^*_A(-v_0)$ and $(-v_0, \gamma) \in \mathrm{epi}\delta^*_A$, which is a contradiction. Therefore, for each $i \in I$ with $\lambda_i \neq 0$, $\delta_i = k_i^{-1}(0)$ and (iii) holds.

Next, we show that (iii) implies (ii). Let $v \in X^*$ with $\inf_{x\in A} \langle v, x\rangle \in \mathbb{R}$. Then, because of condition (iii), there exists $\lambda \in \mathbb{R}^{(I)}_+$ such that

$$\inf_{x\in A} \langle v, x\rangle = -h(-\Sigma_{i\in I}\lambda_i w_i) - \Sigma_{i\in I}\lambda_i k_i^{-1}(0),$$

where h is the Fenchel conjugate of v. Clearly, $v = -\Sigma_{i\in I}\lambda_i w_i$. Therefore, (ii) holds.

Finally, we show that (ii) implies (i). Let $\{(v_k, \alpha_k)\} \subset \mathrm{cone}\,\mathrm{co} \bigcup_{i\in I} \{(w_i, \delta) \in X^* \times \mathbb{R} \mid k_i^{-1}(0) \leq \delta\}$ with (v_k, α_k) converging to (v_0, α_0). Since $(v_0, \alpha_0) \in \mathrm{cl}\,\mathrm{cone}\,\mathrm{co} \bigcup_{i\in I} \{(w_i, \delta) \in X^* \times \mathbb{R} \mid k_i^{-1}(0) \leq \delta\}$, by Theorem 3.28, we have $(v_0, \alpha_0) \in \mathrm{epi}\delta^*_A$. Hence,

$$\inf_{x\in A} -v_0(x) = -\delta^*_A(v_0) \geq -\alpha_0 > -\infty.$$

If $v_0 = 0$, then we can prove that $\alpha_0 \geq 0$. From inclusion (1),

$$(v_0, \alpha_0) \in \mathrm{cone}\,\mathrm{co} \bigcup_{i\in I} \{(w_i, \delta) \in X^* \times \mathbb{R} \mid k_i^{-1}(0) \leq \delta\}.$$

If $v_0 \neq 0$, then by (ii), we show that there exists $\lambda \in \mathbb{R}^{(I)}_+$ such that $v_0 = \Sigma_{i\in I}\lambda_i w_i$ and $\inf_{x\in A} -v_0(x) = -\Sigma_{i\in I}\lambda_i k_i^{-1}(0)$. Therefore $\Sigma_{i\in I}\lambda_i k_i^{-1}(0) \leq \alpha_0$. Because $v_0 \neq 0$, there exists $i_0 \in I$ such that $\lambda_{i_0} > 0$. Then, we set $\delta' \in \mathbb{R}^{(I)}$ as follows. For each

$i \in I$ with $\lambda_i \neq 0$ and $i \neq i_0$, $\delta'_i = k_i^{-1}(0)$, and for each $i \in I$ with $\lambda_i = 0$, $\delta'_i = 0$ and $\delta'_{i_0} = k_{i_0}^{-1}(0) + \frac{\alpha_0 - \Sigma_{i\in I}\lambda_i k_i^{-1}(0)}{\lambda_{i_0}}$. Then $\alpha_0 = \Sigma_{i\in I}\lambda_i \delta'_i$, that is,

$$(v_0, \alpha_0) \in \operatorname{cone} \operatorname{co} \bigcup_{i\in I} \{(w_i, \delta) \in X^* \times \mathbb{R} \mid k_i^{-1}(0) \leq \delta\}.$$

$\square$

Next, we explain the usefulness of the Q-CCCQ. We give the following important example.

Example 4.5. Let $X = \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}$, $g(x) = \frac{1}{2}x_1^2$, and consider the system $\{g(x) \leq 0\}$. Then, we can see that $A = \{x \in \mathbb{R}^n \mid g(x) \leq 0\} = \{x \in \mathbb{R}^n \mid x_1 = 0\}$, $g^* : \mathbb{R}^n \to \mathbb{R}$, $g^*(v) = \frac{1}{2}v_1^2$, and

$$\operatorname{cone} \operatorname{co} \operatorname{epi} g^* = (\mathbb{R}^n \times (0, \infty)) \cup \{x \in \mathbb{R}^n \mid x_1 = 0\},$$

that is, $\{g(x) \leq 0\}$ is not FM. However, we can choose a generator which satisfies the Q-CCCQ. Let $B = \{(1, 0, \cdots, 0), (-1, 0, \cdots, 0)\} \subset \mathbb{R}^n$, $k \in Q$ as follows:

$$k(t) := \begin{cases} \frac{1}{2}t^2 & t > 0, \\ 0 & t \leq 0. \end{cases}$$

Then, $g = \sup_{w\in B} k \circ w$, $k^{-1}(0) = 0$, and

$$\operatorname{cone} \operatorname{co} \cup_{w\in B} \{(w, \delta) \mid k^{-1}(0) \leq \delta\} = \mathbb{R}^n \times [0, \infty)$$

hold, that is, $\{g(x) \leq 0\}$ satisfies the Q-CCCQ w.r.t. $\{(k, w) \mid w \in B\}$. Also, $\{\langle w, x\rangle - k^{-1}(0) \leq 0 \mid w \in B\}$ is FM, hence we can use Theorem 4.1 and Theorem 4.4 to this example.

Finally, we give the following example for quasiconvex but not convex problem.

Example 4.6. Let $X = \mathbb{R}^n$, $a \in \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}$, $g(x) = \sqrt{\|x - a\|} - 2$, and consider the system $\{g(x) \leq 0\}$. Then, we can check that $A = \{x \in \mathbb{R}^n \mid \|x - a\| \leq 4\}$. Let $B = \{w \in \mathbb{R}^n \mid \|w\| = 1\}$, $k_w \in Q$ as follows:

$$k_w(t) := \begin{cases} \sqrt{t - \langle w, a\rangle} - 2 & t > \langle w, a\rangle, \\ 0 & t \leq \langle w, a\rangle. \end{cases}$$

, Then, $g = \sup_{w\in B} k_w \circ w$, $k_w^{-1}(0) = \langle w, a\rangle$, and $\operatorname{epi}\delta_A^* = \operatorname{cone}\operatorname{co} \cup_{w\in B} \{(w, \delta) \mid k_w^{-1}(0) \leq \delta\}$ hold, that is, $\{g(x) \leq 0\}$ satisfies the Q-CCCQ w.r.t. $\{(k_w, w) \mid w \in B\}$. Hence, we can use Theorem 4.4 to this example. Of course, $\{\langle w, x\rangle - k_w^{-1}(0) \leq 0 \mid w \in B\}$ is also FM, hence we can use Theorem 4.1 to this examples.

In any case, it is very profitable and useful to be able to represent constraint functions by using the notion of generator.

4.2. **Optimality conditions and the basic constraint qualification.** The purpose of this section is to generalize Theorem 4.2 for quasiconvex programming. By the notion of generator, we introduce a new subdifferential for quasiconvex functions, and by using this subdifferential, we investigate generalized results reported in previous studies.

At first, we introduce the new subdifferential for quasiconvex functions.

Definition 4.7. Let f be a lsc quasiconvex function with a generator $G = \{(k_s, w_s) \mid s \in S\} \subset Q \times X^*$, and assume that k_s is lower left-hand Dini differentiable for each $s \in S$. Then, we define the subdifferential of f at x_0 with respect to G as follows:

$$\partial_G f(x_0) = \mathrm{clco}\{D_- k_s(\langle w_s, x_0\rangle) w_s \mid s \in S(x_0)\},$$

where $S(x_0) = \{s \in S \mid f(x_0) = k_s \circ w_s(x_0)\}$.

This subdifferential is a generalized notion of the subdifferential for convex functions. Actually, if f is a convex function with the basic generator B_f, then

$$\begin{aligned}
\partial_{B_f} f(x_0) &= \mathrm{cl\,co}\{D_- k_v(\langle v, x_0\rangle) v \mid v \in \mathrm{dom} f^*, f(x_0) = k_v(\langle v, x_0\rangle)\} \\
&= \mathrm{cl\,co}\{v \mid v \in \mathrm{dom} f^*, f(x_0) = \langle v, x_0\rangle - f^*(v)\} \\
&= \partial f(x_0).
\end{aligned}$$

Also, if f is Gâteaux differentiable at x_0, k_s are differentiable at $\langle w_s, x_0\rangle$ for each $s \in S(x_0)$, and $S(x_0) \neq \emptyset$, then we can check $\partial_G f(x_0) = \{f'(x_0)\}$. Actually, for each $s \in S(x_0)$ and $d \in X$,

$$\begin{aligned}
\langle f'(x_0), d\rangle &= \lim_{t \to 0} \frac{f(x_0 + td) - f(x_0)}{t} \\
&\geq \lim_{t \to 0} \frac{k_i \circ w_s(x_0 + td) - k_s \circ w_s(x_0)}{t} \\
&= \langle k_s'(\langle w_s, x_0\rangle), d\rangle .
\end{aligned}$$

Similarly, we can prove that $\langle f'(x_0), -d\rangle \geq \langle k_s'(\langle w_s, x_0\rangle) w_s, -d\rangle$, that is, $f'(x_0) = k_i'(\langle w_s, x_0\rangle) w_s$.

Next, we show a necessary condition for a minimizer of a certain quasiconvex function in a closed convex set.

Theorem 4.8. *Let A be a closed convex subset of X, f be a lsc quasiconvex function with a generator $G = \{(k_s, w_s) \mid s \in S\} \subset Q \times X^*$. Assume that k_s is lower left-hand Dini differentiable for each $s \in S$ and at least one of the following holds:*

(i) *S is finite and k_s is continuous for each $s \in S$,*

(ii) *X is a Banach space, S is a compact topological space, $s \mapsto w_s$ is continuous on S to $(X^*, \|\cdot\|)$, $(s,t) \mapsto k_s(t)$ is usc on $S \times \mathbb{R}$, and $(s,t) \mapsto D_- k_s(t)$ is continuous on $S \times \mathbb{R}$.*

If x_0 is a local minimizer of f in A then,

$$0 \in \partial_G f(x_0) + N_A(x_0).$$

Proof. At first, we show that $\partial_G f(x_0)$ is w^*-compact. It is clear when the condition (i) holds. If the condition (ii) holds, then $S(x_0)$ is compact because $S(x_0) = \{s \in S \mid f(x_0) \leq k_s \circ w_s(x_0)\}$ and $s \mapsto k_s \circ w_s(x_0)$ is usc on S. Thus, $\{D_- k_s(\langle w_s, x_0\rangle) w_s \mid s \in S(x_0)\}$ is bounded since $s \mapsto w_s$ is continuous on S and $(s,t) \mapsto D_- k_s(t)$ is continuous on $S \times \mathbb{R}$. Hence, $\partial_G f(x_0)$ is w^*-compact by the Banach-Alaoglu theorem.

Now we assume that $0 \notin \partial_G f(x_0) + N_A(x_0)$. Since $\partial_G f(x_0) + N_A(x_0)$ is w^*-closed, we can find $d_0 \in X \setminus \{0\}$ satisfying

$$\langle y^*, d_0\rangle < 0 \leq \langle -x^*, d_0\rangle \,,$$

for all $y^* \in \partial_G f(x_0)$ and $x^* \in N_A(x_0)$. If $s \in S(x_0)$, then $D_- k_s(\langle w_s, x_0\rangle) > 0$ and $\langle w_s, d_0\rangle < 0$ since $D_- k_s(\langle w_s, x_0\rangle) w_s \in \partial_G f(x_0)$ and k_s is non-decreasing. From this, we have $\sup_{s\in S(x_0)} \langle w_s, d_0\rangle < 0$ and $d \mapsto \sup_{s \in S(x_0)} \langle w_s, d\rangle$ is usc. Indeed, it is clear when the condition (i) holds. If the condition (ii) holds, we can check them since $S(x_0)$ is compact and $s \mapsto w_s$ is continuous on S.

Therefore, there exists U_{d_0} a neighborhood of d_0 such that $\langle w_s, d\rangle < 0$ for each $s \in S(x_0)$ and $d \in U_{d_0}$. Since x_0 is a local minimizer of f in A, there exists U_{x_0} a neighborhood of x_0 such that for all $x \in U_{x_0} \cap A$, $f(x_0) \leq f(x)$.

Also $d_0 \in T_A(x_0) = \mathrm{cl} \cup_{\lambda > 0} \frac{A - x_0}{\lambda}$ because $\langle x^*, d_0\rangle \leq 0$ for all $x^* \in N_A(x_0)$. Then there exist $d_1 \in U_{d_0}$, $\lambda_0 > 0$ and $x_1 \in A$ such that $d_1 = \frac{x_1 - x_0}{\lambda_0}$. Put $x_n = (1 - \frac{1}{n})x_0 + \frac{1}{n}x_1 = x_0 + \frac{\lambda_0}{n} d_1$, then $x_n \in A \cap U_{x_0}$ for large enough n, therefore $f(x_0) \leq f(x_n)$.

If the condition (i) holds, since S is finite, we can find $s_0 \in S$ and a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $s_0 \in S(x_{n_i})$ for each $i \in \mathbb{N}$, and we have $s_0 \in S(x_0)$ because f and $k_{s_0} \circ w_{s_0}$ are continuous. For large enough $i \in \mathbb{N}$, $k_{s_0} \circ w_{s_0}(x_{n_i}) = f(x_{n_i}) \geq f(x_0) = k_{s_0} \circ w_{s_0}(x_0)$, and then,

$$\frac{k_{s_0}(\langle w_{s_0}, x_0\rangle + \frac{\lambda_0}{n_i}\langle w_{s_0}, d_1\rangle) - k_{s_0}(\langle w_{s_0}, x_0\rangle)}{\frac{\lambda_0}{n_i}\langle w_{s_0}, d_1\rangle} \leq 0,$$

since $d_1 \in U_{d_0}$. Therefore $D_- k_{s_0}(\langle w_{s_0}, x_0\rangle) \leq 0$, it is contradiction.

If the condition (ii) holds, all $S(x_n)$ are not empty because S is compact and $s \mapsto k_s \circ w_s(x_n)$ is usc on S. Let $\{s_n\}$ be a sequence satisfying $s_n \in S(x_n)$ for all $n \in \mathbb{N}$. Then there exists a subsequence $\{s_{n_i}\}$ of $\{s_n\}$ such that $\{s_{n_i}\}$ converges to some $s_0 \in S$. Therefore

$$\begin{aligned} f(x_0) &\leq \liminf_{i\to\infty} f(x_{n_i}) \\ &\leq \limsup_{i\to\infty} k_{s_{n_i}} \circ w_{s_{n_i}}(x_{n_i}) \\ &\leq k_{s_0} \circ w_{s_0}(x_0) \\ &\leq f(x_0), \end{aligned}$$

that is, $s_0 \in S(x_0)$. Then, for sufficiently large $i \in \mathbb{N}$, $k_{s_{n_i}} \circ w_{s_{n_i}}(x_{n_i}) = f(x_{n_i}) \geq f(x_0) \geq k_{s_{n_i}} \circ w_{s_{n_i}}(x_0)$ and $\langle w_{s_{n_i}}, d_1\rangle < 0$, because $s_0 \in S(x_0)$, $d_1 \in U_{d_0}$ and $\{w_{s_{n_i}}\}$ converges w_{s_0}. From this and $k_{s_{n_i}}$ is non-decreasing, $k_{s_{n_i}}$ is constant on interval

$[\langle w_{s_{n_i}}, x_0\rangle + \frac{\lambda_0}{n_i}\langle w_{s_{n_i}}, d_1\rangle, \langle w_{s_{n_i}}, x_0\rangle]$ and hence we have $D_-k_{s_{n_i}}(\langle w_{s_{n_i}}, x_0\rangle) = 0$. Finally we obtain $D_-k_{s_0}(\langle w_{s_0}, x_0\rangle) = 0$, but this is a contradiction. $\square$

On the other hand, in separable Banach space, a similar result was introduced when S is compact, f_s are locally Lipschitz, $f = \sup_{s\in S} f_s$, and certain assumptions hold in [18]. If condition (ii) holds and k_s are differentiable, then $k_s \circ w_s$ are locally Lipschitz. However, in Theorem 4.8, we assume that X is a usual Banach space and k_s are only lower left-hand Dini differentiable, thus, Theorem 4.8 is not a direct consequence of the result in [18]. Also, if f is a proper lsc convex function with basic generator B_f and $\mathrm{dom} f^*$ is compact, then condition (ii) holds. For this reason, it seems that condition (ii) is not so strong for quasiconvex programming.

We define a new constraint qualification and consider an optimality condition for quasiconvex programming with inequality constraints, and we prove that the new constraint qualification is the weakest constraint qualification for the optimality condition. At first, we introduce the following new constraint qualification.

Definition 4.9. Let $\{g_i \mid i \in I\}$ be a family of lsc quasiconvex functions from X to $\overline{\mathbb{R}}$, for each $i \in I$, $\{(k_{(i,j)}, w_{(i,j)}) \mid j \in J_i\} \subset Q \times X^*$ be a generator of g_i, $T = \{t = (i,j) \mid i \in I, j \in J_i\}$, $T(x) = \{t \in T \mid k_t(\langle w_t, x\rangle) = 0,\ k_t^{-1}(0) = \langle w_t, x\rangle\}$, and $A = \{x \in X \mid \forall i \in I, g_i(x) \le 0\}$.

The family $\{g_i \mid i \in I\}$ is said to satisfy the basic constraint qualification for quasiconvex programming (the Q-BCQ) with respect to $\{(k_t, w_t) \mid t \in T\}$ at $x \in A$ if

$$N_A(x) = \mathrm{cone}\,\mathrm{co} \bigcup_{t\in T(x)} \{w_t\}.$$

We can check that one inclusion always holds. Indeed, for each $t \in T(x)$ and $y \in A$, $\langle w_t, y\rangle \le \langle w_t, x\rangle$ because $\langle w_t, x\rangle = k_t^{-1}(0)$. Furthermore, $N_A(x)$ is a convex cone, this shows that $N_A(x) \supset \mathrm{cone}\,\mathrm{co} \bigcup_{t\in T(x)}\{w_t\}$. Therefore, the Q-BCQ is equivalent to the following inclusion

$$N_A(x) \subset \mathrm{cone}\,\mathrm{co} \bigcup_{t\in T(x)} \{w_t\}.$$

In the following theorem, we show an optimality condition for quasiconvex programming and the Q-BCQ is the weakest constraint qualification for this optimality condition. Let $Q_F(X)$ be the set of all quasiconvex functions which have a finite, continuous and lower left-hand Dini differentiable generator, that is,

$$Q_F(X) = \left\{ \sup_{s\in S} k_s \circ w_s \;\middle|\; \begin{array}{l} \{(h_s, u_s) \mid s \in S\} \subset Q \times X^*,\ S : \text{finite}, \\ \forall s \in S, h_s : \text{continuous and lower left-hand Dini diff.} \end{array} \right\}.$$

Theorem 4.10. *Let $\{g_i \mid i \in I\}$ be a family of lsc quasiconvex functions from X to $\overline{\mathbb{R}}$, for each $i \in I$, $\{(k_{(i,j)}, w_{(i,j)}) \mid j \in J_i\} \subset Q \times X^*$ be a generator of g_i, $T = \{t = (i,j) \mid i \in I, j \in J_i\}$, $T(x) = \{t \in T \mid k_t(\langle w_t, x\rangle) = 0,\ k_t^{-1}(0) = \langle w_t, x\rangle\}$, $A = \{x \in X \mid \forall i \in I, g_i(x) \le 0\}$ and $x_0 \in A$. Then, the following statements (i), (ii), (iii) and (iv) are equivalent:*

(i) *$\{g_i(x) \le 0 \mid i \in I\}$ satisfies the Q-BCQ w.r.t. $\{(k_t, w_t) \mid t \in T\}$ at x_0,*

(ii) *for each* $v \in X^*$, x_0 *is a minimizer of* v *in* A *if and only if there exists* $\lambda \in \mathbb{R}_+^{(T)}$ *such that* $\lambda_t = 0$ *for each* $t \in T \setminus T(x_0)$, *the complementarity condition, and*

$$-v = \sum_{t \in T} \lambda_t w_t,$$

(iii) *for each* $f \in \Gamma_0(X)$ *with* $\mathrm{dom} f \cap A \neq \emptyset$ *and* $\mathrm{epi} f^* + \mathrm{epi}\delta_A^*$ *is* w^**-closed,* x_0 *is a minimizer of* f *in* A *if and only if there exists* $\lambda \in \mathbb{R}_+^{(T)}$ *such that* $\lambda_t = 0$ *for each* $t \in T \setminus T(x_0)$, *and*

$$0 \in \partial f(x_0) + \sum_{t \in T} \lambda_t w_t,$$

(iv) *for all* $f \in Q_F(X)$ *with a generator* G, *if* x_0 *is a local minimizer of* f *in* A, *then, there exists* $\lambda \in \mathbb{R}_+^{(T)}$ *such that* $\lambda_t = 0$ *for each* $t \in T \setminus T(x_0)$, *and*

$$0 \in \partial_G f(x_0) + \sum_{t \in T} \lambda_t w_t.$$

Proof. We now first prove (i) implies (iii). By the assumption of f, the subdifferential sum formula holds, that is,

$$\partial(f + \delta_A)(x_0) = \partial f(x_0) + \partial \delta_A(x_0).$$

Because $\partial \delta_A(x_0) = N_A(x_0)$ and condition (i) holds,

$$x_0 \text{ minimizes } f \text{ on } A \iff 0 \in \partial f(x_0) + \mathrm{cone}\,\mathrm{co} \bigcup_{t \in T(x_0)} \{w_t\},$$

this shows that (iii) holds.

Next, it is clear that (iii) implies (ii) and (iv) implies (ii).

We now prove that (ii) implies (i). We want to show that if $x^* \in N_A(x_0)$ then $x^* \in \mathrm{cone}\,\mathrm{co} \bigcup_{t \in T(x_0)} \{w_t\}$. Let $x^* \in N_A(x_0)$. Because $x^* \in N_A(x_0)$, $\delta_A^*(x^*) = \langle x^*, x_0 \rangle$. Therefore, x_0 minimizes $-x^*$ on A. Then by using condition (ii), there exists $\lambda \in \mathbb{R}_+^{(T)}$ such that $x^* = \sum_{t \in T} \lambda_t w_t \in \mathrm{cone}\,\mathrm{co} \bigcup_{t \in T(x_0)} \{w_t\}$.

Finally, by using Theorem 4.8, we can prove (i) implies (iv). This completes the proof. □

In Theorem 4.10, $Q_F(X)$ corresponds to the condition (i) of Theorem 4.8. In the following theorem, we define $Q_C(X)$ which corresponds to the condition (ii) of Theorem 4.8 as follows,

$$Q_C(X) = \left\{ \sup_{s \in S} k_s \circ w_s \;\middle|\; \begin{array}{l} \{(h_s, u_s) \mid s \in S\} \subset Q \times X^*,\ S : \text{compact}, \\ s \mapsto u_s : \text{continuous}, (s,t) \mapsto h_s(t) : \text{usc}, \\ D_- h_s(t) \in \mathbb{R} \text{ and } (s,t) \mapsto D_- h_s(t) : \text{continuous}. \end{array} \right\}.$$

Theorem 4.11. *Let* $\{g_i \mid i \in I\}$ *be a family of lsc quasiconvex functions from* X *to* $\overline{\mathbb{R}}$, *for each* $i \in I$, $\{(k_{(i,j)}, w_{(i,j)}) \mid j \in J_i\} \subset Q \times X^*$ *be a generator of* g_i, $T = \{t = (i,j) \mid i \in I, j \in J_i\}$, $T(x) = \{t \in T \mid k_t(\langle w_t, x \rangle) = 0,\ k_t^{-1}(0) = \langle w_t, x \rangle\}$, $A = \{x \in X \mid \forall i \in I, g_i(x) \leq 0\}$ *and* $x_0 \in A$. *Assume that* X *is a Banach space,* *then, the following statements (v) is equivalent to the statements (i), (ii), (iii) and (iv) in Theorem 4.10.*

(v) *for all $f \in Q_C(X)$ with a generator $G = \{(k_s, w_s) \mid s \in S\} \subset Q \times X^*$, if x_0 is a local minimizer of f in A, then, there exists $\lambda \in \mathbb{R}_+^{(T)}$ such that $\lambda_t = 0$ for each $t \in T \setminus T(x_0)$, and*

$$0 \in \partial_G f(x_0) + \sum_{t \in T} \lambda_t w_t.$$

Proof. By using Theorem 4.8, we can prove (i) implies (v). Also, it is clear that (v) implies (ii). □

Next, we investigate a relation between the Q-BCQ and the BCQ. Let $\{g_i \mid i \in I\}$ be a family of proper lsc convex function with the basic generator, $T = \{(i, v) \mid i \in I, v \in \mathrm{dom} g_i^*\}$, then, for all $x \in A$, we can check

$$\bigcup_{(i,v) \in T(x)} \{v\} = \bigcup_{i \in I(x)} \partial g_i(x),$$

that is, the BCQ and the Q-BCQ w.r.t. the basic generator are equivalent. Furthermore, we can prove Theorem 4.2, by using Theorem 4.10.

Also, we can prove that the conditions (i), (ii) and (iii) in Theorem 4.10 are equivalent by using Theorem 4.2. Let $\{g_i \mid i \in I\}$ be a family of lsc quasiconvex functions from X to $\overline{\mathbb{R}}$, for each $i \in I$, $\{(k_{(i,j)}, w_{(i,j)}) \mid j \in J_i\} \subset Q \times X^*$ be a generator of g_i, and $T = \{t = (i, j) \mid i \in I, j \in J_i\}$. Then, $A = \{x \in X \mid \forall t \in T, w_t(x) - k_t^{-1}(0) \leq 0\}$ and $\partial(w_t - k_t^{-1}(0)) = \{w_t\}$ for each $t \in T$. Since $w_t - k_t^{-1}(0)$ is a continuous linear function, we can prove an equivalence relation of the conditions (i), (ii) and (iii) in Theorem 4.10 by using Theorem 4.2. Hence, we can see that (i), (ii) and (iii) of Theorem 4.10 and Theorem 4.2 are equivalent. However, (iv) of Theorem 4.10 and (v) of Theorem 4.11 are new results which concern quasiconvex programming and we can consider problems whose objective function is quasiconvex by using Theorem 4.10 and 4.11.

In the last of this section, we emphasize the usefulness of optimality conditions and Q-BCQ by some examples. At first, we show the following quasiconvex programming problem that Theorem 4.10 is used effectively.

Example 4.12. Let $X = \mathbb{R}^2$, $I = \{1, 2\}$, $g_1(x) = -(x_1 - 2)^3$, $g_2(x) = -(x_2 - 1)^5$ and $f(x) = \sqrt{|x_1 - 1| + |x_2 - 1|}$, then, f, g_1 and g_2 are continuous quasiconvex, and $A = \{x \in \mathbb{R}^2 \mid x_1 \geq 2, x_2 \geq 1\}$. Also, $G_1 = \{(k_1, (-1, 0)) \mid k(a) = (a + 2)^3\}$ is a generator of g_1, $G_2 = \{(k_2, (0, -1)) \mid k_2(a) = (a + 1)^5\}$ is a generator of g_2 and $G_0 = \{(h_1, (1, 1)), (h_2, (-1, 1)), (h_3, (-1, -1)), (h_4, (1, -1))\}$ is a generator of f, where h_1 be a function from $\mathbb{R}$ to $\mathbb{R}$ as follows:

$$h_1(a) = \begin{cases} \sqrt{a - 2} & a \geq 2, \\ 0 & \text{otherwise.} \end{cases}$$

and $h_2(a) = h_4(a) = h_1(a + 2)$, $h_3(a) = h_1(a + 4)$ for all $a \in \mathbb{R}$. We can check easily that the Q-BCQ w.r.t. $G_1 \cup G_2$ is satisfied at each point of A. We observe whether there exist $x \in A$ and $\lambda \in \mathbb{R}_+^2$ satisfying $0 \in \partial_{G_0} f(x) + \lambda_1(-1, 0) +$

$\lambda_2(0,-1)$ and the complementarity condition or not. If $x \in \mathrm{int}A$, then, $\partial_{G_0} f(x) = \{\frac{1}{2\sqrt{x_1+x_2-2}}(1,1)\}$ and $g_i(x) \neq 0$ $(i \in I)$, this implies $\lambda = 0$ if the complementarity condition holds. Hence, the optimality condition is not satisfied. If $x \in \{y \mid y_1 = 2, y_2 > 1\}$, then, $\lambda_2 = 0$ if the complementarity condition holds. Also, $\partial_{G_0} f(x) = \{\frac{1}{2\sqrt{x_1+x_2-2}}(1,1)\}$, that is, the optimality condition is not satisfied. If $x \in \{y \mid y_1 > 2, y_2 = 1\}$, then, $\partial_{G_0} f(x) = \mathrm{cl}\,\mathrm{co}\left\{\frac{1}{2\sqrt{x_1+x_2-2}}(1,1), \frac{1}{2\sqrt{x_1-x_2}}(1,-1)\right\}$ and $\lambda_1 = 0$ if the complementarity condition holds, that is, the optimality condition is not satisfied. If $x = (2,1)$, then,

$$\begin{aligned}
\partial_{G_0} f(x) &= \mathrm{cl}\,\mathrm{co}\{D_- h_1(\langle (1,1), x_0\rangle)(1,1), D_- h_4(\langle (1,-1), x_0\rangle)(1,-1)\} \\
&= \mathrm{cl}\,\mathrm{co}\left\{\frac{1}{2}(1,1), \frac{1}{2}(1,-1)\right\} \\
&= \left\{v \in \mathbb{R}^2 \,\middle|\, v_1 = \frac{1}{2}, v_2 \in \left[-\frac{1}{2}, \frac{1}{2}\right]\right\}.
\end{aligned}$$

Put $\lambda = \left(\frac{1}{2}, \frac{1}{2}\right)$, then, $0 \in \partial_{G_0} f(x) + \lambda_1(-1,0) + \lambda_2(0,-1)$. Therefore, $(2,1)$ satisfies the necessary condition for a local minimizer. In this case, the other $x \in A$ does not satisfy the optimality condition, hence $(2,1)$ is the global minimizer of f in A.

As stated above, Q-BCQ is used effectively for quasiconvex programming. At the same time, Q-BCQ is useful for convex programming. Now we show the following example that the Q-BCQ is satisfied and the BCQ is not satisfied. This example indicates that the range of functions which satisfies some of constraint qualifications is extended. This extension is very important because the lack of constraint qualifications can cause theoretical and numerical difficulties in applications.

Example 4.13. Let $X = \mathbb{R}^2$, $I = \{1\}$, $g(x) = (x_1 - x_2)^2$. Then, $A = \{y \mid y_1 = y_2\}$, for all $y \in A$, $N_A(y) = \{v \mid v_1 + v_2 = 0\}$, $I(y) = I$. Also, the BCQ is not satisfied at any point $y \in A$ because $\nabla g(y) = 0$. However, we can choose a suitable generator for satisfying the Q-BCQ. Let k be a function from $\mathbb{R}$ to $\mathbb{R}$ as follows:

$$k(t) = \begin{cases} t^2 & t \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

let $J = \{(k,(1,-1)), (k,(-1,1))\}$. Then, J is a generator of g. Furthermore, for all $y \in A$, $k(\langle (1,-1), (y_1,y_2)\rangle) = k(\langle (-1,1), (y_1,y_2)\rangle) = 0$, and

$$N_A(y) = \{v \mid v_1 + v_2 = 0\} = \mathrm{cone}\,\mathrm{co} \bigcup \{(1,-1), (-1,1)\}.$$

Therefore the Q-BCQ w.r.t. J at y is satisfied.

Let $f(x) = (x_1 - 5)^2 + (x_2 - 3)^2$, then, f is a continuous convex function. Since Q-BCQ is satisfied, we can find a minimizer by using an optimality condition in this paper. We observe whether there exist $x \in A$ and $\lambda \in \mathbb{R}^2_+$ satisfying $0 \in \partial f(x) + \lambda_1(1,-1) + \lambda_2(-1,1)$ and the complementarity condition or not. We can check easily that $\partial f(x) = \{\nabla f(x)\} = \{(2(x_1-5), 2(x_2-3))\}$. If there exists λ satisfying the optimality condition, then, we can calculate $x = (4,4)$. Put $\lambda = (0,2)$, then $0 \in \partial f(x) + \lambda_1(1,-1) + \lambda_2(-1,1)$. By using Theorem 4.10, $(4,4)$ is the global minimizer. Also, let $g_1 = \langle (1,-1), x\rangle$ and $g_2 = \langle (-1,1), x\rangle$, then

$A = \{x \in \mathbb{R}^2 \mid g_i(x) \leq 0 (i = 1, 2)\}$ and $\{g_i(x) \leq 0 \mid i = 1, 2\}$ satisfies BCQ. Hence, we can use Theorem 4.2 to this example.

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