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ČECH-COMPLETE MAPS, p-MAPS, M-MAPS AND COMPLETENESS OF METRIZABLE MAPS —— A SURVEY ——

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Abstract. This paper is a summary of the author's doctral dissertation. Firstly, we introduce a new notion "Čech-complete map", and investigate some its basic properties, invariance under perfect maps, characterizations by compactifications of Čech-complete maps and relationships with other maps. Secondly, we introduce new notions of p-maps and M-maps, and investigate some basic properties, which are extensions of corresponding properties of p-spaces and M-spaces. Finally, we prove the existence of fibrewise uniformities on some metrizable maps, and study the relations between the completeness induced by a trivial metric and the one defined by fibrewise uniformities. Further, we discuss the relations between completely metrizable maps and Čech-complete maps.

1. Introduction

The study of General Topology is concerned with the category TOP of topological spaces as objects, and continuous maps as morphisms. The concepts of spaces and maps are equally important and one can even look at a space as a map from this space onto a one-point space and in this manner identify these two concepts. With this in mind, a branch of General Topology which has become known as General Topology of Continuous Maps, or Fibrewise General Topology, was initiated. Fibrewise General Topology is concerned most of all in extending the main notions and results concerning topological spaces to continuous maps. As the generalization of the main spaces in General Topology, (locally) compact maps, compactifications of maps, paracompact maps, metrizable type maps (MT-maps, for short), (completely) trivially metrizable maps (TM-maps, for short) and k-maps are defined and mainly studied by B. A. Pasynkov, I. M. James, D. Buhagiar and T. Miwa in Fibrewise General Topology. In order to generalize the uniformity

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A summary of doctoral thesis at Shimane University.

in TOP, I.M. James ([13]) studied fibrewise uniform space (X, Ω) by using entourage filter Ω and Konami-Miwa ([14, 15]) studied it by using covering system μ (say, fibrewise covering uniform space and denoted by (X, μ)), and proved the equivalence of fibrewise uniform spaces (X, Ω) and $(X, \mu(\Omega))$ where $\mu(\Omega)$ is the fibrewise covering uniformity induced by Ω.

In this paper, we continue to extend some main concepts of General Topology to Fibrewise Topology and study the uniformities on TM-maps, which enrich the study of it. In section 2, we summarize notions and notations and terminologies used in this paper. In section 3, we introduce a new notion "Čech-complete map", and investigate some its basic properties, invariance under perfect maps, characterizations by compactifications of Čech-complete maps and relationships with other maps. In section 4, we introduce new notions of p-maps and M-maps, and investigate some basic properties, which are extensions of corresponding properties of p-spaces and M-spaces. In section 5, we prove the existence of fibrewise uniformities on some metrizable maps, and study the relations between the completeness induced by a trivial metric and the one defined by fibrewise uniformities. Further, we discuss the relations between completely metrizable maps and Čech-complete maps.

Since this paper is a summary, we omit all of the proofs.

2. Preliminaries

In this section, we refer to the notations used in the latter sections, further the notions and notations in Fibrewise Topology.

Throughout this paper, we assume that all *spaces* are topological spaces and all *maps* and *projections* are continuous. Also we use the abbreviation *nbd(s)* for *neighborhood(s)*. We use the notation (B, τ) for a topological space B with a topology τ, which is the fixed base space. For a space X and a point $x \in X$, $N(x)$ is the family of all open nbds of x. The symbols $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{Q}$ and $\mathbf{R}$ are the sets of all natural numbers, all integers, all rational numbers and all real numbers, respectively, and I is the unit interval.

For a projection $p : X \to B$ and each $b \in B$, the *fibre* over b is the subset $X_b = p^{-1}(b)$ of X. Also for each subset W of B, we denote $X_W = p^{-1}(W)$. For $W \subset B$, we use the notation $X_W \times X_W = X_W^2$ and $X \times X = X^2$. For $D, E \subset X^2$, $D \circ E = \{(x, z) |$ there exists $y \in X$ such that $(x, y) \in D, (y, z) \in E\}$ and $D(x) = \{y | (x, y) \in D\}$. For a family $\mathcal{U}$ of subsets of X and $A \subset X$, $\mathcal{U}|_A = \{U \cap A | U \in \mathcal{U}\}$.

Definition 2.1. A projection $p : X \to B$ is called a *T_i-map*, $i = 0, 1, 2$, if for all $x, x' \in X$ such that $x \neq x'$ and $p(x) = p(x')$, the following condition is respectively satisfied:

(1) $i = 0$: at least one of the points x, x' has a nbd in X not containing the other point;
(2) $i = 1$: each of the points x, x' has a nbd in X not containing the other point;
(3) $i = 2$: the points x and x' have disjoint nbds in X.

A T_2-map is also called *Hausdorff.*

Definition 2.2. A projection $p : X \to B$ is called *completely regular* (resp. *regular*), if for every point $x \in X$ and every closed set F in X such that $x \notin F$, there exists a nbd $W \in N(p(x))$ such that the sets $\{x\}$ and F are functionally separated (resp. nbd separated) in X_W. A completely regular (resp. regular) T_0-map is called a *Tychonoff* or a $T_{3\frac{1}{2}}$- (resp. a T_3-) *map.*

It is obvious that every T_j-map is a T_i-map for $j, i = 0, 1, 2, 3, 3\frac{1}{2}$ and $i \leq j$.

Definition 2.3. Let $p : X \to B$ be a projection. The map p is called *normal* if for every $O \in \tau$, every $b \in O$ and every pair of disjoint closed sets F and H in X_O, there exists $W \in N(b)$ with $W \subset O$ such that F and H are nbd separated in X_W. A normal T_3-map is called a T_4-*map.*

Proposition 2.4. If the space X is (a) a T_i-space, $i = 0, 1, 2$, (b) regular, (c) completely regular, then the projection $p : X \to B$ is respectively (a) a T_i-map, $i = 0, 1, 2$, (b) regular, (c) completely regular.

Proposition 2.5. If the space B and the map $p : X \to B$ are: (a) a T_i-space and a T_i-map resp., $i = 0, 1, 2$, (b) regular, (c) completely regular, then the space X is respectively (a) a T_i-space, $i = 0, 1, 2$, (b) regular, (c) completely regular.

Definition 2.6. For a collection of fibrewise spaces $\{(X_\alpha, p_\alpha)|\alpha \in \Lambda\}$, the subspace $X = \{t = \{t_\alpha\} \in \prod\{X_\alpha : \alpha \in \Lambda\} : \ p_\alpha t_\alpha = p_\beta t_\beta \ \forall \alpha, \beta \in \Lambda\}$ of the Tychonoff product $\prod = \prod\{X_\alpha : \alpha \in \Lambda\}$ is called the *fan product* of the spaces X_α with respect to the maps p_α, $\alpha \in \Lambda$.

For the projection $pr_\alpha : \prod \to X_\alpha$ of the product $\prod$ onto the factor X_α, the restriction π_α of pr_α to X is called the projection of the fan product onto the factor $X_\alpha, \alpha \in \Lambda$. From the definition of fan product we have that, $p_\alpha \circ \pi_\alpha = p_\beta \circ \pi_\beta$ for every α and β in Λ. Thus one can define a map $p : X \to B$, called the *product* of the maps p_α, $\alpha \in \Lambda$, by $p = p_\alpha \circ \pi_\alpha$, $\alpha \in \Lambda$, and (X, p) is called the *fibrewise product space* of $\{(X_\alpha, p_\alpha)|\alpha \in \Lambda\}$.

Obviously, the projections p and $\pi_\alpha, \alpha \in \Lambda$, are continuous.

Proposition 2.7. Let $\{(X_\alpha, p_\alpha)|\alpha \in \Lambda\}$ be a collection of fibrewise spaces.
(1) If each p_α is T_i $(i = 0, 1, 2)$ (resp. functionally T_2), then the product p is also T_i $(i = 0, 1, 2)$ (resp. functionally T_2).
(2) If each p_α is a *surjective* T_3- (resp. $T_{3\frac{1}{2}}$-)map, then the product p is also a T_3- (resp. a $T_{3\frac{1}{2}}$-)map.

For a projection $p : X \to B$ and a filter (base) $\mathcal{F}$ in X, we denote that $p_*(\mathcal{F})$ is the filter generated by the set $\{p(F)|F \in \mathcal{F}\}$. For a fibrewise map $\lambda : (X, p) \to (Y, q)$, for a filter (base) $\mathcal{F}$ in X, we define $\lambda_*(\mathcal{F})$ as same. For a filter (base) $\mathcal{G}$ in Y, we define $\lambda^*(\mathcal{G})$ is the filter generated by the set $\{\lambda^{-1}(U)|U \in \mathcal{G}\}$.

Definition 2.8. ([13, Section 4]) For a fibrewise space (X, p), by a *b-filter* (or *tied filter*) on X we mean a pair $(b, \mathcal{F})$, where $b \in B$ and $\mathcal{F}$ is a filter on X such that b is a limit point of the filter $p_*(\mathcal{F})$ on B. By an *adherence point* of a b-filter $\mathcal{F}$

($b \in B$) on X, we mean a point of the fibre X_b which is an adherence point of $\mathcal{F}$ as a filter on X. Points outside X_b are not to be regarded as adherent. The term *limit point* is used similarly.

Definition 2.9. (1) A projection $p : X \to B$ is called a *compact map* if it is perfect (i.e., it is closed and all its fibres $p^{-1}(b)$ are compact). Note that in [13, Definition 3.1], the space X is called *fibrewise compact over* B.
(2) A projection $p : X \to B$ is said to be a *locally compact map* if for each $x \in X_b$, where $b \in B$, there exist a nbd $W \in N(b)$ and a nbd $U \subset X_W$ of x such that $p' : X_W \cap \overline{U} \to W$ is a compact map, where p' is the restriction of p and $X_W \cap \overline{U}$ is the closure of U in X_W.

Proposition 2.10. (1) Compact T_2-map $\Longrightarrow$ T_4-map $\Longrightarrow$ T_3-map.
(2) Locally compact T_2-map $\Longrightarrow$ T_3-map.

Definition 2.11. (1) For a map $p : X \to B$, a map $c(p) : c_pX \to B$ is called a *compactification* of p if $c(p)$ is compact, X is dense in c_pX and $c(p)|X = p$.
(2) A map $p : X \to B$ is called a T_2*-compactifiable map* (or a *Hausdorff-compactifiable map*) (resp. a $T_{3\frac{1}{2}}$*-compactifiable map* (or *a Tyhonoff compactifiable map*)) if p has a compactification $c(p) : c_pX \to B$ and $c(p)$ is a T_2-map (resp. a $T_{3\frac{1}{2}}$-map).

Proposition 2.12. (1) ([13, Section 8]) Every T_2-compactifiable map is a T_3-map.
(2) ([13, Section 8]) Every T_4-map is a T_2-compactifiable map.
(3) ([13, Section 8]) Every locally compact T_2-map is a T_2-compactifiable map.
(4) ([19, Section 1.6]) Every $T_{3\frac{1}{2}}$-map is a $T_{3\frac{1}{2}}$-compactifiable map.

Definition 2.13. (James [13, Definitions 10.1 and 10.3])
(1) Let (X, p) be a fibrewise space. A subset H of X is *quasi-open* (resp. *quasi-closed*) if the following condition is satisfied: for each $b \in B$ and $V \in N(b)$ there exists a nbd $W \in N(b)$ with $W \subset V$ such that whenever $p|K : K \to W$ is compact then $H \cap K$ is open (resp. closed) in K.
(2) Let a projection $p : X \to B$ be a T_2-map. The map p is a *k-map* if every quasi-closed subset of X is closed in X or, equivalently, if every quasi-open subset of X is open in X, where a k-map $p : X \to B$ is same as that X is a fibrewise compactly generated space ([13, Section 8]).

Proposition 2.14. Let $p : X \to B$ be a locally compact T_2-map. If H is quasi-open (resp. quasi-closed) in X then H is open (resp. closed) in X.

Definition 2.15. (D. Buhagiar [6]) A map $p : X \to B$ is said to be *paracompact* if for every point $b \in B$ and every open (in X) cover $\mathcal{U} = \{U_\alpha | \alpha \in \mathcal{A}\}$ of the fibre X_b (i.e., $X_b \subset \cup\{U_\alpha | \alpha \in \mathcal{A}\}$), there exist $W \in N(b)$ and an open (in X) cover $\mathcal{V}$ of X_W such that X_W is covered by $\mathcal{U}$ and $\mathcal{V}$ is a locally finite (in X_W) refinement of $\{X_W\} \wedge \mathcal{U}$.

Definition 2.16. (1) ([8, Definition 2.8]) For a map $p : X \to B$, a sequence $\{\mathcal{U}_n\}_{n \in \mathbf{N}}$ of open (in X) covers of X_b, $b \in B$, is said to be a *b-development* if for every $x \in X_b$ and every $U \in N(x)$, there exist $n \in \mathbf{N}$ and $W \in N(b)$ such that

$x \in st(x, \mathcal{U}_n) \cap X_W \subset U$. The map p is said to have a *p-development* if it has a b-development for every $b \in B$.
(2) ([8, Definition 2.9]) A closed map $p : X \to B$ is said to be an *MT-map* if it is collectionwise normal and has a p-development.

Theorem 2.17. ([8, Proposition 2.20]) Let $p : X \to B$ be an MT-map and $q : Y \to B$ a continuous map. If $\lambda : p \to q$ is a perfect morphism of p onto q, then q is also an MT-map.

Next, according to [20] let us refer to (completely) trivially metrizable maps. For a map $p : X \to B$, a pseudometric ρ on X is called a *trivial metric* (*T-metric*, for short) on p if the restriction of ρ to every fibre $p^{-1}(b)$, $b \in B$, is a metric and $p^{-1}\tau \cup \tau_\rho$, where τ_ρ is the topology on X generated by ρ, is a subbase of the topology of X. A map $p : X \to B$ is called *trivially metrizable map* (a *TM-map*, for short) if there exists a T-metric on p. A T-metric on a map $p : X \to B$ is called *complete* (a *CT-metric*, for short) if

(*) For any b-filter $\mathcal{F}$, $b \in B$, on X containing elements of arbitrary small diameter, $\mathcal{F}$ has adherence points.

A map $p : X \to B$ is called *completely trivially metrizable map* (a *complete TM-map*, for short) if there exists a CT-metric on it.

A map $p : X \to B$ is called (resp. *closedly*) *parallel* to a space Z if there exists an embedding $e : X \to B \times Z$ such that (resp. $e(X)$ is closed in $B \times Z$ and) $p = \pi \circ e$, where $\pi : B \times Z \to B$ is the projection (see [18]). The following are proved in [20]:

Theorem 2.18. A map $p : X \to B$ is a TM-map if and only if p is parallel to a metrizable space, and p is a complete TM-map if and only if it is closedly parallel to a completely metrizable (i.e., metrizable by complete metric) space.

Definition 2.19. (Konami-Miwa [15]) Let $p : X \to B$ be a projection, and Δ be the diagonal of $X \times X$. A *fibrewise entourage uniformity* on X is a filter Ω on $X \times X$ satisfying the following four conditions:

(J1) $\Delta \subset D$ for every $D \in \Omega$.
(J2) Let $D \in \Omega$. Then for each $b \in B$ there exist $W \in N(b)$ and $E \in \Omega$ such that $E \cap X_W^2 \subset D^{-1}$.
(J3) Let $D \in \Omega$. Then for each $b \in B$ there exist $W \in N(b)$ and $E \in \Omega$ such that

$$(E \cap X_W^2) \circ (E \cap X_W^2) \subset D$$

(J4) If $E \subset X \times X$ satisfies that for each $b \in B$ there exist $W \in N(b)$ and $D \in \Omega$ such that $D \cap X_W^2 \subset E$, then $E \in \Omega$.

Note that in [13, Section 12], a filter Ω on $X \times X$ satisfying (J1),(J2) and (J3) is called a *fibrewise uniform structure* on X. So, the notion of a fibrewise entourage uniformity is slightly stronger than one of a fibrewise uniform structure.

For a projection $p : X \to B$ and $W \in \tau$, let μ_W be a non-empty family of coverings of X_W. We say that $\{\mu_W\}_{W \in \tau}$ is a *system of coverings* of $\{X_W\}_{W \in \tau}$. (For this, we briefly use the notations $\{\mu_W\}$ and $\{X_W\}$). Let $\mathcal{U}$ and $\mathcal{V}$ be families of subsets of a set X. If $\mathcal{V}$ refines $\mathcal{U}$ in the usual sense, we denote $\mathcal{V} < \mathcal{U}$. Let us define the notion of fibrewise covering uniformity.

Definition 2.20. ([15]) Let $p : X \to B$ be a projection, and $\mu = \{\mu_W\}$ be a system of coverings of $\{X_W\}$. We say that the system $\mu = \{\mu_W\}$ is a *fibrewise covering uniformity* (and a pair (X, μ) or $(X, \{\mu_W\})$) is a *fibrewise covering uniform space*) if the following conditions are satisfied:

(C1) Let $\mathcal{U}$ be a covering of X_W and for each $b \in W$ there exist $W' \in N(b)$ and $\mathcal{V} \in \mu_{W'}$ such that $W' \subset W$ and $\mathcal{V} < \mathcal{U}$. Then $\mathcal{U} \in \mu_W$.

(C2) For each $\mathcal{U}_i \in \mu_W, i = 1, 2$, there exists $\mathcal{U}_3 \in \mu_W$ such that $\mathcal{U}_3 < \mathcal{U}_i, i = 1, 2$.

(C3) For each $\mathcal{U} \in \mu_W$ and $b \in W$, there exist $W' \in N(b)$ and $\mathcal{V} \in \mu_{W'}$ such that $W' \subset W$ and $\mathcal{V}$ is a star refinement of $\mathcal{U}$.

(C4) For $W' \subset W$, $\mu_{W'} \supset \mu_W|_{X_{W'}}$, where

$$\mu_W|_{X_{W'}} = \{\mathcal{U}|_{X_{W'}} | \mathcal{U} \in \mu_W\} \text{ and } \mathcal{U}|_{X_{W'}} = \{U \cap X_{W'} | U \in \mathcal{U}\}.$$

For a fibrewise entourage uniformity Ω on X, $D \in \Omega$ and $W \in \tau$, let $\mathcal{U}(D, W) = \{D(x) \cap X_W | x \in X_W\}$. Further let $\mu_W(\Omega)$ be the family of coverings $\mathcal{U}$ of X_W satisfying that for each $b \in W$ there exist $W' \in N(b)$ and $D \in \Omega$ such that $W' \subset W$ and $\mathcal{U}(D, W') < \mathcal{U}$. Then the system $\mu(\Omega) = \{\mu_W(\Omega)\}$ is a fibrewise covering uniformity ([15, Proposition 3.7]).

Conversely, for a fibrewise covering uniformity $\mu = \{\mu_W\}$, we can constructed a fibrewise entourage uniformity $\Omega(\mu)$ as follows ([15, Construction 3.8]): For $\mathcal{U} \in \mu_W$, $D(\mathcal{U}) = \bigcup\{U_\alpha \times U_\alpha | U_\alpha \in \mathcal{U}\}$. Let $\Omega(\mu)$ be the family of all subsets $D \subset X \times X$ satisfying the following condition:

> $\Delta \subset D$, and for every $b \in B$ there exist $W \in N(b)$ and $\mathcal{U} \in \mu_W$ such that $D(\mathcal{U}) \subset D$.

Then $\Omega(\mu)$ is a fibrewise entourage uniformity ([15, Proposition 3.10]).

3. Čech-complete maps

In this section, we introduce a new notion "Čech-complete map", and investigate some its basic properties, invariance under perfect maps, characterizations by compactifications of Čech-complete maps and relationships with other maps. For the detail of Čech-completemaps, see Bai-Miwa [2].

Definition 3.1. Let X be a topological space, and A a subset of X. We say that the *diameter of A is less than a family $\mathcal{A} = \{A_s\}_{s\in S}$* of subsets of the space X, and we shall write $\delta(A) < \mathcal{A}$, provided that there exists an $s \in S$ such that $A \subset A_s$.

Definition 3.2. A T_2-compactifiable map $p : X \to B$ is *Čech-complete* if for each $b \in B$, there exists a countable family $\{\mathcal{A}_n\}_{n\in\mathbf{N}}$ of open (in X) covers of X_b with the property that every b-filter $\mathcal{F}$ which contains sets of diameter less than $\mathcal{A}_n$ for every $n \in \mathbf{N}$ has an adherence point.

Theorem 3.3. For a Čech-complete map $p : X \to B$, if F is a closed subset of X, then $p|F : F \to B$ is Čech-complete.

Theorem 3.4. Assume that B is regular. For a Čech-complete map $p : X \to B$, if G is a G_δ-subset of X, then $p|G$: $G \to B$ is Čech-complete.

For the fibrewise product spaces, we have the following.

Theorem 3.5. Let $\{(X_n, p_n)|n \in \mathbf{N}\}$ be a countable family of fibrewise spaces and (X, p) be the fibrewise product space of $\{(X_n, p_n)|n \in \mathbf{N}\}$, where $X = \prod_B X_n$. If each p_n is a surjective Čech-complete map, then the product p is Čech-complete.

We can prove the following about the invariance of Čech-complete maps under perfect maps.

Theorem 3.6. Let a fibrewise map $f : (X, p) \to (Y, q)$ be a perfect map, and p and q be T_2-compactifiable maps. Then p is Čech-complete if and only if q is Čech-complete.

We investigate some characterizations of Čech-complete maps by compactifications of the maps. Further, we give an example in which for a projection $p : X \to B$ each fibre is Čech-complete, but p is not Čech-complete. We can prove the following theorem.

Theorem 3.7. Suppose that B is regular. For a T_2-compactifiable map $p : X \to B$, the following are equivalent:
(1) $p : X \to B$ is Čech-complete.
(2) For every T_2-compactification $p' : X' \to B$ of p and each $b \in B$, X_b is a G_δ-subset of X'_b.
(3) There exists a T_2-compactification $p' : X' \to B$ of p such that X_b is a G_δ-subset of X'_b for each $b \in B$.

In the following example, we shall consider the difference between these characterizations of a Čech-complete map and Čech-completeness of each fibre of the map.

Example 3.8. There exists a map $p : X \to B$ satisfying the following:
(1) p is a T_2-compactifiable map.
(2) Each fibre of p is Čech-complete, but p is not a Čech-complete map.
(3) p is an open map. (Note that p is not a closed map.)

[Construction]: First, note that the space X is the same space constructed in [10, Example 1.6.19].

Let $X = \{0\} \cup (\bigcup_{m \in \mathbf{N}} X_m)$, where $X_m = \{\frac{1}{m}\} \cup \{\frac{1}{m} + \frac{1}{m^2+k} \ |k \in \mathbf{N}\}$, $A_0 = \{0\} \cup \{\frac{1}{m} \mid m \in \mathbf{N}\}$ and $Y_m = X_m - \{\frac{1}{m}\}$ for every $m \in \mathbf{N}$. We denote the n-th element of Y_m by P_{mn}. The topology on X is generated by a nbd system defined as follows: For $x \in Y_m$ for each $m \in \mathbf{N}$, $\mathcal{B}(x) = \{\{x\}\}$; for $x = \frac{1}{m}$, $\mathcal{B}(x) = \{U_n | n \in \mathbf{N}\}$, where $U_n = X_m - \{P_{mi} | i \leq n\}$; for $x = 0$,

$$\mathcal{B}(0) = \{U | \text{there exist a finite set } H \subset \mathbf{N} \text{ and } F \subset X - A_0 \text{ satisfying } F \cap B_m \text{ is finite for each } m \in \mathbf{N} \text{ such that } U = X - (F \cup \bigcup_{m \in H} X_m)\}.$$

This space X is perfectly normal and sequential ([10, Example 1.6.19]).

For each $n \in \mathbf{N}$ let $A_n = \{P_{mn} | m \in \mathbf{N}\}$, then $\{A_n\}_{n \geq 0}$ is a decomposition of X generating a quotient space $B = \{b_n | b_n = A_n \text{ and } n \geq 0\}$, which is a compact T_2-space. We denote the quotient map by $p : X \to B$.

About the relationships between Čech-complete maps and other maps, we shall prove the following implications under some conditions:

Locally compact map $\Longrightarrow$ Čech-complete map $\Longrightarrow$ k-map,

where a k-map $p : X \to B$ is same as that X is a fibrewise compactly generated space ([13, Section 10, Definition 10.3]).

Theorem 3.9. Every locally compact T_2-map is Čech-complete.

From Proposition 2.14, we know that every locally compact T_2-map is a k-map. Further, we can prove the following.

Theorem 3.10. Suppose that B is regular and satisfies the axiom of first countability. Then a Čech-complete map $p : X \to B$ is a k-map.

4. p-maps and M-maps

In this section, we introduce new notions of p-maps and M-maps, and investigate some basic properties, which are extensions of corresponding properties of p-spaces and M-spaces. For the detail of p-maps and M-maps, see Bai-Miwa [3].

Definition 4.1. For a T_2-compactifiable map $f : X \to B$ is a *p-map* if for every $b \in B$, there exists a sequence $\{\mathcal{U}_n\}_{n \in \mathbf{N}}$ of open (in X) covers of X_b satisfying that for every $n \in \mathbf{N}$ and $x \in X_b$, if $x \in U_n \in \mathcal{U}_n$ then
(P1) $(\bigcap_{n \in \mathbf{N}} \overline{U_n}) \cap X_b$ is compact.
(P2) For every open (in X) set U with $(\bigcap_{n \in \mathbf{N}} \overline{U_n}) \cap X_b \subset U$, there exist $n_0 \in \mathbf{N}$ and $W \in N(b)$ such that $(\bigcap_{n \in \mathbf{N}} \overline{U_n}) \cap X_b \subset (\bigcap_{i \leq n_0} \overline{U_i}) \cap X_W \subset U$.

For a p-map $f : X \to B$, we can characterize it by using a compactification of f as follows.

Theorem 4.2. Suppose that B is regular. For a map $f : X \to B$, f is a p-map if and only if there is a compactification $f' : X' \to B$ of f satisfying that for every $b \in B$ there is a sequence $\{\mathcal{P}_n\}_{n \in \mathbf{N}}$ of open families of X' such that:
(1) For every $n \in \mathbf{N}$, $X_b \subset \bigcup \mathcal{P}_n$,
(2) For every $x \in X_b$, $\bigcap_{n \in \mathbf{N}} st(x, \mathcal{P}_n) \cap X'_b \subset X_b$.

For a locally compact T_2-map $f : X \to B$, since it has the Alexandorff-type compactification $f' : X' \to B$ ([13, Section 8]) so X is open in X', we have the following.

Corollary 4.3. Suppose that B is regular. A locally compact T_2-map is a p-map.

For submaps of p-maps, we have the following.

Theorem 4.4. For a p-map $f : X \to B$, we have:
(1) If F is a closed subset of X, then the submap $f|F$ is a p-map.
(2) Suppose that B is regular. If G is a G_δ-subset of X, then the submap $f|G$ is a p-map.

Theorem 4.5. Suppose that B is regular. Let $f_n : X_n \to B$ be a p-map for every $n \in \mathbf{N}$. Then the product map $f = \prod_B f_n : \prod_B X_n \to B$ is a p-map.

Theorem 4.6. Let $f : X \to B$ and $g : Y \to B$ be maps and $\lambda : f \to g$ be a perfect morphism. If g is a p-map, then f is also p-map.

If $f : X \to B$ is a paracompact p-map, the converse of this theorem holds. For this, see Theorem 4.13.

Theorem 4.7. Suppose that B is regular. If $f : X \to B$ is Čech-complete, then f is a p-map.

Theorem 4.8. Suppose that B is regular and satisfies the axiom of first countability. Then a p-map $f : X \to B$ is a k-map.

Definition 4.9. A T_2-compactifiable map $f : X \to B$ is an *M-map* if for every $b \in B$ there is a sequence $\{\mathcal{U}_n\}_{n\in\mathbf{N}}$ of open (in X) covers of X_b satisfying:
(M1) For every $n \in \mathbf{N}$ and $x \in X_b$, if $x_n \in st(x,\mathcal{U}_n) \cap X_b$, then the sequence $\{x_n\}_{n\in\mathbf{N}}$ has an accumulation point in X_b,
(M2) For every $n \in \mathbf{N}$, $\mathcal{U}_{n+1}$ is a b-star refinement of $\mathcal{U}_n$.

For submaps of an M-map, we have the following.

Theorem 4.10. For an M-map $f : X \to B$ and a closed subset F of X, $f|F$ is an M-map.

Theorem 4.11. For maps $f : X \to B$ and $g : Y \to B$, if there is a quasi-perfect morphism $\lambda : f \to g$ and g is an M-map, then f is an M-map.

About the relationships of p-maps and M-maps, there are no implications each other. For these, consider the case B is the singleton set and a p-space (resp. an M-space) is not an M-space (resp. a p-space). In the realm of paracompact maps, we can prove in Theorem 4.12 that an M-map is the same as a p-map which is an analogous theorem of [1, Theorem 16]. Further, we can prove in Theorem 4.13 that a perfect image of a paracompact p-map is also a paracompact p-map which is an analogous theorem of [11, Theorem 1].

Theorem 4.12. For a paracompact map $f : X \to B$, f is an M-map if and only if f is a p-map.

For a perfect morphism image of a p-map, we have the following.

Theorem 4.13. Suppose that B is regular. For T_2-compactifiable maps $f : X \to B$ and $g : Y \to B$, if there exists an onto perfect morphism $\lambda : f \to g$ and f is a paracompact p-map, then g is a paracompact p-map.

In connection with Theorem 4.13, note that, if f is not paracompact, it does not necessarily hold. For this, see [5, Example 2.1] for the case B is the singleton set.

Finally, we investigate the relations of MT-maps and (paracompact) M-maps and some analogous problems of the relations of metrizable spaces and (paracompact) M-spaces. First, we have the following.

Theorem 4.14. Suppose that B is regular. If a T_2-compactifiable map $f : X \to B$ has a f-development, then f is a p-map.

Corollary 4.15. Suppose that B is regular. If a map $f : X \to B$ is an MT-map, then f is a paracompact p-map.

Corollary 4.16. Suppose that B is regular. Let $f : X \to B$ and $g : Y \to B$ be maps and $\lambda : f \to g$ be a perfect morphism. If the map g is an MT-map, then f is a paracompact p-map (so M-map).

For two maps $f : X \to B$ and $g : Y \to B$, f is said to be (resp. *closedly*) *embeddable* to g if there exists a morphism $\lambda : f \to g$ such that $\lambda(X)$ is a (resp. closed) subspace of Y.

Problem 4.17. Let $f : X \to B$ be an M-map (resp. a paracompact M-map). Then is there an MT-map $g : Y \to B$ and a quasi-perfect (resp. a perfect) morphism $\lambda : f \to g$?

In this case, we call f the *preimage-map* of g under λ.

Problem 4.18. Let $f : X \to B$ be a paracompact p-map. Then is f closedly embeddable to the product map of an MT-map and a compact map?

The next theorem is a partial answer of Problem 4.18. If Problem 4.17 is affirmative, Problem 4.18 is affirmative by this theorem.

Theorem 4.19. Let $f : X \to B$ be a map such that f is a preimage-map of an MT-map $g : Y \to B$ under a perfect morphism $\lambda : f \to g$. Then f is closedly embeddable to the product map of g and a T_2-compactification $f' : X' \to B$ of f.

5. Uniformities and completeness on metrizable maps

In this section, we prove the existence of fibrewise uniformities on some metrizable maps, and study the relations between the completeness induced by a trivial metric and the one defined by fibrewise uniformities. Further, we discuss the relations between completely metrizable maps and Čech-complete maps. For the detail of uniformities and completeness on metrizable maps, see Bai-Miwa [4].

Firstly, we shall show that for a TM-map $p : X \to B$ parallel to a metric space (M, ρ), there exists a fibrewise covering uniformity on X induced by ρ. Let $e : X \to B \times M$ be an embedding. For each $n \in \mathbf{N}$, let $\mathcal{U}_n$ be the family $\{U(x, \frac{1}{n}) | x \in M\}$, where $U(x, \frac{1}{n}) = \{y \in M | \rho(x, y) < \frac{1}{n}\}$ and $\mathcal{W}_n = \{e^{-1}(B \times U) | U \in \mathcal{U}_n\}$. Then for each $W \in \tau$, let $\mu_W = \{\mathcal{U} | \bigcup \mathcal{U} = X_W$ and for each $b \in W$ there exist $n \in \mathbf{N}$ and $W' \in N(b)$ with $W' \subset W$ such that $\mathcal{W}_n | X_{W'} < \mathcal{U}\}$.

Since μ_W and μ constructed above are induced by the metric ρ on M (on X), we call this $\mu = \{\mu_W\}$ a *fibrewise covering uniformity on X induced* by the metric ρ, and denoted by $\mu_\rho = \{\mu_W\}_\rho$. Further, by the construction of $\{\mathcal{W}_n | n \in \mathbf{N}\}$ in the above, we say that the family $\{\mathcal{W}_n | n \in \mathbf{N}\}$ is the *standard developable covering* (*sd*-covering, for short) on X induced by ρ. (Note that we exclusively use the notation $\{\mathcal{W}_n | n \in \mathbf{N}\}$ as *sd*-covering induced by ρ in this section.)

We can prove the following theorem.

Theorem 5.1. For a TM-map $p : X \to B$ with a T-metric ρ, the system $\mu_\rho = \{\mu_W\}_\rho$ is a fibrewise covering uniformity on X induced by ρ.

Next, we investigate the equivalence between the completeness of a TM-map in the Pasynkov's sense ([20]) and the one in the James' sense ([13, Section 15]). For convenience' sake, we call them *P-complete(ness)* and *J-complete(ness)*, respectively.

To study it, first we must investigate the completeness by fibrewise covering uniformity (we call it *CU-complete(ness).*) and prove that it is equivalent to J-completeness. For this, using the equivalence between fibrewise entourage uniformity and fibrewise covering uniformity ([15]) and the theory of I. M. James ([13, Chapter 3]), we shall prove in Theorem 5.6 that for a TM-map with T-metric ρ a b-filter is P-Cauchy with respect to ρ (Definition 5.5) if and only if it is J-Cauchy with respect to $\Omega(\mu_\rho)$ (Definition 5.2).

The notion of fibrewise entourage uniformity is slightly stronger than the one of fibrewise uniform structure ([13]), but we use the terminology in [13] in this theory of fibrewise entourage uniformity.

We first recall the definition of Cauchy b-filter in [13].

Definition 5.2. ([13, Definition 14.1]) For a map $p : X \to B$, let Ω be a fibrewise entourage uniformity on X.
(1) A subset M of X is said to be *D-small*, where $D \subset X^2$, if M^2 is contained in D.
(2) A b-filer $\mathcal{F}$, where $b \in B$, is *Cauchy* if $\mathcal{F}$ contains a D-small member for each $D \in \Omega$. (We call $\mathcal{F}$ J-Cauchy with respect to Ω (with respect to Ω, for short), for convenience' sake.)

We shall define a new notion of Cauchy b-filter in fibrewise covering uniformity $\mu = \{\mu_W\}$ on X.

Definition 5.3. For a map $p : X \to B$, let $\mu = \{\mu_W\}$ be a fibrewise covering uniformity on X. A b-filer $\mathcal{F}$, where $b \in B$, is *Cauchy* if for each $W \in N(b)$ and $\mathcal{U} \in \mu_W$ there exist $F \in \mathcal{F}$ and $U \in \mathcal{U}$ such that $F \subset U$. (We call $\mathcal{F}$ CU-Cauchy with respect to μ (with respect to μ, for short), for convenience' sake.)

Theorem 5.4. For a map $p : X \to B$, let Ω be a fibrewise entourage uniformity on X. Then for each $b \in B$, a b-filer $\mathcal{F}$ is J-Cauchy with respect to Ω if and only if it is CU-Cauchy with respect to $\mu(\Omega)$.

In the following we shall give the definitions of P-completion and J-completion of TM-map. For a space X, let $\Upsilon = \{\Phi_\alpha | \alpha \in \Lambda\}$ be a collection of families of subsets of X. We say that a family Ψ of subsets of X is *subordinated* to the family Υ if for each $\alpha \in \Lambda$ there exist $U_\alpha \in \Phi_\alpha$ and $V \in \Psi$ such that $V \subset U_\alpha$.

Definition 5.5. Let $p : X \to B$ be a TM-map with a T-metric ρ.
(1)([20]) The map p is *complete* if for any b-filter $\mathcal{F}$, $b \in B$, on X subordinated to the sd-covering $\{\mathcal{W}_n | n \in \mathbf{N}\}$ induced by ρ, it has adherence points. (We call this "complete" *P-complete*, and also call this b-filter satisfying this condition *P-Cauchy* w.r.t ρ.)
(2)([13, Definition 14.10]) The map p is *complete* if for each $b \in B$ any J-Cauchy b-filter $\mathcal{F}$ with respect to $\Omega(\mu_\rho)$ converges. (We call this "complete" *J-complete.*)

Theorem 5.6. For a TM-map $p : X \to B$ with a T-metric ρ and each $b \in B$, a b-filer $\mathcal{F}$ is a P-Cauchy with respect to ρ if and only if it is a J-Cauchy with respect to Ω_ρ.

Finally, we study the relations between complete TM-maps and Čech-complete maps.

Lemma 5.7. Every TM-map $p : X \to B$ is a $T_{3\frac{1}{2}}$-map.

By this lemma, every TM-map is $T_{3\frac{1}{2}}$-compactifiable. For complete TM-maps, we can prove the following.

Theorem 5.8. If $p : X \to B$ is a complete TM-map, then p is Čech-complete.

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