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RICCI SOLITONS AND LAGRANGIAN SUBMANIFOLDS IN KÄHLER MANIFOLDS

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Abstract. In this paper we will study compact Lagrangian submanifold M in Kähler manifolds and in particular complex space forms, such that the induced metric on the Lagrangian submanifold is a Ricci soliton with respect to potential vector field given by mean curvature vector field and complex structure.

1. Introduction

A Ricci soliton is defined on a Riemannian manifold (M, g) by

$$\frac{1}{2}\pounds_V g + \mathrm{Ric} - \lambda g = 0, \tag{1.1}$$

where V is a vector field (the potential vector field), λ is a constant. Obviously, a trivial Ricci soliton is an Einstein metric with V zero or Killing. Compact Ricci solitons are the fixed points of the Ricci flow: $\partial g/\partial t = -2Ric$ projected from the space of metrics onto its quotient modulo diffeomorphisms and scalings, and often arise as blow-up limits for the Ricci flow on compact manifolds. The Ricci soliton is said to be shrinking, steady, and expanding according as $\lambda > 0$, $\lambda = 0$ and $\lambda < 0$ respectively. If the vector field V is the gradient of a potential function f, then g is called a gradient Ricci soliton. Due to Perelman's result [13, Remark 3.2], we know that in a compact Ricci soliton, the potential vector field is written as the sum of a gradient and a Killing vector field. We refer to [3] for details about Ricci solitons or gradient Ricci solitons.

On the other hand, Lagrangian submanifolds have been important geometric objects of study in symplectic geometry. The problem of minimizing the volume of Lagrangian submanifolds under Hamiltonian deformations was proposed by Oh

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[10], and critical points of the problem, *Hamiltonian minimal submanifolds* are interesting and important objects among Lagrangian submanifolds (cf. [7]). Using Oh's result, we can see that compact Lagrangian submanifold M in a Kähler manifold $\widetilde{M}$ is Hamiltonian minimal if and only if tangent vector field JH is divergence free, where J and H denote complex structure of $\widetilde{M}$ and mean curvature vector field of M, respectively. In this paper, we investigate Lagrangian submanifolds, whose induced metric is a Ricci soliton with potential vector field JH, in Kähler manifolds and in particular in complex space forms. Typical examples of such Lagrangian submanifolds are: (i) Einstein minimal Lagrangian submanifolds, (ii) irreducible Lagrangian submanifolds with *parallel second fundamental form* in Hermitian symmetric spaces (cf. [8], [9]) and (iii) Ricci-flat Lagrangian submanifolds with parallel mean curvature vector field H (cf. [11]).

The primary result is concerned with compact oriented Lagrangian submanifold M in a Kähler manifold $\widetilde{M}$: Suppose that the induced metric g on M is a Ricci soliton with potential vector field JH. If M is Hamiltonian minimal, then M is Einstein and either M is minimal or the mean curvature vector field H of M is non-zero and parallel with respect to the normal connection $\nabla^{\perp}$(Theorem 5). Furthermore if the scalar curvature ρ of (M, g) satisfies either $\rho \geq n\lambda$ or $\rho \leq n\lambda$, then $\rho = n\lambda$ and M is Hamiltonian minimal. Secondly, let M be a compact oriented Lagrangian submanifold M in a *complex space form*, and suppose the induced metric g is a Ricci soliton with potential vector field JH. With respect to the Ricci tensor of M, if $\mathrm{Ric}(JH, JH) \leq 0$, then $\nabla^{\perp} H = 0$ and M is Ricci-flat (Theorem 7). Finally for either 2 or 3-dimensional case, using results of Hamilton [5] and Ivey [6], we obtain (Theorem 8): Let M^n be a compact Lagrangian submanifold in a complex space form with $n = 2, 3$ and let g be the induced metric on M. Suppose (M, g) is a Ricci soliton with $V = JH$. Then (M, g) is either totally geodesic or flat with $\nabla^{\perp} H = 0$.

2. Ricci solitons

On Riemannian manifold (M, g), we have

$$(\pounds_V g)(X, Y) = g(\nabla_X V, Y) + g(X, \nabla_Y V), \tag{2.1}$$

where ∇ denotes the the Levi-Civita connection of M. Hence if (M, g) is a Ricci soliton with potential vector field V, then (1.1) and (2.1) imply

$$\operatorname{div} V + \rho - \lambda n = 0, \tag{2.2}$$

where $\operatorname{div} V = \mathrm{trace}_g(X \mapsto \nabla_X V)$ and ρ denote divergence of V and the scalar curvature of M, respectively. By (2.2) and Green's Theorem $\int_M \operatorname{div} V \mu_g = 0$ for a compact oriented Riemannian manifold M (μ_g is the volume form of (M, g)), we obtain:

Proposition 1. *Let (M, g) be a compact oriented Ricci soliton satisfying* (1.1) *with respect to potential vector field V on M. Then*

(i) $\int_M \rho \mu_g = \lambda n \operatorname{vol}(M)$. *Hence λ is uniquely determined by g.*
(ii) *The scalar curvature ρ of M is constant if and only if* $\operatorname{div} V = 0$.

(iii) *If either* $\rho \leq \lambda n$ *or* $\rho \geq \lambda n$ *on* M, *then* $\rho = \lambda n$ *and* $\operatorname{div} V = 0$.

Now we recall the following results due to Hamilton and Ivey:

Theorem 2. [5], [6] *Let* (M, g) *is either* 2 *or* 3*-dimensional compact Ricci soliton with potential vector field* V. *Then sectional curvatures of* M *are constant.*

Also the following is known:

Proposition 3. *On compact Ricci soliton* (M, g), *if the scalar curvature is constant, then* (M, g) *is Einstein.*

3. Lagrangian submanifolds in complex space forms

First we recall about Hamiltonian deformation of Lagrangian submanifolds in Kähler manifolds, defined by Oh [10]. Let $\widetilde{M}$ be a complex n-dimensional Kähler manifold with Kähler form ω, Riemannian metric $\langle \ , \ \rangle$, and complex structure J. Let $x : M \to \widetilde{M}$ be a Lagrangian immersion from a real n-dimensional manifold M to $\widetilde{M}$, i.e., $\omega|_{TM} = 0$. For a vector field V along x, we define a 1-form α_V on M as $\alpha_V = \langle JV, \cdot \rangle|_{TM}$. Smooth family of embeddings $\iota_t : M \to P$ is called *Hamiltonian deformation* if for the variational vector field V, the 1-form α_V is exact. A Lagrangian submanifold M is *Hamiltonian minimal* (or *H-minimal*) if M is stationary for any Hamiltonian deformation. Oh [10] showed that when M is *compact*, M is H-minimal if and only if α_H is co-closed, i.e., $\delta\alpha_H = 0$ where H is the mean curvature vector field of M. We have

$$M \text{ is Hamiltonian minimal } \Leftrightarrow \operatorname{div} JH = 0. \tag{3.1}$$

With respect to Lagrangian submanifold M in a Kähler manifold $\widetilde{M}$ and the *induced metric* g on M, the following relations hold:

$$\nabla\sigma = 0 \Rightarrow \nabla^{\perp} H = 0 \Rightarrow \pounds_{JH} g = 0 \Rightarrow \operatorname{div} JH = 0, \tag{3.2}$$

where σ and $\nabla^{\perp}$ denote second fundamental form and normal connection of M in $\widetilde{M}$ respectively, and $\nabla\sigma$ is defined by $(\nabla_X\sigma)(Y, Z) = \nabla^{\perp}_X \sigma(Y, Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z)$ for tangent vector fields X, Y, Z on M. We note that $\nabla_X(JH) = J\nabla^{\perp}_X H$.

Let $\widetilde{M}^n(4c)$ be an n-dimensional complex space form with constant holomorphic sectional curvature $4c$ and let $M = M^n$ be a Lagrangian submanifold in $\widetilde{M}^n(4c)$. Then Gauss equation is

$$R(X, Y)Z = c\{g(Y, Z)X - g(X, Z)Y\} + A_{\sigma(Y,Z)}X - A_{\sigma(X,Z)}Y,$$

where R and A denote curvature tensor and shape operator of M with $\langle \sigma(X, Y), \xi \rangle = g(A_\xi X, Y)$ for tangent vector field X, Y and normal vector field ξ on M, respectively. The Ricci tensor of M is then given by

$$\operatorname{Ric}(Y, Z) = (n-1)cg(Y, Z) + \langle \sigma(Y, Z), H \rangle - \operatorname{trace}_g(X \mapsto A_{\sigma(X,Z)}Y). \tag{3.3}$$

The scalar curvature ρ of M is

$$\rho = n(n-1)c + \|H\|^2 - \|\sigma\|^2. \tag{3.4}$$

Let T be a symmetric $(0,3)$-tensor field on M defined by $T(X,Y,Z) = \langle\sigma(X,Y), JZ\rangle$. Then Codazzi equation

$$(\nabla_X\sigma)(Y,Z) = (\nabla_Y\sigma)(X,Z) \tag{3.5}$$

implies that

$$\nabla T \text{ is a symmetric } (0,4)\text{-tensor field on } M.$$

Combining with (3.2), we can easily obtain the following *local* result:

Proposition 4. *Let M be a Lagrangian submanifold in a complex space form of constant holomorphic sectional curvature. Then the mean curvature vector field H is parallel with respect to the normal connection $\nabla^\perp$ if and only if JH is a Killing vector field on M.*

4. Ricci soliton of Lagrangian submanifolds

Let $M = M^n$ be a Lagrangian submanifolds in a Kähler manifold $\widetilde{M}^n$, and let g be the induced metric on M. Suppose (M,g) is a Ricci soliton (1.1) with $V = JH$. By Proposition 1, Proposition 3, (2.2) and (3.1), we obtain:

Theorem 5. *Let M^n be a compact oriented Lagrangian submanifold in a Kähler manifold $\widetilde{M}^n$ and let g be the induced metric on M. Suppose (M,g) is a Ricci soliton with $V = JH$. Then we have:*
- (i) *If M is Hamiltonian minimal, then M is Einstein and JH is a Killing vector field on M.*
- (ii) *If the scalar curvature ρ of (M,g) satisfies either $\rho \geq n\lambda$ or $\rho \leq n\lambda$, then $\rho = n\lambda$ and M is Einstein and the same conclusion as (i) holds.*

We note that Lagrangian submanifolds M^n with parallel second fundamental form in complex space forms are classified by Naitoh [8], [9]. By (3.1) and (3.2), they are Hamiltonian minimal. Also they are locally symmetric and for irreducible ones, they are Einstein. Hence *irreducible* Lagrangian submanifolds in complex space forms satisfying $\nabla\sigma = 0$ are considered as Ricci solitons with Killing potential vector field JH.

When $\dim_{\mathbb{R}} M = 2$ or 3, by Theorem 2, (3.1) and (3.2), we get:

Proposition 6. *Let M be a compact Lagrangian submanifold in a Kähler manifold $\widetilde{M}$ with $\dim_{\mathbf{C}} \widetilde{M} = 2$ or 3, and let g be the induced metric on M. Suppose (M,g) is a Ricci soliton with $V = JH$. Then (M,g) is constant curvature and $\nabla^\perp H = 0$.*

Let $\widetilde{M}^n(4c)$ be an n-dimensional complex space form with constant holomorphic sectional curvature $4c$. For Lagrangian submanifold (M,g) in $\widetilde{M}^n(4c)$, using (2.1) and (3.3) we obtain that Ricci soliton equation (1.1) with $V = JH$ is written as

$$\begin{aligned}\frac{1}{2}\{g(\nabla_X(JH),Y) + g(X,\nabla_Y(JH))\} - \langle\sigma(X,Y),H\rangle \\ + \operatorname{trace}_g(Z \mapsto A_{\sigma(Z,Y)}X) + ((n-1)c-\lambda)g(X,Y) = 0.\end{aligned} \tag{4.1}$$

By contracting with respect to X and Y, we obtain

$$\operatorname{div} JH - \|H\|^2 + \|\sigma\|^2 + n((n-1)c - \lambda) = 0, \tag{4.2}$$

which is clearly equivalent to (2.2) with $V = JH$ by (3.4).

By taking covariant differentiation of (4.1) by W, we get

$$\begin{aligned}\frac{1}{2}g(\nabla_W\nabla_X(JH) - \nabla_{\nabla_W X}(JH), Y) + \frac{1}{2}g(X, \nabla_W\nabla_Y(JH) - \nabla_{\nabla_W Y}(JH))\\ -\langle(\nabla_W\sigma)(X,Y), H\rangle - \langle\sigma(X,Y), \nabla^\perp_W H\rangle\\ + \operatorname{trace}_g(Z \mapsto (\nabla_W A)_{\sigma(Z,Y)}X) + \operatorname{trace}_g(Z \mapsto A_{(\nabla_W\sigma)(Z,Y)}X) = 0.\end{aligned}$$

Then Codazzi equation (3.5) yields that

$$\begin{aligned}\frac{1}{2}g(R(W,X)JH, Y) + \frac{1}{2}g(X, \nabla_W\nabla_Y(JH) - \nabla_{\nabla_W Y}(JH))\\ -\frac{1}{2}g(W, \nabla_X\nabla_Y(JH) - \nabla_{\nabla_X Y}(JH)) - \langle\sigma(X,Y), \nabla^\perp_W H\rangle + \langle\sigma(W,Y), \nabla^\perp_X H\rangle\\ + \operatorname{trace}_g(Z \mapsto (\nabla_W A)_{\sigma(Z,Y)}X) + \operatorname{trace}_g(Z \mapsto A_{(\nabla_W\sigma)(Z,Y)}X)\\ - \operatorname{trace}_g(Z \mapsto (\nabla_X A)_{\sigma(Z,Y)}W) - \operatorname{trace}_g(Z \mapsto A_{(\nabla_X\sigma)(Z,Y)}W) = 0.\end{aligned}$$

By contracting with respect to W and Y, and using (3.5) and (4.2) we get

$$\operatorname{Ric}(X, JH) + g(X, \Delta(JH)) = 0.$$

Because of

$$\begin{aligned}\frac{1}{2}\Delta\|H\|^2 = \frac{1}{2}\Delta\|JH\|^2 &= g(\Delta(JH), JH) + \|\nabla(JH)\|^2\\ &= -\operatorname{Ric}(JH, JH) + \|\nabla^\perp H\|^2,\end{aligned}$$

and $\int_M \Delta\|H\|^2\mu_g = 0$ when M is compact and oriented, the following result holds:

Theorem 7. *Let M^n be a compact oriented Lagrangian submanifold in a complex space form $\widetilde{M}^n(4c)$, and let g be the induced metric on M. Suppose (M,g) is a Ricci soliton with $V = JH$. If* $\operatorname{Ric}(JH,JH) \le 0$ *on M, then $\nabla^\perp H = 0$ and (M,g) is Ricci-flat.*

With respect to either 2 or 3-dimensional Lagrangian submanifolds in complex space forms, Proposition 6 is reduced to:

Theorem 8. *Let M^n be a compact Lagrangian submanifold in a complex space form $\widetilde{M}^n(4c)$ with $n = 2, 3$ and let g be the induced metric on M. Suppose (M,g) is a Ricci soliton with $V = JH$. Then (M,g) is either totally geodesic or flat with parallel mean curvature vector field.*

In fact the assumption yields that, by Propositions 4 and 6, $\nabla^\perp H = 0$. When $H = 0$, Ejiri's result [4] implies that (M,g) is either totally geodesic or flat. When $H \neq 0$, (M,g) is constant curvature and has parallel non-zero tangent vector field JH. Hence M is flat.

For Lagrangian flat surfaces in $\widetilde{M}^2(4c) = \mathbf{CP}^2$: complex projective plane with $c > 0$, Ogata [11] proved the following: Let $x : \mathbb{R}^2 \to \mathbf{CP}^2$ be an isometric immersion with non-zero parallel mean curvature vector field H. Then $x(\mathbb{R}^2)$ is an orbit of the Abelian Lie subgroup G of $U(3)$.

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