Mem. Fac. Sci. Eng. Shimane Univ.
Series B: Mathematical Science
43 (2010), pp. 13–26

ORDER AND TOPOLOGICAL STRUCTURES OF POSETS OF THE FORMAL BALLS ON METRIC SPACES

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(Received: January 4, 2010)

Abstract. In the present note, we present a brief survey on the order-theoretic structures on the spaces of formal balls on metric spaces from the topological point of view. Some open questions are also included.

1. Introduction

The concept of formal balls was introduced by Weihrauch and Schreiber [19] to represent a metric space in a domain, and several authors studies the posets of the formal balls as a computational model for a metric space [6, 2, 11, 12].

Let $\mathbb{R}$ and $\mathbb{R}_+$ denote the sets of real numbers and non-negative real numbers, respectively. Let (X, d) be a metric space and $\mathbf{B}^+(X, d) = X \times \mathbb{R}_+$. An element of $\mathbf{B}^+(X, d)$ is called a *formal ball*.

In [17], Tsuiki and Hattori extended the notion of formal balls to balls having negative radii, say generalized formal balls, i.e., let $\mathbf{B}(X, d) = X \times \mathbb{R}$ and we call an element of $\mathbf{B}(X, d)$ a *generalized formal ball*.

We induce an partial order in $\mathbf{B}^+(X, d)$ ($\mathbf{B}(X, d)$) as $(x, r) \sqsubseteq (y, s)$ if $d(x, y) \leq r - s$. Then $(\mathbf{B}^+(X, d), \sqsubseteq)$ and $(\mathbf{B}(X, d), \sqsubseteq)$ are continuous posets, and they have the Lawson, and the Martin topologies. Tsuiki and Hattori ([10], [17]) investigated the Lawson topology of $\mathbf{B}^+(X, d)$ and $\mathbf{B}(X, d)$, and Hashiriura [11] investigated the Martin topology of $\mathbf{B}^+(X, d)$ and $\mathbf{B}(X, d)$ from the topological point of view. In the present note, we sketch these recent studies of the relationship between order-theoretic structures and topological structures in $\mathbf{B}^+(X, d)$ ($\mathbf{B}(X, d)$).

We give some preliminaries from the domain theory and spaces of (generalized) formal balls in section 2. In section 3, we discuss on the relation between the Lawson topology and the product topology on $\mathbf{B}^+(X, d)$ ($\mathbf{B}(X, d)$) which were obtained in [17] and [10]. The relation between the Martin topology and the product topology on $\mathbf{B}^+(X, d)$ ($\mathbf{B}(X, d)$) which were obtained in [11] are presented in section 4. In

2000 *Mathematics Subject Classification.* Primary 54E35; Secondary 46B20, 54B10, 54B20.

Key words and phrases. Metric space, domain, Lawson topology, Martin topology, formal ball, hyperbolic topology, Sorgenfrey line.

The author was partially supported by Grant-in-Aid for Scientific Research (No.19540086) from Japan Society for the Promotion of Science.

section 5, we discuss on topologies of the set of real numbers $\mathbb{R}$ which are between the Euclidean topology and the topology of the Sorgenfrey line. We also ask some open questions.

2. Preliminaries and notation

We present preliminary results from domain theory. The reader would refer [8] for the details of the domain theory.

2.1. **Fundamental concepts from the domain theory.** Let $(L, \leq)$ be a partially ordered set (abbrev. *poset*). Then a non-empty subset D of $(L, \leq)$ is called a *directed set* if every finite subset of D has an upper bound. For elements $x, y \in L$, we say that x is *way below* y, and we write $x \ll y$, if for every directed subset D of L for which $\sup D$ exists and $y \leq \sup D$, there exists $d \in D$ such that $x \leq d$. For a poset $(L, \leq)$, $x \in L$ and $A \subset L$ we use the following notation:

$\Uparrow x = \{y \in L : x \ll y\}$,
$\Downarrow x = \{y \in L : x \gg y\}$,
$\Uparrow A = \{y \in L : x \ll y$ for some $x \in A\}$, and
$\Downarrow A = \{y \in L : x \gg y$ for some $x \in A\}$.

Similarly, we write

$\uparrow x = \{y \in L : x \leq y\}$,
$\downarrow x = \{y \in L : x \geq y\}$,
$\uparrow A = \{y \in L : x \leq y$ for some $x \in A\}$, and
$\downarrow A = \{y \in L : x \geq y$ for some $x \in A\}$.

A poset L is called a *directed complete poset* (abbrev. *dcpo*) if every directed subset of L has a least upper bound, and L is said to be *continuous*, if $\Downarrow x$ is directed and $x = \sup \downarrow x$ for each $x \in L$. A poset L is called a *domain* if L is a continuous dcpo.

Let L be a poset and U a subset of L. Then U is said to be *Scott open* if $U = \uparrow U$ and, for every directed set D of L with $\sup D \in U$, $D \cap U \neq \emptyset$. The family $\sigma(L)$ of all Scott open sets of L is a topology of L and we say it the *Scott topology*. It is well known that $\Uparrow x$ is a Scott open set for every $x \in L$ and $\{\Uparrow x : x \in L\}$ forms an open base for the Scott topology if L is continuous (cf. [8] and [13]).

We call the topology of a poset L generated by $\{L - \uparrow x : x \in L\}$ the *lower topology* and we denote it by $\omega(L)$. The join $\sigma(L) \vee \omega(L)$ of the Scott topology $\sigma(L)$ and the lower topology $\omega(L)$ is called the *Lawson topology*. The Lawson topology of L is denoted by $\lambda(L)$. If L is continuous, then the Lawson topology of L is generated by $\{\Uparrow x : x \in L\} \cup \{L - \uparrow x : x \in L\}$. The Scott topology and the lower topology satisfy only the T_0-separation axiom. On the other hand, the Lawson topology of a continuous poset is regular T_1.

Let L be a continuous poset. The *Martin topology* $\mu(L)$ of L is a topology on L generating by the sets of the form $\downarrow x \cap \Uparrow y$ for $x, y \in L$. The Martin topology was introduced by K. Martin [14] to study a quantitative statements on programs. It is easy to see that the Martin topology is stronger than the Lawson topology, and the sets $\downarrow x \cap \Uparrow y$, $x, y \in L$ are clopen in $\mu(L)$. Hence the Martin topology $\mu(L)$ is regular and zero-dimensioanl.

2.2. **Domains of formal balls of metric spaces.** Let (X, d) be a metric space. For each point x of (X, d) and each $r \in \mathbb{R}_+$ we denote the *r-open ball* at x by $S_r(x) = \{y \in X : d(x, y) < r\}$ and the *r-closed ball* at x by $B_r(x) = \{y \in X : d(x, y) \leq r\}$.

For a metric space (X, d) let $\mathbf{B}^+(X, d) = X \times \mathbb{R}_+$ and $\mathbf{B}(X, d) = X \times \mathbb{R}$. An element of $\mathbf{B}^+(X, d)$ ($\mathbf{B}(X, d)$) is called a *formal ball* (*generalized formal ball*). We usually denote $\mathbf{B}^+(X, d)$ and $\mathbf{B}(X, d)$ by $\mathbf{B}^+X$ and $\mathbf{B}X$, respectively.

We induce an partial order in $\mathbf{B}^+X$ ($\mathbf{B}X$) as follows:

$$(x, r) \sqsubseteq (y, s) \text{ if } d(x, y) \leq r - s.$$

Roughly speaking, for formal balls (x, r) and (y, s) the relation $(x, r) \sqsubseteq (y, s)$ implies that the closed ball $B_s(y)$ is contained in the closed ball $B_r(x)$. Furthermore, $(x, r) \ll (y, s)$ implies that the closed ball $B_s(y)$ is strictly included in the closed ball $B_r(x)$ (see Lemma 2.1 below). The concept of formal balls was introduced by Weihrauch and Schreiber [19], and Tsuiki and Hattori [17] extended them to generalized formal balls. Edalat and Heckmann [6] gave the following fundamental results on the posets of formal balls.

Lemma 2.1 ([6]). *Let (X, d) be a metric space and $(x, r), (y, s) \in \mathbf{B}^+X$. Then $(x, r) \ll (y, s)$ if and only if $d(x, y) < r - s$.*

Theorem 2.2 ([6]). *For a metric space (X, d) $\mathbf{B}^+X$ is a continuous poset. Furthermore, $\mathbf{B}^+X$ is a dcpo if and only if (X, d) is complete.*

Let $\mathrm{Max}\,(L)$ denote the set of all maximal elements of a poset L. A poset L is said to be a *computational model* for a topological space X if the relative Scott and the relative Lawson topologies on $\mathrm{Max}\,(L)$ coincide and X is homeomorphic to $\mathrm{Max}\,(L)$. The following theorem implies that the posets of formal balls have approximation structures for metric spaces.

Theorem 2.3 ([6]). *For a metric space (X, d) $\mathbf{B}^+X$ is a computational model for (X, d).*

We refer the reader to [8] for the details of the domain theory, and [7] for topology.

3. The Lawson topology on the space of formal balls

In this section, we describe several properties of the Lawson topology of the posets of formal balls.

3.1. **The Lawson topology and the product topology.** Let λ_d denote the Lawson topology of the set $\mathbf{B}^+X$ ($\mathbf{B}X$) of formal balls (generalized formal balls) of a metric space (X, d). The sets $\mathbf{B}^+X$ and $\mathbf{B}X$ also naturally have the product topology of the metric topology of X and the Euclidean topology of $\mathbb{R}_+$ and $\mathbb{R}$, respectively. We denote π by the product topology of $\mathbf{B}^+X$ and $\mathbf{B}X$. It is natural to ask whether the Lawson topologies λ_d of $\mathbf{B}^+X$ and $\mathbf{B}X$ coincide with the product topology π. We can easily see that the Lawson topology λ_d is weaker than the product topology π in $\mathbf{B}^+X$ and $\mathbf{B}X$ (cf. [17, Corollary 7]). Since the Lawson and the product topologies of $\mathbf{B}^+X$ are the subspace topologies of the Lawson topology and the product topology of $\mathbf{B}X$ respectively, if the Lawson and the product topologies coincide in the poset of the generalized formal balls $\mathbf{B}X$ then they coincide in $\mathbf{B}^+X$. In [17], we have the following.

Theorem 3.1 ([17, Theorem 9]). *If (X,d) is a totally bounded metric space (in particular, (X,d) is compact), then the Lawson topology λ_d and the product topology π coincide in $\mathbf{B}X$.*

For the case of $\mathbf{B}^+X$, we can generalize the theorem above.

Theorem 3.2 ([17, Theorem 14]). *Let (X,d) be a metric space. If for each bounded subset A of X the restriction of d on A is totally bounded, then the Lawson topology λ_d and the product topology π coincide in $\mathbf{B}^+X$.*

Now, we have the examples concerning the theorems above.

Example 3.3 ([17, Examples 12 and 15]). Let $\mathbb{N}$ be the set of natural numbers. Then there are metrics d_1, d_2 and d_3 in $\mathbb{N}$ which induce the discrete topology in $\mathbb{N}$ such that

(a) $\lambda_{d_1} \neq \pi$ neither in $\mathbf{B}^+\mathbb{N}$ nor $\mathbf{B}\mathbb{N}$,
(b) $\lambda_{d_2} = \pi$ in $\mathbf{B}^+\mathbb{N}$, but $\lambda_{d_2} \neq \pi$ in $\mathbf{B}\mathbb{N}$,
(c) $\lambda_{d_3} = \pi$ in $\mathbf{B}^+\mathbb{N}$ and $\mathbf{B}\mathbb{N}$.

Proof. We define the metrics d_1, d_2 and d_3 on $\mathbb{N}$ as follows:

$$d_1(x,y) = \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } 1 \in \{x,y\} \textit{ and } x \neq y, \\ 2, & \text{otherwise}, \end{cases}$$

$$d_2(x,y) = \begin{cases} 0, & \text{if } x = y, \\ x+y, & \text{if } x \neq y, \end{cases}$$

$$d_3(x,y) = \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } x \neq y. \end{cases}$$

Then one can show that d_1, d_2 and d_3 are desired. □

By Example 3.3 (b), it follows that the total boundedness in Theorem 3.1 can not be generalized to the totally boundedness for bounded sets as in Theorem 3.2. It also follows from Example 3.3 that the Lawson topology of the poset of (generalized) formal balls depends on a metric function, not an induced topology.

Now, we turn our attention to normed linear spaces. As we have shown in Example 3.3, even when the Lawson and the product topologies coincide on $\mathbf{B}^+X$, they may not coincide on $\mathbf{B}X$ in general. Now, we show that the Lawson and the product topologies of $\mathbf{B}X$ coincide if X is a normed linear space.

Proposition 3.4 ([17, Proposition 16]). *Let $(X, \|\cdot\|)$ be a normed linear space and d the metric induced by the norm $\|\cdot\|$. If the Lawson topology and the product topology coincide on $\mathbf{B}^+X$, then they coincide also on $\mathbf{B}X$.*

Since every finite dimensional normed linear space satisfies the condition in Theorem 3.2, it follows that for every finite dimensional normed linear space X the Lawson and the product topologies coincide on $\mathbf{B}^+X$ and $\mathbf{B}X$. Hence, $(\mathbf{B}^+\mathbb{R}^n, \lambda_d) = (\mathbb{R}^n \times \mathbb{R}_+, \pi)$ and $(\mathbf{B}\mathbb{R}^n, \lambda_d) = (\mathbb{R}^{n+1}, \pi)$, where d is the Euclidean metric on $\mathbb{R}^n$.

Let dim denote the covering dimension of a topological space (cf. [7]). Since $\dim(X \times \mathbb{I}) = \dim X + 1$ for every space X (cf. [16]), by Theorem 3.1, it follows that if a metric space (X, d) is totally bounded, then $\dim(\mathbf{B}X, \lambda_d) = \dim X + 1$. Now, we may ask the following.

Question 3.5. Let (X, d) a metric space. Does the equality $\dim(\mathbf{B}X, \lambda_d)) = \dim X + 1$ (or the inequality $\dim(\mathbf{B}X, \lambda_d) \leq \dim X + 1$) hold?

We give some comments from the topological point of view. Let (X, d) be a metric space and $\mathcal{B}(X) = \{B_r(x) : x \in X, r > 0\}$ denote the set of "real" closed balls in X. Then $\mathcal{B}(X)$ is a subset of the set $\mathcal{F}(X)$ of all (non-empty) closed subsets of X. It is well known that $\mathcal{F}(X)$ has several hyperspace topologies, say the Hausdorff metric topology, the Wijsman topology, the Fell topology and the hit-and miss topology, etc (cf. [1]). It might be interested in topological properties of $\mathcal{B}(X)$ as a subspace of hyperspace topology. Now, we naturally introduce the *Lawson topology* λ in $\mathcal{B}(X)$ as follows:

Definition 3.6. Let (X, d) a metric space. For each $B_r(x) \in \mathcal{B}(X)$, we put

$$(B_r(x)^c)^+ = \{B \in \mathcal{B}(X) : B \setminus B_r(x) \neq \emptyset\},$$

$$B_r(x)^{++} = \{B \in \mathcal{B}(X) : S_\varepsilon(B) \subset B_r(x) \text{ for some } \varepsilon > 0\}.$$

We call the topology τ_λ of $\mathcal{B}(X)$ generated by the sets of the form $(B_r(x)^c)^+$ and $B_r(x)^{++}$ for $B_r(x) \in \mathcal{B}(X)$, the *Lawson topology* of $\mathcal{B}(X)$.

For each $F \in \mathcal{F}(X)$, we put

$$(F^c)^+ = \{E \in \mathcal{F}(X) : E \setminus F \neq \emptyset\},$$

$$F^{++} = \{E \in \mathcal{F}(X) : S_\varepsilon(E) \subset F \text{ for some } \varepsilon > 0\}.$$

We call the topology sup-τ_λ of $\mathcal{F}(X)$ generated by the sets of the form $(F^c)^+$ and F^{++} for $F \in \mathcal{F}(X)$ the *sup-Lawson topology* of $\mathcal{F}(X)$.

It is clear that the subspace topology of $\mathcal{B}(X)$ of sup-Lawson topology of $\mathcal{F}(X)$ is stronger than the Lawson topology of $\mathcal{B}(X)$. One can show easily the following.

Proposition 3.7. *Let $(X, \|\cdot\|)$ be a normed linear space and d the metric induced by the norm $\|\cdot\|$. Then $(\mathbf{B}^+X, \lambda_d)$ is homeomorphic to $(\mathcal{B}(X), \tau_\lambda)$.*

We may ask the following.

Question 3.8. Investigate the topological properties of the Lawson topology in $\mathcal{B}(X)$. In particular, how are the relations between the Lawson topology and other hyperspace topologies in $\mathcal{B}(X)$?

3.2. **The hyperbolic topology of metric spaces.** In [17], we introduced the hyperbolic topology in a metric space (X, d) motivated by an argument of the Lawson topology of the space of generalized formal balls. In this section, we describe several properties of the hyperbolic topology which are given in [17] and [10].

Let (X, d) be a metric space. We consider a new topology derived from by a subspace topology of certain subspaces of the space of generalized formal balls $(\mathbf{B}X, \lambda_d)$. Let $p : \mathbf{B}X \to X$ be the natural projection, $p(x, s) = x$ for each $(x, s) \in \mathbf{B}X$. It is clear that the restriction of p over $X \times \{t\}$ is a homeomorphism onto X for each $t \in \mathbb{R}$. Now, we consider the subspace $\mathrm{Bd}(a, u) = \{(y, s) \in \mathbf{B}X : d(a, y) = u - s\}$ of $(\mathbf{B}X, \lambda_d)$ for $(a, u) \in \mathbf{B}X$. Since the restriction of p over $\mathrm{Bd}(a, u)$ is a bijection, this derives a topology $\theta_{(a,u)}$ of X, i.e., $\theta_{(a,u)} = \{p(\mathrm{Bd}(a, u) \cap U) : U \in \lambda_d\}$. What is the topology $\theta_{(a,u)}$? It is easy to see that the subspaces $\mathrm{Bd}(a, u)$ and $\mathrm{Bd}(a, v)$ drive the same topology of X by the projection p for each $u, v \in \mathbb{R}$. Hence, it suffices to consider about $Bd(a, 0)$, i.e., $u = 0$.

Since the Lawson topology of $\mathbf{B}X$ is generated by sets of the form $\Uparrow(x, r) = \{(y, s) : d(x, y) < r - s\}$ and $\mathbf{B}X - \uparrow(x, r) = \{(y, s) : d(x, y) > r - s\}$ for $(x, r) \in \mathbf{B}X$, the subspace topology of the set $\mathrm{Bd}(a, 0) = \{(y, s) \in \mathbf{B}X : d(a, y) = -s\}$ is generated by sets of the form $\{(y, s) \in \mathrm{Bd}(a, 0) : d(y, x) < r - s\} = \{(y, -d(a, y)) \in \mathbf{B}X : d(y, x) - d(a, y) < r\}$ and $\{(y, s) \in \mathrm{Bd}(a, 0) : d(y, x) > r - s\} = \{(y, -d(a, y)) \in \mathbf{B}X : d(y, x) - d(a, y) > r\}$ for $(x, r) \in \mathbf{B}X$. Hence, θ_a is generated by sets of the form $\{y : d(y, x) - d(a, y) < r\}$ and $\{y : d(y, x) - d(a, y) > r\}$ for $x \in X$ and $r \in \mathbb{R}$. Since $d(y, x) - d(a, y) \geq -d(a, x)$, if $r \leq -d(a, x)$ then $\{y : d(y, x) - d(a, y) < r\} = \emptyset$ and $\{y : d(y, x) - d(a, y) > r\} = X$. Similarly, since $d(y, x) - d(a, y) \leq d(a, x)$, if $r \geq d(a, x)$ then $\{y : d(y, x) - d(a, y) < r\} = X$ and $\{y : d(y, x) - d(a, y) > r\} = \emptyset$. Hence, we can assume that $-d(a, x) < r < d(x, a)$. In [17], we obtained that θ_a does not depend on the choice of the point $a \in X$ ([17, Theorem 25]). Hence, we write the topology θ_a by θ, and call this topology the *hyperbolic topology* of a metric space (X, d). Since the hyperbolic topology does not depend on the choice of the point $a \in X$, θ can be generated by $\{y : d(y, b) - d(a, y) < r\}$ and $\{y : d(y, b) - d(a, y) > r\}$ for $a, b \in X$ and $-d(a, b) < r < d(a, b)$. Hence, it follows that θ is generated by sets of the form $\{y : d(a, y) - d(b, y) < r\}$ for $a, b \in X$ with $-d(a, b) < r < d(a, b)$.

It is natural to ask how different is the hyperbolic topology from the metric topology. At first we present an example which shows that the hyperbolic topology depends on the metric, but it does not depend on the induced topology.

Example 3.9 ([17, Example 30])**.** Let X be an uncountable set and $x_0 \in X$. Then we define a metric d_1 of X as follows (cf. Example 3.1):

$$d_1(x, y) = \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } x_0 \in \{x, y\} \text{ and } x \neq y, \\ 2, & \text{otherwise,} \end{cases}$$

Then the hyperbolic topology of (X, d_1) is generated by those sets $\{x\}$ for $x \in X - \{x_0\}$ and $X - A$, where A is a finite subset of X with $x_0 \notin A$. Hence, since X

is uncountable, the hyperbolic topology of (X, d_1) is not first countable at x_0. On the other hand, let d be the discrete metric on X, i.e.,

$$d(x,y) = \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } x \neq y. \end{cases}$$

Then the hyperbolic topology of (X, d) is the discrete topology, and hence it is first countable.

It is easy to see the following.

Proposition 3.10 ([17, Proposition 23]). *For a metric space (X, d) the hyperbolic topology is weaker than the topology induced by the metric d.*

In [17], we obtained the following result, which shows that the Lawson topology of the poset of formal balls is closely related to the hyperbolic topology of a metric space.

Theorem 3.11 ([17, Theorem 29]). *For a metric space (X, d), the following are equivalent.*

(1) The Lawson topology λ_d coincides with the product topology π in $\mathbf{B}X$.

(2) The hyperbolic topology θ induced by the metric d coincides with the metric topology in X.

Example 3.9 also shows that the hyperbolic topology and the metric topology do not coincide in general. As a corollary to Theorems 3.1 and 3.11, we have the following.

Corollary 3.12 ([10, Proposition 2.3]). *If (X, d) is a totally bounded metric space (in particular, (X, d) is a compact metric space), then the hyperbolic topology and the metric topology coincide.*

Now, we turn our attention to normed linear spaces. For normed linear spaces, we have the following.

Theorem 3.13 ([10, Theorem 2.7]). *The hyperbolic topology and the norm topology coincide for finite-dimensional normed linear spaces.*

We consider about the normed linear spaces $L_p(\Omega, \Sigma, \mu)$, where $1 \leq p \leq \infty$.

We recall some definitions. For an infinite sequence $x = (x_1, x_2, x_3, \dots)$ of real numbers, we define the p-norm $\|x\|_p = \sqrt[p]{\sum_{i=1}^{\infty} |x_i|^p}$ for $1 \leq p < \infty$ and $\|x\|_\infty = \sup\{|x_i| : i = 1, 2, 3, \dots\}$. For $1 \leq p \leq \infty$, we define the normed space ℓ_p to be the set of all infinite sequences $x = (x_1, x_2, x_3, \dots)$ with $\|x\|_p < \infty$.

Let Σ be a σ-algebra of subsets of a set Ω and μ a positive measure on Σ. We assume that $\Omega \in \Sigma$. For a real number p with $1 \leq p < \infty$ let $\mathcal{L}^p(\Omega, \Sigma, \mu)$ be the set of all measurable functions f such that $|f|^p$ is integrable. For each $f \in \mathcal{L}^p(\Omega, \Sigma, \mu)$ we define the p-norm $\|f\|_p$ as

$$\|f\|_p = \left(\int_\Omega |f|^p d\mu \right)^{1/p}.$$

A measurable function f is said to be *essentially bounded* if there is a constant t such that $|f| \leq t$ a.e. Let $\mathcal{L}^\infty(\Omega, \Sigma, \mu)$ be the set of all essentially bounded measurable functions. For $f \in \mathcal{L}^\infty(\Omega, \Sigma, \mu)$, we define

$$\|f\|_\infty = \inf\{t : t > 0, \mu(\{x : x \in \Omega, |f(x)| > t\}) = 0\}.$$

For $1 \leq p \leq \infty$ and $f, g \in \mathcal{L}^p(\Omega, \Sigma, \mu)$, we say that f and g are *equivalent* if $f = g$ a.e. Let $\tilde{f}$ denote the class of all functions in $\mathcal{L}^p(\Omega, \Sigma, \mu)$ which are equivalent to f. We define the normed space $L_p(\Omega, \Sigma, \mu)$ to be the space of all equivalence classes of functions in $\mathcal{L}^p(\Omega, \Sigma, \mu)$ with the p-norm. We notice that we use $\|\cdot\|$ for the p-norm in $L_p(\Omega, \Sigma, \mu)$ instead of $\|\cdot\|_p$ without any confusions.

If we consider the interval $I = [0, 1]$ as Ω, the collection of Lebesgue-measurable subsets of I as Σ, and the Lebesgue measure on I as μ, then $L_p(\Omega, \Sigma, \mu)$ is denoted by $L_p(I)$. We notice that if Ω is $\mathbb{N}$, Σ is the collection of all subsets of $\mathbb{N}$, and μ is the counting measure on Σ, then $L_p(\Omega, \Sigma, \mu)$ coincides with ℓ_p. Finally, we define $C(I)$ to be the Banach space of real-valued continuous functions on the interval $I = [0, 1]$ with the norm $\|f\| = \sup\{|f(x)| : x \in I\}$.

Further, we refer the reader to [5] for the details about normed linear spaces.

Now, we consider the relation between the hyperbolic topology and the norm topology for normed linear spaces $L_p(\Omega, \Sigma, \mu)$, where Σ is a σ-algebra of subsets of a set Ω, and μ is a positive measure on Σ.

Definition 3.14 ([4])**.** A normed linear space $(X, \|\cdot\|)$ is said to be *uniformly convex* if for every $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ such that for each $x, y \in X$ with $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \varepsilon$, $\frac{\|x+y\|}{2} < 1 - \delta(\varepsilon)$.

A normed linear space $(X, \|\cdot\|)$ is said to be *locally uniformly convex* if for each $x \in X$ with $\|x\| = 1$ and $\varepsilon > 0$ there is $\delta(x, \varepsilon) > 0$ such that for each $y \in X$ with $\|y\| = 1$ and $\|x - y\| \geq \varepsilon$, $\frac{\|x+y\|}{2} < 1 - \delta(x, \varepsilon)$.

We notice that the sum norm and the max norm on $\mathbb{R}^2$ are not locally uniformly convex. It is known that $L_p(\Omega, \Sigma, \mu)$ for $1 < p < \infty$ are uniformly convex ([4]), and hence it is locally uniformly convex.

Theorem 3.15 ([10, Theorem 3.2])**.** *If $(X, \|\cdot\|)$ is a locally uniformly convex normed linear space, then the hyperbolic topology coincides with the norm topology on X.*

As we noticed above, for $1 < p < \infty$, the normed space $L_p(\Omega, \Sigma, \mu)$ is uniformly convex, we have the following.

Corollary 3.16 ([10, Corollary 3.3])**.** *Let $1 < p < \infty$, Σ a σ-algebra of subsets of a set Ω, and μ a positive measure on Σ. The hyperbolic topology and the norm topology coincide on $L_p(\Omega, \Sigma, \mu)$.*

Corollary 3.17 ([10, Corollary 3.4])**.** *If $1 < p < \infty$, then the hyperbolic topology coincides with the norm topology on ℓ_p and $L_p[0, 1]$.*

Next, we consider the cases $p = 1$ and $p = \infty$. Let (Ω, Σ, μ) be a measure space. A set $A \in \Sigma$ is said to be an *atom* if $\mu(A) > 0$ and for each $B \subset A$ with $B \in \Sigma$ we have $\mu(B) = 0$ or $\mu(B) = \mu(A)$.

We say that a measure space (Ω, Σ, μ) *has a finite partition by atoms* if there are finitely many atoms $A_i \in \Sigma$, $i = 1, \ldots, n$ such that $\Omega = A_1 \cup \cdots \cup A_n$ and $A_i \cap A_j = \emptyset$ if $i \neq j$.

We notice that the measure space $(\mathbb{N}, 2^{\mathbb{N}}, \mu)$, where μ is the counting measure, contains atoms, but it does not have a finite partition by atoms.

Then we have the following.

Theorem 3.18 ([10, Corollary 3.6]). *Let (Ω, Σ, μ) be a measure space and $1 \leq p \leq \infty$. If $\mu(\Omega) = 0$, or (Ω, Σ, μ) has a finite partition by atoms, then the hyperbolic topology coincides with the norm topology in $L_p(\Omega, \Sigma, \mu)$.*

Now we have results for the cases $p = 1$ and $p = \infty$.

Theorem 3.19 ([10, Theorems 3.8 and 3.10]). *Let (Ω, Σ, μ) be a measure space. Then we have the following.*

(1) The hyperbolic topology coincides with the norm topology in $L_1(\Omega, \Sigma, \mu)$ if and only if $\mu(\Omega) = 0$ or (Ω, Σ, μ) has a finite partition by atoms.

(2) The hyperbolic topology coincides with the norm topology in $L_\infty(\Omega, \Sigma, \mu)$ if and only if $\mu(\Omega) = 0$ or (Ω, Σ, μ) has an atom.

Since the measure space $(\mathbb{N}, 2^{\mathbb{N}}, \mu)$, where μ is the counting measure, does not have a finite partition by atoms, the following are direct consequences of Theorem 3.19 (1) and (2), respectively.

Corollary 3.20 ([10, Corollary 3.9]). *The hyperbolic topology is strictly weaker than the norm topology in ℓ_1.*

Corollary 3.21 ([10, Corollary 3.11]). *The hyperbolic topology coincides with the norm topology in ℓ_∞.*

We can also have the following.

Corollary 3.22 ([10, Corollary 3.12]). *The hyperbolic topology does not coincide with the norm topology in $C([0, 1])$.*

4. The Martin topology on the space of formal balls

In this section, we describe several properties of the Martin topology of the posets of generalized formal balls in the same direction as the previous section. In the sequel, we present certain relationships between the Martin topology μ on continuous posets of generalized formal balls and certain product topology.

Let (X, d) be a metric space. It is clear that the subspace $X \times \{t\}$ of $(\mathbf{B}X, \mu)$ is a discrete space for each $t \in \mathbb{R}$, and the subspace $\{x\} \times \mathbb{R}$ of $(\mathbf{B}X, \mu)$ is the Sorgenfrey line $\mathbb{S}$ for each $x \in X$. This suggests that the Martin topology μ of $\mathbf{B}X$ is relating to the product topology of the discrete topology of X and the Sorgenfrey line topology. We denote the product topology of the metric space (X, d) and the Sorgenfrey line by $\mathcal{T}_{X \times \mathbb{S}}$. In [11], Hashiriura obtained the following.

Proposition 4.1 ([11, Proposition 3.1]). *Let (X, d) be a metric space. Then, $\mathcal{T}_{X\times\mathbb{S}} = \mu$ if and only if d induces the discrete topology.*

The proposition above implies that for almost all metric spaces the Martin topology differs from the product topology of X and $\mathbb{S}$. However, we have some possibility that μ is homeomorphic to the product topology of X and $\mathbb{S}$. Now, we consider the homeomorphic relation between $(\mathbf{B}X, \mu)$ and $X \times \mathbb{S}$. As we noticed in section 2, the Martin topology is zero-dimensional. Hence, we consider only zero-dimensional metric spaces. For a special subspace of the real line, the Martin topology may homeomorphic to the product topology with the Sorgenfrey line.

Proposition 4.2 ([11, Proposition 3.2]). *Let A be a subspace of $\mathbb{R}$. If for each $a \in A$, there exists $\varepsilon > 0$ such that $(a-\varepsilon,\ a)\cap A = \emptyset$, then $(\mathbf{B}A, \mu)$ is homeomorphic to $A \times \mathbb{S}$.*

Corollary 4.3. *Let c_0 be a convergent sequence in $\mathbb{R}$, then $(\mathbf{B}c_0, \mu)$ is homeomorphic to $c_0 \times \mathbb{S}$.*

On the other hand, the space of irrational numbers and the Cantor set are not the case.

Theorem 4.4 ([11, Proposition 3.3]). *If X is an uncountable separable metric space, then $X \times \mathbb{S}$ is not homeomorphic to $(\mathbf{B}X, \mu)$.*

Corollary 4.5 ([11, Corollary 3.1]). *Let $\mathbb{P}$ and $\mathbb{C}$ be the space of irrational numbers and the Cantor set respectively. Then $\mathbb{P} \times \mathbb{S}$ is not homeomorphic to $(\mathbf{B}\mathbb{P}, \mu)$, and $\mathbb{C} \times \mathbb{S}$ is not homeomorphic to $(\mathbf{B}\mathbb{C}, \mu)$.*

Now, we may ask the following.

Question 4.6 ([11, Question 3.2]). Let X be a countable metric space. Is $(\mathbf{B}X, \mu)$ homeomorphic to $X \times \mathbb{S}$? In particular, how about the space of rational numbers $\mathbb{Q}$?

As we mentioned above, for the real line $\mathbb{R}$ the Martin topology of the poset of generalized formal balls is not homeomorphic to the product space $\mathbb{R} \times \mathbb{S}$. Furthermore, we can describe on $(\mathbf{B}\mathbb{R}, \mu)$ as follows.

Theorem 4.7 ([11, Proposition 3.4]). *$(\mathbf{B}\mathbb{R}, \mu)$ is homeomorphic to $\mathbb{S}^2$.*

Concerning the theorem above, we may ask the following questions.

Question 4.8. Is $(\mathbf{B}\mathbb{R}^n, \mu)$ homeomorphic to $\mathbb{S}^{n+1}$ for each natural number $n \geq 2$?

Question 4.9 ([11, Question 3.3]). For each subset X of $\mathbb{R}$, $X_{\mathbb{S}}$ denotes the subspace X of $\mathbb{S}$. Then, is $(\mathbf{B}X, \mu)$ homeomorphic to $X_{\mathbb{S}} \times \mathbb{S}$? In particular, how about $\mathbb{Q}_{\mathbb{S}}$ or $\mathbb{P}_{\mathbb{S}}$?

Remark 4.10. It is clear that the usual topology of $\mathbb{Q}$ is strictly weaker than the topology of $\mathbb{Q}_{\mathbb{S}}$, however it follows from the topological characterization of the space of the rationals (e.g., see [15, Theorem 1.9.6]) that $\mathbb{Q}$ is homeomorphic to $\mathbb{Q}_{\mathbb{S}}$ (cf. [3]). Hence, the positive answer to to Question 4.9 for the space $\mathbb{Q}$ gives the positive answer to the second part of Question 4.6.

Now, we consider special subspaces of $(\mathbf{B}\mathbb{R}, \mu)$ as we considered in section 3.2.

Let (X, d) be a metric space. Let $p : \mathbf{B}X \to X$ be the natural projection, $p(x, s) = x$ for each $(x, s) \in \mathbf{B}X$. We consider the subsets $\mathrm{Bd}(a, u) = \{(y, s) \in \mathbf{B}X : d(a, y) = u - s\}$ and $\mathrm{Bd}^-(a, u) = \{(y, s) \in \mathbf{B}X : d(a, y) = s - u\}$ of $\mathbf{B}X$ for $(a, u) \in \mathbf{B}X$. Since the restriction of p over $\mathrm{Bd}(a, u)$ and $\mathrm{Bd}^-(a, u)$ are bijections, these derive topologies $\tau_{(a,u)}$ $\tau^-_{(a,u)}$ of X respectively, i.e., $\tau_{(a,u)} = \{p(\mathrm{Bd}(a, u) \cap U) : U \in \mu\}$ and $\tau^-_{(a,u)} = \{p(\mathrm{Bd}^-(a, u) \cap U) : U \in \mu\}$. What are the topologies $\tau_{(a,u)}$ and $\tau^-_{(a,u)}$? It is easy to see that the subspaces $\mathrm{Bd}(a, u)$ and $\mathrm{Bd}(a, v)$ ($\mathrm{Bd}^-(a, u)$ and $\mathrm{Bd}^-(a, v)$) drive the same topology of X by the projection p for each $u, v \in \mathbb{R}$. Hence, it suffices to consider about $Bd(a, 0)$ and $Bd^-(a, 0)$, i.e., $u = 0$, and we write $\tau_a = \tau_{(a,0)}$ and $\tau^-_a = \tau^-_{(a,0)}$.

Since the Martin topology of $\mathbf{B}X$ is generated by sets of the form $\Uparrow(x, r) = \{(y, s) : d(x, y) < r - s\}$ and $\downarrow(x, r) = \{(y, s) : d(x, y) \leq s - r\}$ for $(x, r) \in \mathbf{B}X$, the subspace topology of the set $\mathrm{Bd}(a, 0)$ is generated by sets of the form $\{(y, s) \in \mathrm{Bd}(a, 0) : d(y, x) < r - s\} = \{(y, -d(a, y)) \in \mathbf{B}X : d(y, x) - d(a, y) < r\}$ and $\{(y, s) \in \mathrm{Bd}(a, 0) : d(y, x) \leq s - r\} = \{(y, -d(a, y)) \in \mathbf{B}X : d(y, x) + d(a, y) \leq -r\}$ for $(x, r) \in \mathbf{B}X$. Hence, τ_a is generated by sets of the form $\{y : d(y, x) - d(a, y) < r\}$ and $\{y : d(y, x) + d(a, y) \leq -r\}$ for $x \in X$ and $r \in \mathbb{R}$. Since $d(y, x) - d(a, y) \geq -d(a, x)$, if $r \leq -d(a, x)$ then $\{y : d(y, x) - d(a, y) < r\} = \emptyset$. Similarly, if $r > -d(a, x)$ then $\{y : d(y, x) + d(a, y) \leq -r\} = \emptyset$. Hence, τ_a is generated by $\{\{y : d(y, x) - d(a, y) < r\} : x \in X, r > -d(a, x)\} \cup \{\{y : d(y, x) + d(a, y) \leq -r\} : x \in X, r \leq -d(a, x)\}$.

Similarly, it follows that τ^-_a is generated by $\{\{y : d(y, x) + d(a, y) < r\} : x \in X, r > d(a, x)\} \cup \{\{y : d(y, x) - d(a, y) \leq -r\} : x \in X, r \leq d(a, x)\}$.

We notice that a is an isolated point of (X, τ_a) for each $a \in X$. Hence, τ_a may differ from τ_b if $a \neq b$. However, we do not know whether (X, τ_a) may be homeomorphic to (X, τ_b). Further, we do not know whether τ^-_a depend on the choice of the point $a \in X$.

Lemma 4.11. *(1) For each $a, b \in \mathbb{R}$ with $a < b$, the subspace $[a, b)$ of the Sorgenfrey line $\mathbb{S}$ is homeomorphic to $\mathbb{S}$.*

(2) The topological sum of countably many copies of the Sorgenfrey line is homeomorphic to the Sorgenfrey line $\mathbb{S}$.

Proof. (1) The Sorgenfrey line $\mathbb{S}$ can be represented as a topological sum of countably many half-open intervals $[n, n + 1), n \in \mathbb{Z}$, and the subspace $[a, b)$ of the Sorgenfrey line can be also represented as a topological sum of countably many half-open intervals $[x_n, x_{n+1})$, $n \in \mathbb{N}$, where $a = x_1 < x_2 \cdots < b$ and $\lim x_n = b$. Further, it obvious that the half-open intervals $[n, n + 1)$ and $[x_n, x_{n+1})$ are homeomorphic. Hence, $\mathbb{S}$ is homeomorphic to the subspace $[a, b)$.

(2) Since the Sorgenfrey line is a topological sum of countably many copies of half-open intervals $[n, n+1), n \in \mathbb{Z}$ which is homeomorphic to $\mathbb{S}$ by (1), a topological sum of countably many copies of the Sorgenfrey line is also a topological sum of countably many copies of half-open intervals. As we showed in the proof of (1), a

topological sum of countably many copies of half-open intervals is homeomorphic to $\mathbb{S}$. $\square$

Proposition 4.12. *Let $a \in \mathbb{R}$. Then we have the following.*
(1) $(\mathbb{R}, \tau_a)$ is homeomorphic to the subspace $(-\infty, 0]$ of the Sorgenfrey line.
(2) $(\mathbb{R}, \tau_a^-)$ is homeomorphic to the Sorgenfrey line.

Proof. (1) It is easy to see that the following is a basic neighborhood system of each point x of $(\mathbb{R}, \tau_a)$: $\{(y, x] : y < x\}$ if $x < a$, $\{\{a\}\}$ if $x = a$, and $\{[x, y) : y > x\}$ if $x > a$. Then $(\mathbb{R}, \tau_a)$ is a topological sum of two copies of the Sorgenfrey line and an isolated point. Since a topological sum of two copies of the Sorgenfrey line is homeomorphic to the Sorgenfrey line, $(\mathbb{R}, \tau_a)$ is homeomorphic to the topological sum of the Sorgenfrey line and an isolated point, and hence it is homeomorphic to the subspace $(-\infty, 0]$ of the Sorgenfrey line.

(2) It is easy to see that the following is a basic neighborhood system of each point x of $(\mathbb{R}, \tau_a^-)$: $\{(y, x] : y < x\}$ if $x < a$, $\{(y, z) : y < x < z\}$ if $x = a$, and $\{[x, y) : y > x\}$ if $x > a$. Hence $(\mathbb{R}, \tau_a^-) = (-\infty, a) \cup \{a\} \cup (a, \infty)$, where $(-\infty, a)$ and (a, ∞) are homeomorphic to the Sorgenfrey line and a has the usual neighborhood system. We put $A_1 = (-\infty, a - \frac{1}{2}]$ and $B_1 = [a + \frac{1}{2}, \infty)$. Then for each $n \geq 2$ we put $A_n = (a - \frac{1}{n}, a - \frac{1}{n+1}]$ and $B_n = [a + \frac{1}{n+1}, a + \frac{1}{n})$. Now, we can define a homeomorphism $f : (\mathbb{R}, \tau_a^-) \to [0, 1) (\subset \mathbb{S})$ such as

(i) $f(A_n) = [\frac{1}{2n}, \frac{1}{2n-1})$ for each $n = 1, 2, \ldots$,
(ii) $f(B_n) = [\frac{1}{2n+1}, \frac{1}{2n})$ for each $n = 1, 2, \ldots$ and,
(iii) $f(a) = 0$.

By Lemma 4.11, it follows that $[0, 1)$ is homeomorphic to the Sorgenfrey line, and hence $(\mathbb{R}, \tau_a^-)$ is homeomorphic to the Sorgenfrey line. $\square$

5. Several topologies on the set of the real numbers

Concerning the observation in the previous section, we may be interested in the relation between several topologies on the set $\mathbb{R}$ of real numbers and the topology of the Sorgenfrey line. The results and the questions described in this section may be already known or stupid. However, I could not find the results in the literature, so we shall mention them.

Let τ_E and τ_S denote the usual (Euclidean) topology of $\mathbb{R}$ and the topology of the Sorgenfrey line $\mathbb{S}$, respectively.

As we mentioned in Proposition 4.12, it follows that τ_a^- is homeomorphic to τ_S, but it is obvious that $\tau_E \subsetneq \tau_a^- \subsetneq \tau_S$.

Definition 5.1. Let A be a subset of the set $\mathbb{R}$ of the real numbers. We define the topology $\tau(A)$ on the set $\mathbb{R}$ as follows:

(1) For each $x \in A$, $\{(x - \varepsilon, x + \varepsilon) : \varepsilon > 0\}$ is the neighborhood base at x.
(2) For each $x \in \mathbb{R} - A$, $\{[x, x + \varepsilon) : \varepsilon > 0\}$ is the neighborhood base at x.

It is clear that $\tau(\mathbb{R}) = \tau_E$ and $\tau(\emptyset) = \tau_S$, and it follows from Proposition 4.12 that $\tau(\{a\})$ is homeomorphic to τ_S for each $a \in \mathbb{R}$. Now, we may ask the following question.

Question 5.2. How different are the topologies $\tau(A)$ for $A \subset \mathbb{R}$?

Remark 5.3. If $F \subset \mathbb{R}$ is finite, then $(\mathbb{R}, \tau(\mathbb{R} - F))$ has $|F|$-many components, and each components are half-open intervals with the usual topology. Hence $\tau(\mathbb{R} - F_1)$ and $\tau(\mathbb{R} - F_2)$ is not homeomorphic for every finite subsets F_1 and F_2 of $\mathbb{R}$ with $|F_1| \neq |F_2|$.

Remark 5.4. By an argument similar to Proposition 4.12, we have that $\tau(A)$ is homeomorphic to τ_S if $A \subset \mathbb{R}$ is a discrete subspace of $(\mathbb{R}, \tau_E)$.

It is clear that if A has a non-empty interior in $(\mathbb{R}, \tau_E)$, $\tau(A)$ is not homeomorphic to τ_S. We ask the following.

Question 5.5. Let A be a nowhere dense subset of $(\mathbb{R}, \tau_E)$. Is $\tau(A)$ homeomorphic to τ_S?

We may also ask the following question.

Question 5.6. Let $\mathbb{P}_{\mathbb{S}}$ be the subspace of irrational numbers of the Sorgenfrey line. Is $\mathbb{P}_{\mathbb{S}}$ homeomorphic to $\mathbb{S}$?

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