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p-HARMONIC FUNCTIONS ON A RESISTIVE TREE

HISAYASU KURATA

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Abstract. We discuss the p-harmonicity of a linear combination of p-harmonic functions on a tree. If $p \neq 2$, the p-harmonicity is non-linear, i.e., the linear combination of p-harmonic functions need not be p-harmonic. In spite of this non-linear nature, we find some p-harmonic functions whose linear combinations become p-harmonic.

Also we discuss the quasi-symmetricity of p-Green functions. It is well known that two 2-Green functions g_a and g_b are symmetric, i.e., $g_a(b) = g_b(a)$. However it is not known whether two p-Green functions are symmetric or not if $p \neq 2$. Moreover it is not known even whether those are quasi-symmetric or not, which means that $g_a(b)/g_b(a)$ is bounded or not. In this article we show that, for every tree, there exists a resistance such that two p-Green functions are quasi-symmetric; also we show that, for every tree, there exists a resistance such that two p-Green functions are not quasi-symmetric.

This paper is rewritten from the doctor's thesis.

1. Introduction

Let $1 < p < \infty$. We consider p-harmonic functions on a tree. Let $\mathcal{T} = (V, E, r)$ be a locally finite connected tree with a resistance r, where $V = V(\mathcal{T})$ is the vertex set and $E = E(\mathcal{T})$ is the edge set. An edge $(x, y) \in E$ is an ordered pair of vertices such that $(x, y) \in E$ if and only if $(y, x) \in E$. If $(x, y) \in E$, then we say that x is adjacent to y and write $x \sim y$. A resistance r is a positive function on E such that $r(y, x) = r(x, y)$. We define the discrete derivative ∇u and the discrete p-Laplacian $\Delta_p u$ for a function u on V by

$$\nabla u(x, y) = r(x, y)^{-1}(u(y) - u(x)),$$
$$\Delta_p u(x) = \sum_{\substack{y \in V \\ y \sim x}} |\nabla u(x, y)|^{p-2} \nabla u(x, y).$$

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A summary of doctoral thesis at Shimane University.

Let $\varphi_p(t) = |t|^{p-2}t$. Then we can write

$$\Delta_p u(x) = \sum_{\substack{y\in V\\ y\sim x}} \varphi_p(\nabla u(x,y)) = -\sum_{\substack{y\in V\\ y\sim x}} \varphi_p(\nabla u(y,x)).$$

Let $D \subset V$. If $\Delta_p u = 0$ in D, then we say that u is p-harmonic in D.

Let $x, y \in V$. A path joining x to y is a sequence $\{x = x_0, x_1, \dots, x_{l-1}, x_l = y\}$ of distinct vertices such that $x_0 \sim x_1 \sim \cdots \sim x_{l-1} \sim x_l$. Since $\mathcal{T}$ is a tree, the path joining x to y is unique. The number l is called the length of the path and is denoted by $\rho(x,y)$. For $x \in V$ let $\deg(x) = \#\{y \in V; \rho(x,y) = 1\}$. This is the number of neighbors of x. Let A be a subset of V. We say that A is connected if any two vertices of A are joined by a path whose vertices are still in A. By $\overline{A}$ we denote the minimal connected set including A. Let $B \subset E$ and $x \in V$. We remove B from E, then we obtain some components. We denote by $\mathcal{S}(\mathcal{T}, B, x)$ the component which contains x.

We define the Dirichlet sum $D_p[u]$ of order p by

$$D_p[u] = \frac{1}{2} \sum_{(x,y)\in E} r(x,y)|\nabla u(x,y)|^p.$$

Denote by $\mathbf{D}^{(p)}(\mathcal{T})$ the set of functions on V with finite Dirichlet sum of order p. Then $\mathbf{D}^{(p)}(\mathcal{T})$ is a Banach space with the norm $\|u\|_p = (D_p[u] + |u(x_0)|)^{1/p}$, where x_0 is a fixed vertex. Let $L_0(\mathcal{T})$ be the set of functions on V with finite support. Also let $\mathbf{D}_0^{(p)}(\mathcal{T})$ be the closure of $L_0(\mathcal{T})$ in $\mathbf{D}^{(p)}(\mathcal{T})$ with respect to the norm $\|\cdot\|_p$. A tree $\mathcal{T}$ is said to be of hyperbolic type of order p if $1 \notin \mathbf{D}_0^{(p)}(\mathcal{T})$; a tree $\mathcal{T}$ is said to be of parabolic type of order p otherwise. Consider the discrete boundary value problem

$$\Delta_p u = -\delta_a, \qquad u \in \mathbf{D}_0^{(p)}(\mathcal{T}), \tag{1}$$

where δ_a is the characteristic function of $\{a\}$, i.e., $\delta_a(x) = 1$ if $x = a$ and $\delta_a(x) = 0$ otherwise. The solution u to (1) uniquely exists if and only if the tree is of hyperbolic type of order p. We call the solution u the p-Green function with pole at a and denote it by g_a. For these accounts see Kayano-Yamasaki [1], Nakamura-Yamasaki [4], Soardi-Yamasaki [5], Yamasaki [6, 7, 8].

We discuss the p-harmonicity of a linear combination of p-harmonic functions on a tree. If $p \neq 2$, the p-harmonicity is non-linear, i.e., the linear combination of p-harmonic functions need not be p-harmonic. In spite of this non-linear nature, we find some p-harmonic functions whose linear combinations become p-harmonic. By definition a constant is a p-harmonic function and the linear combination of an arbitrary p-harmonic function and a constant is p-harmonic. We shall find other p-harmonic linear combinations of p-harmonic functions. Let $\{u_1, \dots, u_m\}$ be an m-tuple of p-harmonic functions in $D \subset V$ such that $\{1, u_1, \dots, u_m\}$ is linearly independent. We say that $\{u_1, \dots, u_m\}$ has a linear relation in D if every linear combination $\sum_{j=1}^m t_j u_j$ is p-harmonic in D. Also we say that $\{u_1, \dots, u_m\}$ has a partial linear relation in D if $\sum_{j=1}^m t_j u_j$ is p-harmonic in D for some $t_1, \dots, t_m \in \mathbb{R} \setminus \{0\}$. This problem has studied in [2].

Theorem 1. *Let $D \subset V$ and let $\{u_1, \dots, u_m\}$ be an m-tuple of p-harmonic functions in D. Suppose that, for each $(x,y) \in E$ with $x \in D$ or $y \in D$, there is $j_0(x,y) \in \{1,\dots,m\}$ such that $u_j(x) = u_j(y)$ whenever $j \neq j_0(x,y)$. Then $\{u_1,\dots,u_m\}$ has a linear relation in D.*

Example 2. Let $\mathcal{T}$ be a tree formed by m half lines meeting at a vertex x_0, i.e., $V = \{x_0\} \cup \bigcup_{i=1}^m \{x_{i,k}\}_{k=1}^\infty$ and $E = \bigcup_{i=1}^m \{(x_{i,k-1}, x_{i,k}), (x_{i,k}, x_{i,k-1})\}_{k=1}^\infty$, where $x_{i,0} = x_0$ for $i = 1,\dots,m$. Let r be an arbitrary resistance on E. We define a function u_i on V by

$$u_i(x_0) = 0,$$
$$u_i(x_{j,k}) = \begin{cases} r(x_0, x_{j,1}) + r(x_{j,1}, x_{j,2}) + \cdots + r(x_{j,k-1}, x_{j,k}) & \text{if } j = i, \\ 0 & \text{otherwise} \end{cases}$$

for $i = 1,\dots,m$. Let $1 < p < \infty$. Then the following statements hold:

(i) Every u_i is p-harmonic in $V \setminus \{x_0\}$ and the m-tuple $\{u_1,\dots,u_m\}$ has a linear relation in $V \setminus \{x_0\}$.

(ii) Every difference $v_{i,j} = u_i - u_j$ is p-harmonic in V. Moreover, let $\Lambda := \{(i_1,j_1),\dots,(i_\mu,j_\mu)\} \subset \{1,\dots,m\} \times \{1,\dots,m\}$ such that $i_1, j_1, \dots, i_\mu, j_\mu$ are distinct integers. Then $\{v_{i,j}\}_{(i,j)\in\Lambda}$ has a linear relation in V.

(iii) The $(m-1)$-tuple $\{v_{1,2}, v_{1,3}, \dots, v_{1,m}\}$ has a partial linear relation in V.

We give different types of p-harmonic functions with linear relation.

Theorem 3. *Suppose that $\deg(x) \geq 2$ for every $x \in V$. For $a \in V$ we define a function h_a on V by*

$$h_a(a) = 0, \qquad h_a(x) = \sum_{k=0}^{l-1} r(x_k, x_{k+1}) \prod_{j=0}^{k} (\deg(x_j) - 1)^{1/(1-p)},$$

where $\{a = x_0, x_1, \dots, x_{l-1}, x_l = x\}$ is the path joining a to x. Then the function h_a is p-harmonic in $V \setminus \{a\}$. If A is a finite subset of V, then $\{h_a\}_{a\in A}$ has a linear relation in $V \setminus \overline{A}$.

Theorem 4. *Suppose that $\deg(x) \geq 2$ for every $x \in V$. Let h_a be the function on V defined in Theorem 3. If $a, b \in V$ with $\rho(a,b) = 2$, then $\{h_a, h_b\}$ has a partial linear relation in $V \setminus \{a,b\}$.*

We show that the set of p-Green functions with poles $a \in A$ has a linear relation outside $\overline{A}$.

Theorem 5. *Suppose that the tree $\mathcal{T}$ is of hyperbolic type of order p. Then the p-Green function g_a is p-harmonic in $V \setminus \{a\}$. If A is a finite subset of V, then $\{g_a\}_{a\in A}$ has a linear relation in $V \setminus \overline{A}$.*

Next we discuss the quasi-symmetricity of p-Green functions. It is well known that two 2-Green functions g_a and g_b are symmetric, i.e., $g_a(b) = g_b(a)$. However it is not known whether two p-Green functions are symmetric or not if $p \neq 2$. Moreover it is not known even whether those are quasi-symmetric or not, which means that $g_a(b)/g_b(a)$ is bounded or not. Let $\mathcal{T}$ be a tree of hyperbolic type of order p. Let

$$H(x,y) = \frac{g_x(y)}{g_y(x)} \qquad \text{for } x,y \in V,$$
$$M(\mathcal{T}) = \sup_{x,y\in V} H(x,y).$$

We consider the problem whether $M(\mathcal{T})$ is finite or not for $p \neq 2$. A tree $\mathcal{T}$ is said to have a symmetric p-Green function if $M(\mathcal{T}) = 1$; a tree $\mathcal{T}$ is said to have a quasi-symmetric p-Green function if $M(\mathcal{T})$ is finite. This problem has studied in [3].

Theorem 6. *Let $\mathcal{T} = (V,E,r)$ be a tree of hyperbolic type of order p. Let $(a_1,a_2) \in E$. Let $\mathcal{T}_1 = \mathcal{S}(\mathcal{T},\{(a_1,a_2)\},a_1)$ and $\mathcal{T}_2 = \mathcal{S}(\mathcal{T},\{(a_1,a_2)\},a_2)$. Then $\mathcal{T}$ has a quasi-symmetric p-Green function if and only if each of $\mathcal{T}_1$ and $\mathcal{T}_2$ has a quasi-symmetric p-Green function.*

Theorem 7. *Let $p \neq 2$. Let (V,E,r) be a tree.*

(i) *Suppose that there are only finitely many $x \in V$ such that* $\deg(x) \geq 3$. *Then (V,E,r) has a quasi-symmetric p-Green function whenever (V,E,r) is of hyperbolic type of order p.*

(ii) *Suppose that there are infinitely many $x \in V$ such that* $\deg(x) \geq 3$. *Then we find two resistances r_1 and r_2 with the following conditions.*

   (a) *The tree (V,E,r_1) is of hyperbolic type of order p and has a quasi-symmetric p-Green function.*

   (b) *The tree (V,E,r_2) is of hyperbolic type of order p and does not have a quasi-symmetric p-Green function.*

2. Proof of Theorems 1, 3, 4 and 5

Note that the function $\varphi_p(t) = |t|^{p-2}t$ satisfies $\varphi_p(st) = \varphi_p(s)\varphi_p(t)$.

Proof of Theorem 1. Let $u = \sum_{j=1}^m t_j u_j$. We shall prove $\Delta_p u(x) = 0$ for every $x \in D$. Take $y \in V$ with $y \sim x$. By the assumption we have $\nabla u_j(x,y) = 0$ for $j \neq j_0(x,y)$. Therefore

$$\varphi_p(\nabla u(x,y)) = \varphi_p(t_{j_0(x,y)})\varphi_p(\nabla u_{j_0(x,y)}(x,y)) = \sum_{j=1}^m \varphi_p(t_j)\varphi_p(\nabla u_j(x,y)).$$

Hence

$$\Delta_p u(x) = \sum_{j=1}^m \varphi_p(t_j)\Delta_p u_j(x) = 0.$$

This means u is p-harmonic at x, and hence in D. □

Proof of Example 2. (i). We observe that

$$\nabla u_i(x_{j,k-1}, x_{j,k}) = \begin{cases} 1 & \text{if } j = i, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

It is clear that $\Delta_p u_i(x_{j,k}) = 0$ if $j \neq i$. If $j = i$, then

$$\nabla u_i(x_{j,k}, x_{j,k+1}) = 1, \qquad \nabla u_i(x_{j,k}, x_{j,k-1}) = -1,$$

and therefore $\Delta_p u_i(x_{i,k}) = 0$. Hence u_i is p-harmonic in $V \setminus \{x_0\}$ for every p. Also (2) means that $\{u_1, \dots, u_m\}$ satisfies the condition of Theorem 1 for $D = V \setminus \{x_0\}$. Hence $\{u_1, \dots, u_m\}$ has a linear relation in $V \setminus \{x_0\}$.

(ii). We observe from (i) that $v_{i,j}$ are p-harmonic in $V \setminus \{x_0\}$ and so is their arbitrary linear combination. By definition

$$\Delta_p v_{i,j}(x_0) = \varphi_p(\nabla u_i(x_0, x_{i,1})) - \varphi_p(\nabla u_j(x_0, x_{j,1})) = 0,$$

and hence $v_{i,j}$ is p-harmonic at $\{x_0\}$ as well. We see that $\{v_{i,j}\}_{(i,j)\in\Lambda}$ satisfies the condition of Theorem 1 for $D = V$, and therefore $\{v_{i,j}\}_{(i,j)\in\Lambda}$ has a linear relation in V.

(iii). Let $u = \sum_{j=2}^m t_j v_{1,j}$. Then $u = \big(\sum_{j=2}^m t_j\big)u_1 - \big(\sum_{j=2}^m t_j u_j\big)$, so that

$$\Delta_p u(x_0) = \varphi_p\Big(\sum_{j=2}^m t_j\Big) - \sum_{j=2}^m \varphi_p(t_j).$$

Therefore, u is p-harmonic at x_0 if and only if $\varphi_p\big(\sum_{j=2}^m t_j\big) = \sum_{j=2}^m \varphi_p(t_j)$. This shows that $\{v_{1,2}, v_{1,3}, \dots, v_{1,m}\}$ has a partial linear relation in V. □

We shall prove Theorems 3 and 4. For simplicity we let $w(x) = (\deg(x)-1)^{1/(1-p)}$. Then

$$h_a(a) = 0, \qquad h_a(x) = \sum_{k=0}^{l-1} r(x_k, x_{k+1}) \prod_{j=0}^{k} w(x_j),$$

where $\{a = x_0, x_1, \dots, x_{l-1}, x_l = x\}$ is the path joining a to x.

Proof of Theorem 3. Let $x \in V \setminus \{a\}$. Take the path $\{a = x_0, x_1, \dots, x_{l-1}, x_l = x\}$ joining a to x. Also let $y = x_{l-1}$ and let $z_1, \dots, z_{\deg(x)-1}$ be the other neighbors of x. Since the path joining a to z_k is $\{a, x_1, \dots, x_{l-1}, x, z_k\}$, we have

$$\nabla h_a(x, y) = -\frac{r(x_{l-1}, x_l)}{r(x, y)} \prod_{j=0}^{l-1} w(x_j) = -\prod_{j=0}^{l-1} w(x_j),$$

$$\nabla h_a(x, z_k) = \frac{r(x_l, z_k)}{r(x, z_k)} \prod_{j=0}^{l} w(x_j) = \prod_{j=0}^{l} w(x_j).$$

Therefore

$$\nabla h_a(x, z_k) = -w(x)\nabla h_a(x, y). \tag{3}$$

Since $\varphi_p(w(x)) = (\deg(x) - 1)^{-1}$,

$$\Delta_p h_a(x) = \varphi_p(\nabla h_a(x, y)) + \sum_{k=1}^{\deg(x)-1} \varphi_p(\nabla h_a(x, z_k)) = 0.$$

Next let A be a finite subset of V and let $u = \sum_{a\in A} t_a h_a$. Let $x \in V \setminus \overline{A}$. Let y be the neighbor of x which is on the path joining $a \in A$ to x. Note that y is independent of the choice of a since $x \notin \overline{A}$. Let $z_1, \dots, z_{\deg(x)-1}$ be the other neighbors of x. Then (3) holds for each $a \in A$. Therefore $\nabla u(x, z_k) = -w(x)\nabla u(x, y)$, and consequently

$$\Delta_p u(x) = \varphi_p(\nabla u(x, y)) + \sum_{k=1}^{\deg(x)-1} \varphi_p(\nabla u(x, z_k)) = 0.$$

This means u is p-harmonic at x, and hence in $V \setminus \overline{A}$. □

Proof of Theorem 4. Let $u = sh_a + th_b$. Let x be the vertex between a and b and let $z_1, \dots, z_{\deg(x)-2}$ be the other neighbors of x. Then u is p-harmonic in $V \setminus \{a, x, b\}$ by Theorem 3.

Now we consider the p-harmonicity of u at x. We have $h_a(a) = 0$ and $h_b(a) = r(b, x)w(b) + r(x, a)w(b)w(x)$. Therefore

$$u(a) = tr(b, x)w(b) + tr(x, a)w(b)w(x).$$

Similarly we have

$$\begin{aligned} u(b) &= sr(a, x)w(a) + sr(x, b)w(a)w(x), \\ u(z_k) &= sr(a, x)w(a) + sr(x, z_k)w(a)w(x) + tr(b, x)w(b) + tr(x, z_k)w(b)w(x), \\ u(x) &= sr(a, x)w(a) + tr(b, x)w(b). \end{aligned}$$

Hence

$$\begin{aligned} \nabla u(x, a) &= tw(b)w(x) - sw(a), \\ \nabla u(x, b) &= sw(a)w(x) - tw(b), \\ \nabla u(x, z_k) &= sw(a)w(x) + tw(b)w(x). \end{aligned}$$

If we take s and t such that $sw(a) + tw(b) = 0$, then

$$\nabla u(x, a) = -\nabla u(x, b), \qquad \nabla u(x, z_k) = 0,$$

and therefore $\Delta_p u(x) = 0$. This means that $\{h_a, h_b\}$ has a partial linear relation. □

Proof of Theorem 5. It is evident that g_a is p-harmonic in $V \setminus \{a\}$. Let A be a finite subset of V. Let $x \in V \setminus \overline{A}$. Take the path $\{a, x_1, \dots, x_{l-1}, x\}$ joining $a \in A$ to x. Let k be the number such that $x_k \in \overline{A}$ and $x_{k+1} \notin \overline{A}$. Let $y = x_k$. Then y is independent of the choice of a.

Let $\tilde{\mathcal{T}}$ be the subtree whose vertex set is the union of $\{y\}$ and the connected component of $V \setminus \overline{A}$ including x. Let $\tilde{\Delta}_p$ be the p-Laplacian with respect to $\tilde{\mathcal{T}}$. Let $\tilde{u}_a$ be the restriction of g_a to $\tilde{\mathcal{T}}$. Then it is easy to see that

$$\tilde{\Delta}_p \tilde{u}_a = \Delta_p g_a = 0 \quad \text{in } \tilde{\mathcal{T}} \setminus \{y\}, \qquad \tilde{u}_a \in \mathbf{D}_0^{(p)}(\tilde{\mathcal{T}}).$$

Let $c_a = -\varphi_q(\tilde{\Delta}_p \tilde{u}_a(y))$, where q is the number such that $1/p + 1/q = 1$. Note that $\varphi_p(\varphi_q(t)) = t$. We see that $\tilde{v}_a = \tilde{u}_a / c_a$ satisfies

$$\tilde{\Delta}_p \tilde{v}_a = \frac{\tilde{\Delta}_p \tilde{u}_a}{\varphi_p(c_a)} = -\delta_y \quad \text{in } \tilde{\mathcal{T}}, \qquad \tilde{v}_a \in \mathbf{D}_0^{(p)}(\tilde{\mathcal{T}}).$$

Therefore $\tilde{v}_a$ is the p-Green function $\tilde{g}_y$ with pole at y with respect to $\tilde{\mathcal{T}}$. The uniqueness of the p-Green function implies that $\tilde{v}_a = \tilde{g}_y$, especially that $\tilde{v}_a$ is independent of the choice of a. This means that

$$g_a = \tilde{u}_a = c_a \tilde{v}_a = c_a \tilde{g}_y \qquad \text{in } \tilde{\mathcal{T}}.$$

If we set $u = \sum_{a \in A} t_a g_a$, then

$$u = (\sum_{a \in A} t_a c_a) \tilde{g}_y \qquad \text{in } \tilde{\mathcal{T}}.$$

Hence

$$\Delta_p u(x) = \varphi_p(\sum_{a \in A} t_a c_a) \Delta_p \tilde{g}_y(x) = \varphi_p(\sum_{a \in A} t_a c_a) \tilde{\Delta}_p \tilde{g}_y(x) = 0.$$

Thus u is p-harmonic at x. Since $x \in V \setminus \overline{A}$ is arbitrary, u is p-harmonic in $V \setminus \overline{A}$. Therefore $\{g_a\}_{a \in A}$ has a linear relation in $V \setminus \overline{A}$. □

3. Proof of Theorem 6

First note that $cu|_{V(\mathcal{T}')} \in \mathbf{D}_0^{(p)}(\mathcal{T}')$ for any $u \in \mathbf{D}_0^{(p)}(\mathcal{T})$, for any subtree $\mathcal{T}'$ of $\mathcal{T}$ and for any constant c. This fact is applied repeatedly.

Lemma 8. *Let $\mathcal{T} = (V, E, r)$ be a tree of hyperbolic type of order p. Let $(a, b) \in E$. Let $\mathcal{T}' = \mathcal{S}(\mathcal{T}, \{(a, b)\}, a)$ and $\mathcal{T}'' = \mathcal{S}(\mathcal{T}, \{(a, b)\}, b)$. Let $x \in V(\mathcal{T}') \cup \{b\}$.*

(i) *If $\mathcal{T}''$ is of hyperbolic type of order p, then $g_x|_{V(\mathcal{T}'')}$ is constant times of $g_b^{\mathcal{T}''}$;*

(ii) *if $\mathcal{T}''$ is of parabolic type of order p, then $g_x|_{V(\mathcal{T}'')}$ is a constant function in $V(\mathcal{T}'')$.*

Proof. We denote the p-Laplacian (resp. p-Green function, and so on) with respect to $\mathcal{T}'$ by $\Delta_p^{\mathcal{T}'}$ (resp. $g_x^{\mathcal{T}'}$, and so on).

First suppose $x \in V(\mathcal{T}')$. Let $\overline{xb} = \{x = x_0, x_1, \dots, x_{l-1}, x_l = b\}$. Let

$$\begin{aligned} \mathcal{T}_j &= \mathcal{S}(\mathcal{T}, \{(x_{j-1}, x_j)\}, x_j) && \text{for } j = 1, \dots, l, \\ \mathcal{S}_j &= \mathcal{S}(\mathcal{T}, \{(x_{j-1}, x_j), (x_j, x_{j+1})\}, x_j) && \text{for } j = 1, \dots, l-1, \\ \mathcal{S}_0 &= \mathcal{S}(\mathcal{T}, \{(x_0, x_1)\}, x_0). \end{aligned}$$

If $\mathcal{T}_l$ is of hyperbolic type of order p, then we let $u = g_{x_l}^{\mathcal{T}_l}$ in $V(\mathcal{T}_l)$; if $\mathcal{T}_l$ is of parabolic type of order p, then we let $u = 1$ in $V(\mathcal{T}_l)$. Then u is p-harmonic in $V(\mathcal{T}_l) \setminus \{x_l\}$ and $u \in \mathbf{D}_0^{(p)}(\mathcal{T}_l)$. Let

$$u(x_{l-1}) = u(x_l) - r(x_{l-1}, x_l) \varphi_q(\Delta_p^{\mathcal{T}_l} u(x_l)),$$

where q is the number with $1/p + 1/q = 1$. Since φ_q is the inverse function of φ_p, we have that u is p-harmonic at x_l.

If $\mathcal{S}_{l-1}$ is of hyperbolic type of order p, then we let

$$u = \frac{u(x_{l-1})}{g_{x_{l-1}}^{\mathcal{S}_{l-1}}(x_{l-1})} g_{x_{l-1}}^{\mathcal{S}_{l-1}} \qquad \text{in } V(\mathcal{S}_{l-1});$$

if $\mathcal{S}_{l-1}$ is of parabolic type of order p, then we let $u = u(x_{l-1})$ in $V(\mathcal{S}_{l-1})$. Then u is p-harmonic in $V(\mathcal{T}_{l-1}) \setminus \{x_{l-1}\}$ and $u \in \mathbf{D}_0^{(p)}(\mathcal{T}_{l-1})$.

Repeat this argument and obtain a function u which is p-harmonic in $V \setminus \{x\}$ and $u \in \mathbf{D}_0^{(p)}(\mathcal{T})$. Therefore u is a constant times of g_x. Since $\mathcal{T}_l = \mathcal{T}''$ and $x_l = b$, we have the result in this case.

Next suppose $x = b$. If $\mathcal{T}'$ is of hyperbolic type of order p, then we let $u = g_a^{\mathcal{T}'}$ in $V(\mathcal{T}')$; if $\mathcal{T}'$ is of parabolic type of order p, then we let $u = 1$ in $V(\mathcal{T}')$. Then u is p-harmonic in $V(\mathcal{T}') \setminus \{a\}$ and $u \in \mathbf{D}_0^{(p)}(\mathcal{T}')$. Let

$$u(b) = u(a) - r(a,b)\varphi_q(\Delta_p^{\mathcal{T}'} u(a)).$$

Then u is p-harmonic at a.

If $\mathcal{T}''$ is of hyperbolic type of order p, then we let

$$u = \frac{u(b)}{g_b^{\mathcal{T}''}(b)} g_b^{\mathcal{T}''} \qquad \text{in } V(\mathcal{T}'');$$

if $\mathcal{T}''$ is of parabolic type of order p, then we let $u = u(b)$ in $V(\mathcal{T}'')$. Then u is p-harmonic in $V \setminus \{b\}$ and $u \in \mathbf{D}_0^{(p)}(\mathcal{T})$. Therefore u is a constant times of g_x, and the result follows. □

Lemma 9. *For any $x, y, z \in V$ we have*

$$H(x,z) = H(x,y)H(y,z).$$

Proof. First assume that y is a vertex on the path $\overline{xz}$. Let y' be the vertex adjacent to y and on the path $\overline{yx}$. Let $\mathcal{T}' = \mathcal{S}(\mathcal{T}, \{(y,y')\}, y)$. By Lemma 8 both $g_x|_{V(\mathcal{T}')}$ and $g_y|_{V(\mathcal{T}')}$ are constant times of $g_y^{\mathcal{T}'}$ if $\mathcal{T}'$ is of hyperbolic type of order p, or both are constant functions on $V(\mathcal{T}')$ if $\mathcal{T}'$ is of parabolic type. Therefore

$$g_x|_{V(\mathcal{T}')} = \frac{g_x(y)}{g_y(y)} g_y|_{V(\mathcal{T}')}.$$

Especially

$$g_x(z) = \frac{g_x(y)}{g_y(y)} g_y(z).$$

Similarly we have

$$g_z(x) = \frac{g_z(y)}{g_y(y)} g_y(x).$$

Hence the result follows in this case.

Next we consider the general case. Let w be the intersection vertex among x, y and z, i.e., the vertex which is simultaneously on the three paths $\overline{xy}$, $\overline{yz}$ and $\overline{zx}$ (see Figure 1). Then the first part implies that

$$H(x,z) = H(x,w)H(w,z),$$
$$H(x,y) = H(x,w)H(w,y),$$
$$H(y,z) = H(y,w)H(w,z).$$

Since $H(w,y)$ is the reciprocal of $H(y,w)$, we have the result in the general case. □

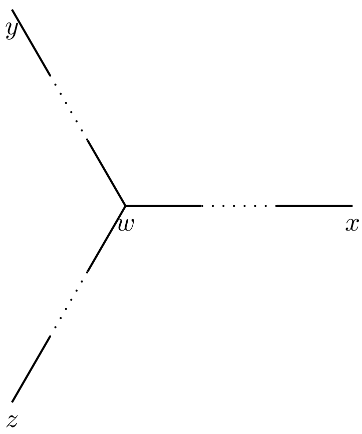

FIGURE 1. The intersection vertex

Let $\mathcal{T} = (V, E, r)$ be a tree of hyperbolic type of order p. Let $(y, z) \in E$ and $\mathcal{S} = \mathcal{S}(\mathcal{T}, \{(y, z)\}, z)$. If $\mathcal{S}$ is of parabolic type, then we call $\mathcal{S}$ a parabolic end of $\mathcal{T}$.

Lemma 10. *Let $\mathcal{T} = (V, E, r)$ be a tree of hyperbolic type of order p. Let $\{\mathcal{S}_j\}_j$ be the set of maximal parabolic ends of $\mathcal{T}$. Let $\mathcal{T}' = (V', E', r')$ be the subtree which is obtained by removing $\bigcup_j \mathcal{S}_j$ from $\mathcal{T}$. Then*

$$M(\mathcal{T}) = M(\mathcal{T}').$$

Proof. For each j we can take an edge $(y_j, z_j) \in E$ such that $\mathcal{S}_j = \mathcal{S}(\mathcal{T}, \{(y_j, z_j)\}, z_j)$. Let $x \in V$. We denote the p-Green function with respect to $\mathcal{T}'$ by $g_x^{\mathcal{T}'}$ Then it is easy to verify that the p-Green function g_x is represented as

$$\begin{aligned}
g_x(y) &= g_x^{\mathcal{T}'}(y) && \text{if } x \in V' \text{ and } y \in V', \\
g_x(y) &= g_x^{\mathcal{T}'}(y_j) && \text{if } x \in V' \text{ and } y \in V(\mathcal{S}_j), \\
g_x(y) &= g_{y_j}^{\mathcal{T}'}(y) && \text{if } x \in V(\mathcal{S}_j) \text{ and } y \in V', \\
g_x(y) &= g_{y_j}^{\mathcal{T}'}(y_i) && \text{if } x \in V(\mathcal{S}_j) \text{ and } y \in V(\mathcal{S}_i) \text{ with } i \neq j, \\
g_x(y) &= g_w(w) && \text{if } x \in V(\mathcal{S}_j) \text{ and } y \in V(\mathcal{S}_j),
\end{aligned}$$

where w is the intersection vertex among x, y and z_j. Therefore

$$\begin{aligned}
H(x,y) &= H^{\mathcal{T}'}(x,y) && \text{if } x \in V' \text{ and } y \in V', \\
H(x,y) &= H^{\mathcal{T}'}(x,y_j) && \text{if } x \in V' \text{ and } y \in V(\mathcal{S}_j), \\
H(x,y) &= H^{\mathcal{T}'}(y_j,y) && \text{if } x \in V(\mathcal{S}_j) \text{ and } y \in V', \\
H(x,y) &= H^{\mathcal{T}'}(y_j,y_i) && \text{if } x \in V(\mathcal{S}_j) \text{ and } y \in V(\mathcal{S}_i) \text{ with } i \neq j, \\
H(x,y) &= 1 && \text{if } x \in V(\mathcal{S}_j) \text{ and } y \in V(\mathcal{S}_j).
\end{aligned}$$

This means $M(\mathcal{T}) = M(\mathcal{T}')$. ∎

Proof of Theorem 6. Let $\mathcal{T}_1 = (V_1, E_1, r_1)$ and $\mathcal{T}_2 = (V_2, E_2, r_2)$. By means of Lemma 10 we may assume that both $\mathcal{T}_1$ and $\mathcal{T}_2$ are of hyperbolic type of order p.

First we observe that the function $g_{a_1}|_{V_1}$ is p-harmonic in $V_1 \setminus \{a_1\}$ and

$$\Delta_p^{\mathcal{T}_1}(g_{a_1}|_{V_1})(a_1) = \Delta_p g_{a_1}(a_1) - \varphi_p(\nabla g_{a_1}(a_1, a_2)) = -1 + \varphi_p(\nabla g_{a_1}(a_2, a_1)).$$

Let q be the number with $1/p + 1/q = 1$. Then we have that φ_q is the inverse function of φ_p. Therefore

$$\frac{1}{c_0} g_{a_1}|_{V_1} = g_{a_1}^{\mathcal{T}_1},$$

where $c_0 = \varphi_q(1 - \varphi_p(\nabla g_{a_1}(a_2, a_1)))$. Especially

$$g_{a_1}(x) = c_0 g_{a_1}^{\mathcal{T}_1}(x) \qquad \text{for } x \in V_1. \tag{4}$$

Note that c_0 is independent of x.

Let $\{b_1, \ldots, b_{d-1}\}$ be the neighbors of a_1 in $\mathcal{T}_1$, where $d = \deg(a_1)$. Let $\mathcal{T}_{1i} = \mathcal{S}(\mathcal{T}_1, \{(a_1, b_i)\}, b_i)$. Let $x \in V_1 \setminus \{a_1\}$ and $\{a_1 = x_0, x_1, \ldots, x_{l-1}, x_l = x\}$ be the path $\overline{a_1 x}$. Then $x_1 = b_{i_0}$ for some i_0. For $i \neq i_0$ the function $g_{x_1}|_{V(\mathcal{T}_{1i})}$ is p-harmonic in $V(\mathcal{T}_{1i}) \setminus \{b_i\}$ and

$$\Delta_p^{\mathcal{T}_{1i}}(g_{x_1}|_{V(\mathcal{T}_{1i})})(b_i) = \Delta_p g_{x_1}(b_i) - \varphi_p(\nabla g_{x_1}(b_i, a_1)) = -\varphi_p(\nabla g_{x_1}(b_i, a_1)).$$

Therefore

$$\frac{1}{\nabla g_{x_1}(b_i, a_1)} g_{x_1}|_{V(\mathcal{T}_{1i})} = g_{b_i}^{\mathcal{T}_{1i}},$$

and hence

$$\begin{aligned} g_{x_1}(a_1) &= g_{x_1}(b_i) + r(a_1, b_i)\nabla g_{x_1}(b_i, a_1) \\ &= \nabla g_{x_1}(b_i, a_1)(g_{b_i}^{\mathcal{T}_{1i}}(b_i) + r(a_1, b_i)). \end{aligned}$$

Similarly we have

$$\begin{aligned} g_x(a_1) &= \nabla g_x(b_i, a_1)(g_{b_i}^{\mathcal{T}_{1i}}(b_i) + r(a_1, b_i)), \\ g_{x_1}^{\mathcal{T}_1}(a_1) &= \nabla g_{x_1}^{\mathcal{T}_1}(b_i, a_1)(g_{b_i}^{\mathcal{T}_{1i}}(b_i) + r(a_1, b_i)), \\ g_x^{\mathcal{T}_1}(a_1) &= \nabla g_x^{\mathcal{T}_1}(b_i, a_1)(g_{b_i}^{\mathcal{T}_{1i}}(b_i) + r(a_1, b_i)). \end{aligned}$$

Therefore

$$\frac{g_{x_1}(a_1)}{\nabla g_{x_1}(b_i, a_1)} = \frac{g_x(a_1)}{\nabla g_x(b_i, a_1)} = \frac{g_{x_1}^{\mathcal{T}_1}(a_1)}{\nabla g_{x_1}^{\mathcal{T}_1}(b_i, a_1)} = \frac{g_x^{\mathcal{T}_1}(a_1)}{\nabla g_x^{\mathcal{T}_1}(b_i, a_1)}. \tag{5}$$

Also we have

$$\frac{1}{\nabla g_{x_1}(a_2, a_1)} g_{x_1}|_{V_2} = g_{a_2}^{\mathcal{T}_2},$$

and hence

$$g_{x_1}(a_1) = \nabla g_{x_1}(a_2, a_1)(g_{a_2}^{\mathcal{T}_2}(a_2) + r(a_1, a_2)).$$

Similarly we have

$$g_x(a_1) = \nabla g_x(a_2, a_1)(g_{a_2}^{\mathcal{T}_2}(a_2) + r(a_1, a_2)).$$

Therefore

$$\frac{g_{x_1}(a_1)}{\nabla g_{x_1}(a_2, a_1)} = \frac{g_x(a_1)}{\nabla g_x(a_2, a_1)}. \tag{6}$$

Since $\Delta_p g_x(a_1) = 0$ and $\Delta_p g_{x_1}(a_1) = 0$, we have

$$\varphi_p(\nabla g_x(a_1, x_1)) = \sum_{i \neq i_0} \varphi_p(\nabla g_x(b_i, a_1)) + \varphi_p(\nabla g_x(a_2, a_1)),$$
$$\varphi_p(\nabla g_{x_1}(a_1, x_1)) = \sum_{i \neq i_0} \varphi_p(\nabla g_{x_1}(b_i, a_1)) + \varphi_p(\nabla g_{x_1}(a_2, a_1)).$$

Using (5) and (6), we have

$$\nabla g_x(b_i, a_1) = \frac{g_x(a_1)}{g_{x_1}(a_1)} \nabla g_{x_1}(b_i, a_1),$$
$$\nabla g_x(a_2, a_1) = \frac{g_x(a_1)}{g_{x_1}(a_1)} \nabla g_{x_1}(a_2, a_1),$$

and hence

$$\frac{\nabla g_x(a_1, x_1)}{\nabla g_{x_1}(a_1, x_1)} = \frac{g_x(a_1)}{g_{x_1}(a_1)}. \tag{7}$$

Similarly, since $\Delta_p^{\mathcal{T}_1} g_x^{\mathcal{T}_1}(a_1) = 0$ and $\Delta_p^{\mathcal{T}_1} g_{x_1}^{\mathcal{T}_1}(a_1) = 0$, we have

$$\varphi_p(\nabla g_x^{\mathcal{T}_1}(a_1, x_1)) = \sum_{i \neq i_0} \varphi_p(\nabla g_x^{\mathcal{T}_1}(b_i, a_1)),$$
$$\varphi_p(\nabla g_{x_1}^{\mathcal{T}_1}(a_1, x_1)) = \sum_{i \neq i_0} \varphi_p(\nabla g_{x_1}^{\mathcal{T}_1}(b_i, a_1)).$$

Using (5), we have

$$\frac{\nabla g_x^{\mathcal{T}_1}(a_1, x_1)}{\nabla g_{x_1}^{\mathcal{T}_1}(a_1, x_1)} = \frac{g_x^{\mathcal{T}_1}(a_1)}{g_{x_1}^{\mathcal{T}_1}(a_1)}. \tag{8}$$

Combining (7) and (8), since $x_1 = b_{i_0}$, we have

$$\frac{\nabla g_x(a_1, x_1)}{\nabla g_x^{\mathcal{T}_1}(a_1, x_1)} = c_{i_0} \frac{g_x(a_1)}{g_x^{\mathcal{T}_1}(a_1)}, \tag{9}$$

where

$$c_j = \frac{g_{b_j}^{\mathcal{T}_1}(a_1)}{g_{b_j}(a_1)} \frac{\nabla g_{b_j}(a_1, b_j)}{\nabla g_{b_j}^{\mathcal{T}_1}(a_1, b_j)}.$$

If we put $c = \max(c_1, \dots, c_{d-1}, c_1^{-1}, \dots, c_{d-1}^{-1})$, then c is independent of x. Now we obtain by (9)

$$\begin{aligned} g_x(x_1) &= g_x(a_1) + r(a_1, x_1) \nabla g_x(a_1, x_1) \\ &= g_x(a_1) + r(a_1, x_1) c_{i_0} \frac{g_x(a_1)}{g_x^{\mathcal{T}_1}(a_1)} \nabla g_x^{\mathcal{T}_1}(a_1, x_1) \end{aligned}$$

$$\leq c\frac{g_x(a_1)}{g_x^{\mathcal{T}_1}(a_1)}(g_x^{\mathcal{T}_1}(a_1)+r(a_1,x_1)\nabla g_x^{\mathcal{T}_1}(a_1,x_1)) = c\frac{g_x(a_1)}{g_x^{\mathcal{T}_1}(a_1)}g_x^{\mathcal{T}_1}(x_1).$$

Therefore

$$\frac{g_x(x_1)}{g_x^{\mathcal{T}_1}(x_1)} \leq c\frac{g_x(a_1)}{g_x^{\mathcal{T}_1}(a_1)}. \tag{10}$$

Similarly we have

$$\frac{g_x(x_1)}{g_x^{\mathcal{T}_1}(x_1)} \geq c^{-1}\frac{g_x(a_1)}{g_x^{\mathcal{T}_1}(a_1)}. \tag{11}$$

Next let $y \in V_1$ with $y \sim x_1$ and $y \neq a_1, x_2$. Let $\mathcal{T}_{1y} = \mathcal{S}(\mathcal{T}_1, \{(x_1, y)\}, y)$. Then we have

$$\frac{1}{\nabla g_x(y,x_1)} g_x|_{V(\mathcal{T}_{1y})} = g_y^{\mathcal{T}_{1y}},$$

and therefore

$$g_x(x_1) = \nabla g_x(y,x_1)(g_y^{\mathcal{T}_{1y}}(y)+r(y,x_1)).$$

Similarly we have

$$g_x^{\mathcal{T}_1}(x_1) = \nabla g_x^{\mathcal{T}_1}(y,x_1)(g_y^{\mathcal{T}_{1y}}(y)+r(y,x_1)).$$

Combined with (10) and (11), we have

$$c^{-1}\frac{g_x(a_1)}{g_x^{\mathcal{T}_1}(a_1)} \leq \frac{\nabla g_x(y,x_1)}{\nabla g_x^{\mathcal{T}_1}(y,x_1)} \leq c\frac{g_x(a_1)}{g_x^{\mathcal{T}_1}(a_1)}. \tag{12}$$

Since $\Delta_p g_x(x_1) = 0$ and $\Delta_p^{\mathcal{T}_1} g_x^{\mathcal{T}_1}(x_1) = 0$, we have

$$\varphi_p(\nabla g_x(x_1,x_2)) = \sum_{\substack{y\sim x_1\\ y\neq a_1,x_2}} \varphi_p(\nabla g_x(y,x_1)) + \varphi_p(\nabla g_x(a_1,x_1)),$$

$$\varphi_p(\nabla g_x^{\mathcal{T}_1}(x_1,x_2)) = \sum_{\substack{y\sim x_1\\ y\neq a_1,x_2}} \varphi_p(\nabla g_x^{\mathcal{T}_1}(y,x_1)) + \varphi_p(\nabla g_x^{\mathcal{T}_1}(a_1,x_1)).$$

Formulas (9) and (12) imply that

$$c^{-1}\frac{g_x(a_1)}{g_x^{\mathcal{T}_1}(a_1)} \leq \frac{\nabla g_x(x_1,x_2)}{\nabla g_x^{\mathcal{T}_1}(x_1,x_2)} \leq c\frac{g_x(a_1)}{g_x^{\mathcal{T}_1}(a_1)}.$$

Using (10), we have

$$\begin{aligned} g_x(x_2) &= g_x(x_1) + r(x_1,x_2)\nabla g_x(x_1,x_2) \\ &\leq c\frac{g_x(a_1)}{g_x^{\mathcal{T}_1}(a_1)}(g_x^{\mathcal{T}_1}(x_1)+r(x_1,x_2)\nabla g_x^{\mathcal{T}_1}(x_1,x_2)) \\ &= c\frac{g_x(a_1)}{g_x^{\mathcal{T}_1}(a_1)}g_x^{\mathcal{T}_1}(x_2), \end{aligned}$$

and therefore

$$\frac{g_x(x_2)}{g_x^{\mathcal{T}_1}(x_2)} \leq c\frac{g_x(a_1)}{g_x^{\mathcal{T}_1}(a_1)}.$$

Also we have by (11)

$$\frac{g_x(x_2)}{g_x^{\mathcal{T}_1}(x_2)} \geq c^{-1}\frac{g_x(a_1)}{g_x^{\mathcal{T}_1}(a_1)}.$$

We repeat these arguments and obtain

$$c^{-1}\frac{g_x(a_1)}{g_x^{\mathcal{T}_1}(a_1)} \leq \frac{\nabla g_x(x_{l-1},x)}{\nabla g_x^{\mathcal{T}_1}(x_{l-1},x)} \leq c\frac{g_x(a_1)}{g_x^{\mathcal{T}_1}(a_1)}, \tag{13}$$

$$c^{-1}\frac{g_x(a_1)}{g_x^{\mathcal{T}_1}(a_1)} \leq \frac{g_x(x)}{g_x^{\mathcal{T}_1}(x)} \leq c\frac{g_x(a_1)}{g_x^{\mathcal{T}_1}(a_1)},$$

$$c^{-1}\frac{g_x(a_1)}{g_x^{\mathcal{T}_1}(a_1)} \leq \frac{\nabla g_x(z,x)}{\nabla g_x^{\mathcal{T}_1}(z,x)} \leq c\frac{g_x(a_1)}{g_x^{\mathcal{T}_1}(a_1)} \qquad \text{for } z \sim x \text{ with } z \neq x_{l-1}. \tag{14}$$

We have

$$\sum_{\substack{z\sim x\\ z\neq x_{l-1}}} \varphi_p(\nabla g_x(z,x)) + \varphi_p(\nabla g_x(x_{l-1},x)) = -\Delta_p g_x(x) = 1,$$

$$\sum_{\substack{z\sim x\\ z\neq x_{l-1}}} \varphi_p(\nabla g_x^{\mathcal{T}_1}(z,x)) + \varphi_p(\nabla g_x^{\mathcal{T}_1}(x_{l-1},x)) = -\Delta_p^{\mathcal{T}_1} g_x^{\mathcal{T}_1}(x) = 1.$$

Equations (13) and (14) imply that

$$\varphi_p\Big(c^{-1}\frac{g_x(a_1)}{g_x^{\mathcal{T}_1}(a_1)}\Big) \leq 1 \leq \varphi_p\Big(c\frac{g_x(a_1)}{g_x^{\mathcal{T}_1}(a_1)}\Big).$$

This means that

$$c^{-1} \leq \frac{g_x(a_1)}{g_x^{\mathcal{T}_1}(a_1)} \leq c.$$

Hence, combining with (4), we have

$$c^{-1}c_0^{-1}H^{\mathcal{T}_1}(x,a_1) \leq H(x,a_1) \leq cc_0^{-1}H^{\mathcal{T}_1}(x,a_1).$$

We obtain similarly that there are constants c' and c_0' such that

$$c'^{-1}c_0'^{-1}H^{\mathcal{T}_2}(x,a_2) \leq H(x,a_2) \leq c'c_0'^{-1}H^{\mathcal{T}_2}(x,a_2) \qquad \text{for } x \in V_2.$$

Therefore Lemma 9 implies that, if $x,y \in V_1$, then

$$\begin{aligned} H(x,y) &= H(x,a_1)H(a_1,y) \leq cc_0^{-1}H^{\mathcal{T}_1}(x,a_1)cc_0H^{\mathcal{T}_1}(a_1,y) \\ &= c^2H^{\mathcal{T}_1}(x,y), \end{aligned}$$

and similarly

$$H(x,y) \geq c^{-2}H^{\mathcal{T}_1}(x,y);$$

if $x \in V_1$ and $y \in V_2$, then

$$\begin{aligned} H(x,y) &= H(x,a_1)H(a_1,a_2)H(a_2,y) \\ &\leq cc_0^{-1}H^{\mathcal{T}_1}(x,a_1) \times H(a_1,a_2) \times c'c_0'H^{\mathcal{T}_2}(a_2,y), \end{aligned}$$

and similarly

$$H(x,y) \geq c^{-1}c'^{-1}c_0^{-1}c_0'H^{\mathcal{T}_1}(x,a_1)H(a_1,a_2)H^{\mathcal{T}_2}(a_2,y).$$

These imply the result. □

4. Proof of Theorem 7

Lemma 11. *Let $\mathcal{T} = (V, E, r)$ be a tree. Let $a, x \in V$ and $\{a = x_0, x_1, \dots, x_{l-1}, x_l = x\}$ the path $\overline{ax}$. Let $r_j = r(x_{j-1}, x_j)$ and*

$$\mathcal{T}_j^- = \mathcal{S}(\mathcal{T}, \{(x_j, x_{j+1})\}, x_j) \qquad \text{for } j = 0, \dots, l-1,$$
$$\mathcal{T}_j^+ = \mathcal{S}(\mathcal{T}, \{(x_{j-1}, x_j)\}, x_j) \qquad \text{for } j = 1, \dots, l.$$

Suppose that $\mathcal{T}_j^-$ and $\mathcal{T}_j^+$ are of hyperbolic type of order p. Let

$$\lambda_j = g_{x_j}^{\mathcal{T}_j^-}(x_j), \qquad \rho_j = g_{x_j}^{\mathcal{T}_j^+}(x_j).$$

Then

$$H(a,x)^{p-1} = \prod_{j=1}^{l} \frac{(\lambda_{j-1} + r_j)^{p-1} + \rho_j^{p-1}}{\lambda_{j-1}^{p-1} + (\rho_j + r_j)^{p-1}}.$$

Proof. Since

$$g_a|_{V(\mathcal{T}_j^+)} = \frac{g_a(x_j)}{g_{x_j}^{\mathcal{T}_j^+}(x_j)} g_{x_j}^{\mathcal{T}_j^+},$$

we have

$$\nabla g_a(x_j, y) = \frac{g_a(x_j)}{\rho_j} \nabla g_{x_j}^{\mathcal{T}_j^+}(x_j, y) \qquad \text{for } y \in V(\mathcal{T}_j^+) \text{ with } y \sim x_j.$$

Since $\Delta_p g_a(x_j) = 0$ and $\Delta_p^{\mathcal{T}_j^+} g_{x_j}^{\mathcal{T}_j^+}(x_j) = -1$, we have

$$\varphi_p(\nabla g_a(x_j, x_{j-1})) + \sum_{\substack{y \in V(\mathcal{T}_j^+) \\ y \sim x_j}} \varphi_p(\nabla g_a(x_j, y)) = 0,$$

$$\sum_{\substack{y \in V(\mathcal{T}_j^+) \\ y \sim x_j}} \varphi_p(\nabla g_{x_j}^{\mathcal{T}_j^+}(x_j, y)) = -1.$$

Therefore

$$\varphi_p(\nabla g_a(x_j, x_{j-1})) - \varphi_p(\frac{g_a(x_j)}{\rho_j}) = 0,$$

or

$$\frac{g_a(x_{j-1}) - g_a(x_j)}{r_j} = \frac{g_a(x_j)}{\rho_j},$$

and hence

$$g_a(x_j) = \frac{\rho_j}{\rho_j + r_j} g_a(x_{j-1}).$$

Therefore

$$g_a(x) = (\prod_{j=1}^{l} \frac{\rho_j}{\rho_j + r_j}) g_a(a).$$

Similarly we have

$$g_x(a) = (\prod_{j=1}^{l} \frac{\lambda_{j-1}}{\lambda_{j-1} + r_j}) g_x(x).$$

Therefore

$$H(a,x) = \frac{g_a(a)}{\lambda_0(\rho_1 + r_1)} (\prod_{j=1}^{l-1} \frac{\rho_j(\lambda_{j-1} + r_j)}{\lambda_j(\rho_{j+1} + r_{j+1})}) \frac{(\lambda_{l-1} + r_l)\rho_l}{g_x(x)}. \tag{15}$$

Since

$$g_{x_j}|_{V(\mathcal{T}_{j-1}^-)} = \frac{g_{x_j}(x_{j-1})}{g_{x_{j-1}}^{\mathcal{T}_{j-1}^-}(x_{j-1})} g_{x_{j-1}}^{\mathcal{T}_{j-1}^-},$$

we have

$$\nabla g_{x_j}(x_{j-1}, y) = \frac{g_{x_j}(x_{j-1})}{\lambda_{j-1}} \nabla g_{x_{j-1}}^{\mathcal{T}_{j-1}^-}(x_{j-1}, y)$$

for $y \in V(\mathcal{T}_{j-1}^-)$ with $y \sim x_{j-1}$. Therefore $\Delta_p g_{x_j}(x_{j-1}) = 0$ and $\Delta_p^{\mathcal{T}_{j-1}^-} g_{x_{j-1}}^{\mathcal{T}_{j-1}^-}(x_{j-1}) = -1$ imply that

$$\varphi_p(\nabla g_{x_j}(x_{j-1}, x_j)) - \varphi_p(\frac{g_{x_j}(x_{j-1})}{\lambda_{j-1}}) = 0,$$

that is

$$g_{x_j}(x_{j-1}) = \frac{\lambda_{j-1}}{\lambda_{j-1} + r_j} g_{x_j}(x_j).$$

Hence

$$\nabla g_{x_j}(x_j, x_{j-1}) = -\frac{1}{\lambda_{j-1} + r_j} g_{x_j}(x_j). \tag{16}$$

Next since

$$g_{x_j}|_{V(\mathcal{T}_j^+)} = \frac{g_{x_j}(x_j)}{g_{x_j}^{\mathcal{T}_j^+}(x_j)} g_{x_j}^{\mathcal{T}_j^+},$$

we have similarly

$$\nabla g_{x_j}(x_j, y) = \frac{g_{x_j}(x_j)}{\rho_j} \nabla g_{x_j}^{\mathcal{T}_j^+}(x_j, y) \qquad \text{for } y \in V(\mathcal{T}_j^+) \text{ with } y \sim x_j. \tag{17}$$

Since $\Delta_p g_{x_j}(x_j) = -1$ and $\Delta_p^{\mathcal{T}_j^+} g_{x_j}^{\mathcal{T}_j^+}(x_j) = -1$, we have by (16) and (17)

$$-\varphi_p(\frac{g_{x_j}(x_j)}{\lambda_{j-1} + r_j}) - \varphi_p(\frac{g_{x_j}(x_j)}{\rho_j}) = -1.$$

Therefore

$$\frac{1}{g_{x_j}(x_j)^{p-1}} = \frac{1}{(\lambda_{j-1} + r_j)^{p-1}} + \frac{1}{\rho_j^{p-1}} \qquad \text{for } j = 1, \ldots, l.$$

Similarly we have

$$\frac{1}{g_{x_j}(x_j)^{p-1}} = \frac{1}{\lambda_j^{p-1}} + \frac{1}{(\rho_{j+1} + r_{j+1})^{p-1}} \qquad \text{for } j = 0, \ldots, l-1.$$

Hence

$$(\frac{g_a(a)}{\lambda_0(\rho_1+r_1)})^{p-1} = \frac{1}{\lambda_0^{p-1}+(\rho_1+r_1)^{p-1}},$$

$$(\frac{\rho_j(\lambda_{j-1}+r_j)}{\lambda_j(\rho_{j+1}+r_{j+1})})^{p-1} = \frac{\rho_j^{p-1}+(\lambda_{j-1}+r_j)^{p-1}}{\lambda_j^{p-1}+(\rho_{j+1}+r_{j+1})^{p-1}},$$

$$(\frac{(\lambda_{l-1}+r_l)\rho_l}{g_x(x)})^{p-1} = (\lambda_{l-1}+r_l)^{p-1}+\rho_l^{p-1}.$$

Combining these and (15) we have the result. □

Lemma 12. *Let* $V = \{x_j\}_{j=0}^{\infty}$, $E = \{(x_j, x_{j+1})\}_{j=0}^{\infty}$ *and* r *a resistance.*

(i) *If* $\sum_{j=0}^{\infty} r(x_j, x_{j+1}) = \infty$, *then* (V, E, r) *is of parabolic type of order* p*;*

(ii) *If* $\sum_{j=0}^{\infty} r(x_j, x_{j+1}) < \infty$, *then* (V, E, r) *is of hyperbolic type of order* p *and has a symmetric* p*-Green function.*

Proof. We shall show only (ii). It is easy to see that the p-Green function g_{x_m} is represented as

$$g_{x_m}(x_l) = \sum_{j=\max(l,m)}^{\infty} r(x_j, x_{j+1}).$$

Therefore $H(x_m, x_l) = 1$. □

Lemma 13. *For* $0 \le s, t \le M$ *we have*

$$2^{-|p-2|} \le \frac{t^{p-1}+(M-t)^{p-1}}{s^{p-1}+(M-s)^{p-1}} \le 2^{|p-2|}.$$

Proof. It is clearly that, if $p < 2$, then we have $M^{p-1} \le t^{p-1} + (M-t)^{p-1} \le 2^{2-p}M^{p-1}$; if $p \ge 2$, then we have $2^{2-p}M^{p-1} \le t^{p-1} + (M-t)^{p-1} \le M^{p-1}$. This leads to the result. □

Lemma 14. *Let* $V = \{x_j\}_{j=-\infty}^{\infty}$, $E = \{(x_j, x_{j+1})\}_{j=-\infty}^{\infty}$ *and* r *a resistance. Let* $S^+ = \sum_{j=0}^{\infty} r(x_j, x_{j+1})$ *and* $S^- = \sum_{j=-\infty}^{-1} r(x_j, x_{j+1})$.

(i) *If both* S^+ *and* S^- *diverge, then* (V, E, r) *is of parabolic type of order* p*;*

(ii) *If one of* S^+ *and* S^- *diverges and the other converges, then* (V, E, r) *is of hyperbolic type of order* p *and has a symmetric* p*-Green function;*

(iii) *If both* S^+ *and* S^- *converge, then* (V, E, r) *is of hyperbolic type of order* p *and has a quasi-symmetric* p*-Green function.*

Proof. Lemma 10 reduces (ii) to Lemma 12 (ii). We shall show (iii). We use the same notation as in Lemma 11. Then we easily have $\rho_m = \sum_{j=m}^{\infty} r(x_j, x_{j+1})$ and $\lambda_m = \sum_{j=-\infty}^{m-1} r(x_j, x_{j+1})$. Also we have that $\lambda_{m-1} + r_m = \lambda_m$ and $\rho_m + r_m = \rho_{m-1}$. Therefore Lemma 11 implies that, if $l > 0$, then

$$H(x_0, x_l)^{p-1} = \frac{\rho_l^{p-1}+\lambda_l^{p-1}}{\rho_0^{p-1}+\lambda_0^{p-1}}.$$

Since $M = \rho_m + \lambda_m$ is independent of m, we have by Lemma 13

$$2^{-|p-2|} \le H(x_0, x_l)^{p-1} \le 2^{|p-2|}.$$

The case $l < 0$ can be treated similarly. □

Proof of Theorem 7 (i). Let $\mathcal{T}$ be a tree as in Theorem 7 (i). Using Lemma 10, we may assume that there are no parabolic ends. Then $\mathcal{T}$ is represented as the union of finitely many trees in Lemmas 12 and 14. Since they have quasi-symmetric p-Green functions for any resistances, $\mathcal{T}$ also has quasi-symmetric p-Green functions by Theorem 6. □

Lemma 15. *Let q be the number with $1/p + 1/q = 1$. Let $\mathcal{T} = (V, E, r)$ be a tree such that $\deg(x) \ge 3$ for each x and*

$$r(x, y) = \frac{\psi(x)\psi(y) - 1}{(\psi(x) + 1)(\psi(y) + 1)},$$

where $\psi(x) = (\deg(x) - 1)^{q-1}$. Then $\mathcal{T}$ has a quasi-symmetric p-Green function.

Proof. Let x, y be distinct vertices and let $\{x = x_0, x_1, \ldots, x_{l-1}, x_l = y\}$ be the path $\overline{xy}$. It is easy to see that the p-Green function g_x is represented as

$$g_x(y) = \frac{1}{\deg(x)^{q-1}} \frac{1}{\psi(x_1) \cdot \cdots \cdot \psi(x_{l-1})} \frac{1}{\psi(y) + 1},$$

$$g_x(x) = \frac{1}{\deg(x)^{q-1}} \frac{\psi(x)}{\psi(x) + 1}.$$

Therefore

$$H(x, y) = \frac{\deg(y)^{q-1}}{\deg(x)^{q-1}} \frac{\psi(x) + 1}{\psi(y) + 1} = \frac{(1 - \deg(x)^{-1})^{q-1} + \deg(x)^{1-q}}{(1 - \deg(y)^{-1})^{q-1} + \deg(y)^{1-q}}.$$

Using Lemma 13 for q instead of p, we have

$$2^{-|q-2|} \le H(x, y) \le 2^{|q-2|}.$$

Hence the result follows. □

Lemma 16. *Let $\mathcal{T} = (V, E, r)$ be a tree of hyperbolic type of order p. Let $x_0, y_0 \in V$ and $\{x_0, x_1, \ldots, x_{l-1}, x_l = y_0\}$ the path $\overline{x_0 y_0}$. Suppose that $\deg(x_0) \ge 3$, $\deg(y_0) \ge 3$ and $\deg(x_j) = 2$ for $j = 1, \ldots, l - 1$. Then*

$$2^{-|q-2|} H(x_0, y_0) \le H(x_0, x_j) \le 2^{|q-2|} H(x_0, y_0)$$

for $j = 1, \ldots, l - 1$, where q is the number with $1/p + 1/q = 1$.

Proof. Let $\mathcal{T}_1 = \mathcal{S}(\mathcal{T}, \{(x_0, x_1)\}, x_0)$ and $\mathcal{T}_2 = \mathcal{S}(\mathcal{T}, \{(x_{l-1}, x_l)\}, y_0)$. We denote

$$u_1(j) = g_{x_0}^{\mathcal{T}_1}(x_0) + \sum_{i=1}^{j} r(x_{i-1}, x_i),$$

$$u_2(j) = g_{y_0}^{\mathcal{T}_2}(y_0) + \sum_{i=j+1}^{l} r(x_{i-1}, x_i).$$

Then it is easy to see that the p-Green function with pole at x_j is given by

$$\begin{aligned} g_{x_j}(x_j) &= \frac{1}{(u_1(j)^{1-p}+u_2(j)^{1-p})^{q-1}}, \\ g_{x_j}(x_k) &= \frac{u_1(k)}{u_1(j)} g_{x_j}(x_j) \qquad \text{if } 0 \le k < j, \\ g_{x_j}(x) &= \frac{g^{\mathcal{T}_1}_{x_0}(x)}{u_1(j)} g_{x_j}(x_j) \qquad \text{if } x \in V(\mathcal{T}_1), \\ g_{x_j}(x_k) &= \frac{u_2(k)}{u_2(j)} g_{x_j}(x_j) \qquad \text{if } j < k \le l, \\ g_{x_j}(x) &= \frac{g^{\mathcal{T}_2}_{y_0}(x)}{u_2(j)} g_{x_j}(x_j) \qquad \text{if } x \in V(\mathcal{T}_2). \end{aligned}$$

Similarly we have

$$\begin{aligned} g_{x_0}(x_0) &= \frac{1}{(u_1(0)^{1-p}+u_2(0)^{1-p})^{q-1}}, \\ g_{x_0}(x_j) &= \frac{u_2(j)}{u_2(0)} g_{x_0}(x_0). \end{aligned}$$

Therefore

$$H(x_0,x_j)^{p-1} = \frac{u_1(j)^{p-1}+u_2(j)^{p-1}}{u_1(0)^{p-1}+u_2(0)^{p-1}}.$$

Similarly we have

$$H(x_0,y_0)^{p-1} = \frac{u_1(l)^{p-1}+u_2(l)^{p-1}}{u_1(0)^{p-1}+u_2(0)^{p-1}}.$$

Hence

$$\Big(\frac{H(x_0,x_j)}{H(x_0,y_0)}\Big)^{p-1} = \frac{u_1(j)^{p-1}+u_2(j)^{p-1}}{u_1(l)^{p-1}+u_2(l)^{p-1}}.$$

Since $u_1(j)+u_2(j)$ is independent of j, Lemma 13 implies that

$$2^{-|p-2|} \le \Big(\frac{H(x_0,x_j)}{H(x_0,y_0)}\Big)^{p-1} \le 2^{|p-2|}.$$

Hence the result follows. □

Proof of Theorem 7 (iia). Let $\mathcal{T}=(V,E,r)$ be a tree which has infinitely many $x \in V$ such that $\deg(x) \ge 3$. If there is a subtree such that either

$$\deg(x_0) \ge 3, \deg(x_1)=2, \dots, \deg(x_l)=2, \deg(x_{l+1})=1$$

for some $l \ge 0$, or

$$\deg(x_0) \ge 3, \deg(x_1)=2, \dots, \deg(x_l)=2, \dots,$$

then we may remove it from $\mathcal{T}$ since we can make it a parabolic end.

Let $\{\{z^i_0, z^i_1, \dots, z^i_{m_i-1}, z^i_{m_i}\}\}_i$ be all of paths such that

$$\deg(z^i_0) \ge 3, \deg(z^i_1)=2, \dots, \deg(z^i_{m_i-1})=2, \deg(z^i_{m_i}) \ge 3$$

for some $m_i \geq 2$. Let $V' = V \setminus \{z^i_j\}_{i,j}$ and $E' = E \cup \{(z^i_0, z^i_{l_i})\}_i \setminus \{(z^i_{j-1}, z^i_j)\}_{i,j}$. Then $\deg^{(V',E')}(x) \geq 3$ for all $x \in V'$. Therefore Lemma 15 shows that there is a resistance r' on E' such that $\mathcal{T}' = (V', E', r')$ has a quasi-symmetric p-Green function. Let r be a resistance on E such that $r = r'$ on $E \cap E'$ and

$$r'(z^i_0, z^i_{m_i}) = \sum_{j=1}^{m_i} r(z^i_{j-1}, z^i_j) \qquad \text{for each } i.$$

Let $x \in V'$. Then it is easy to see that the p-Green function g_x is

$$\begin{aligned} g_x =& g^{\mathcal{T}'}_x \qquad \text{on } V', \\ g_x(z^i_k) =& g^{\mathcal{T}'}_x(z^i_0) + \nabla g^{\mathcal{T}'}_x(z^i_0, z^i_{m_i}) \sum_{j=1}^{k} r(z^i_{j-1}, z^i_j). \end{aligned}$$

Therefore

$$H(x,y) = H^{\mathcal{T}'}(x,y) \quad \text{for } x, y \in V'.$$

Hence Lemma 16 implies that, if $x \in V'$ and $y = z^i_j$, then

$$\begin{aligned} H(x,y) &= H(x, z^i_0) H(z^i_0, y) \leq H^{\mathcal{T}'}(x, z^i_0) \cdot 2^{|q-2|} H^{\mathcal{T}'}(z^i_0, z^i_{m_i}) \\ &= 2^{|q-2|} H^{\mathcal{T}'}(x, z^i_{m_i}); \end{aligned}$$

if $x = z^i_j$ and $y = z^k_l$, then

$$\begin{aligned} H(x,y) &= H(x, z^i_{m_i}) H(z^i_{m_i}, z^k_0) H(z^k_0, y) \\ &\leq 2^{|q-2|} H^{\mathcal{T}'}(z^i_0, z^i_{m_i}) \cdot H^{\mathcal{T}'}(z^i_{m_i}, z^k_0) \cdot 2^{|q-2|} H^{\mathcal{T}'}(z^k_0, z^k_{m_k}) \\ &= 2^{2|q-2|} H^{\mathcal{T}'}(z^i_0, z^k_{m_k}). \end{aligned}$$

Therefore

$$M(\mathcal{T}) \leq 2^{2|q-2|} M(\mathcal{T}').$$

This completes the proof. □

Lemma 17. *Let $\mathcal{T}_0 = (V_0, E_0, r_0)$, $\mathcal{T}_j = (V_j, E_j, r_j)$ and $\mathcal{T}'_j = (V'_j, E'_j, r'_j)$ for $j \geq 1$. Let $a_j \in V_0$, $b_j \in V_j$ and $b'_j \in V'_j$. Let ρ_j be positive numbers. Let $r = r_0$ on E_0, $r = r_j$ on E_j and $r(a_j, b_j) = \rho_j$. Let $r' = r_0$ on E_0, $r' = r'_j$ on E_j and $r'(a_j, b'_j) = \rho_j$. Let*

$$\begin{aligned} \mathcal{T} &= (V_0 \cup \bigcup_j V_j, E_0 \cup \bigcup_j E_j \cup \{(a_j, b_j)\}_j, r), \\ \mathcal{T}' &= (V_0 \cup \bigcup_j V'_j, E_0 \cup \bigcup_j E'_j \cup \{(a_j, b'_j)\}_j, r'). \end{aligned}$$

If $g^{\mathcal{T}_j}_{b_j}(b_j) = g^{\mathcal{T}'_j}_{b'_j}(b'_j)$ for all j, then $g_x(y) = g^{\mathcal{T}'}_x(y)$ for $x, y \in V_0$.

Proof. For $x \in V_0$ we have

$$g_x|_{V_j} = \frac{g_x(b_j)}{g^{\mathcal{T}_j}_{b_j}(b_j)} g^{\mathcal{T}_j}_{b_j}.$$

Therefore

$$\nabla g_x(b_j, y) = \frac{g_x(b_j)}{g_{b_j}^{\mathcal{T}_j}(b_j)} \nabla g_{b_j}^{\mathcal{T}_j}(b_j, y) \qquad \text{for } y \in V_j \text{ with } y \sim b_j.$$

Since $\Delta_p g_x(b_j) = 0$ and $\Delta_p^{\mathcal{T}_j} g_{b_j}^{\mathcal{T}_j}(b_j) = -1$, we have

$$\varphi_p(\nabla g_x(b_j, a_j)) - \varphi_p(\frac{g_x(b_j)}{g_{b_j}^{\mathcal{T}_j}(b_j)}) = 0,$$

and hence

$$\frac{g_x(a_j) - g_x(b_j)}{\rho_j} = \frac{g_x(b_j)}{g_{b_j}^{\mathcal{T}_j}(b_j)},$$

that is

$$g_x(b_j) = \frac{g_{b_j}^{\mathcal{T}_j}(b_j)}{\rho_j + g_{b_j}^{\mathcal{T}_j}(b_j)} g_x(a_j).$$

Therefore it is easy to see that the p-Green function $g_x^{\mathcal{T}'}$ is

$$g_x^{\mathcal{T}'} = g_x \qquad \text{in } V_0,$$
$$g_x^{\mathcal{T}'} = \frac{g_x(b_j)}{g_{b'_j}^{\mathcal{T}'_j}(b'_j)} g_{b'_j}^{\mathcal{T}'_j} \qquad \text{in } V'_j.$$

Hence the result follows. □

Lemma 18. *Let α, β, and γ be the numbers with $0 < \alpha, \beta, \gamma < 1$. Let a, b, and c be the positive numbers such that*

$$a = \left(\frac{(1-\beta)^{p-1} + \beta^{p-1}(1-\alpha)^{p-1}}{(1-\alpha)^{p-1} + \alpha^{p-1}(1-\beta)^{p-1}} \right)^{q-1} \frac{\alpha}{\beta}, \tag{18}$$
$$b = \left(\frac{(1-\beta)^{p-1} + \beta^{p-1}(1-\alpha)^{p-1}}{1 - \alpha^{p-1}\beta^{p-1}} \right)^{q-1} \frac{(1-\gamma)\alpha}{(1-\beta)(1-\alpha)},$$
$$c = \frac{\alpha}{((1-\alpha)^{p-1} + \alpha^{p-1}(1-\beta)^{p-1})^{q-1}},$$

where q is the number with $1/p + 1/q = 1$.

Let $\mathcal{T} = (V, E, r)$ be a tree (as shown in Figure 2) such that

$$V = \{x_l, y_{l,k}\}_{l \in \mathbb{Z}, k \in \mathbb{N}},$$
$$E = \{(x_{l-1}, x_l), (x_l, y_{l,1}), (y_{l,k}, y_{l,k+1})\}_{l \in \mathbb{Z}, k \in \mathbb{N}},$$
$$r(x_{l-1}, x_l) = a^l, \qquad r(x_l, y_{l,1}) = a^l b, \qquad r(y_{l,k}, y_{l,k+1}) = a^l b \gamma^k.$$

If $p \neq 2$ and $\alpha \neq \beta$, then $\mathcal{T}$ does not have a quasi-symmetric p-Green function. More precisely, we have

$$\sup_{l \in \mathbb{Z}} H(x_0, x_l) = \infty.$$

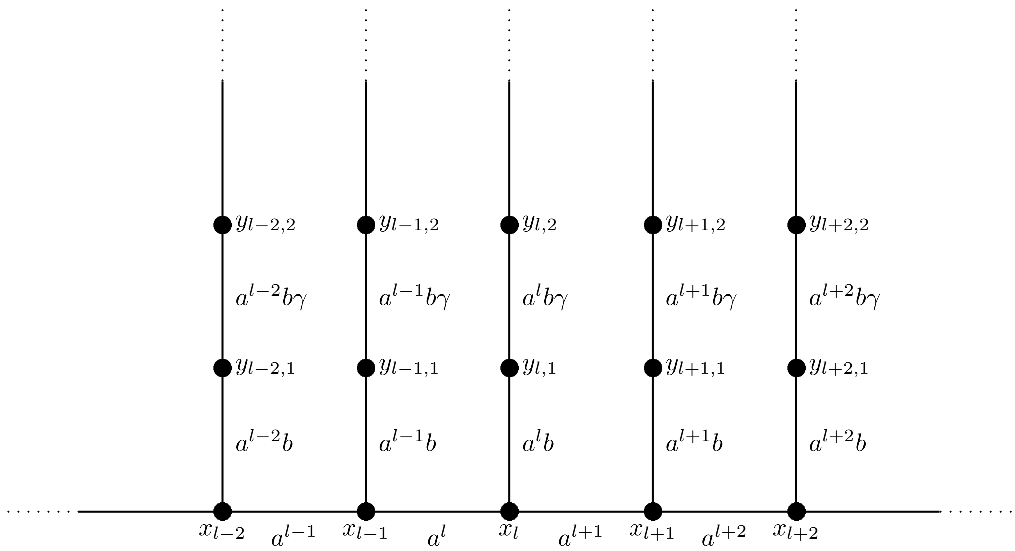

FIGURE 2. The tree of Lemma 18

Proof. First we observe that a, b, and c satisfy

$$\begin{aligned}(\alpha^{-1}-1)^{p-1} - a^{1-p}(1-\alpha)^{p-1} - b^{1-p}(1-\gamma)^{p-1} &= 0,\\ -(1-\beta)^{p-1} + a^{1-p}(\beta^{-1}-1)^{p-1} - b^{1-p}(1-\gamma)^{p-1} &= 0,\\ c^{p-1}(a^{1-p}(1-\alpha)^{p-1} + (1-\beta)^{p-1} + b^{1-p}(1-\gamma)^{p-1}) &= 1.\end{aligned}$$

These imply that the p-Green function g_{x_m} satisfies

$$\begin{aligned} g_{x_m}(x_l) &= ca^m\alpha^{l-m} && \text{if } l \ge m,\\ g_{x_m}(x_l) &= ca^m\beta^{m-l} && \text{if } l < m,\\ g_{x_m}(y_{l,k}) &= \gamma^k g_{x_m}(x_l). \end{aligned}$$

Especially

$$\begin{aligned} g_{x_0}(x_l) &= c\alpha^l, \qquad & g_{x_l}(x_0) &= ca^l\beta^l \qquad && \text{if } l > 0,\\ g_{x_0}(x_l) &= c\beta^{-l}, \qquad & g_{x_l}(x_0) &= ca^l\alpha^{-l} \qquad && \text{if } l < 0. \end{aligned}$$

Therefore, using (18), we have

$$H(x_0, x_l) = \left(\frac{\alpha}{a\beta}\right)^l = \left(\frac{(1-\alpha)^{p-1} + \alpha^{p-1}(1-\beta)^{p-1}}{(1-\beta)^{p-1} + \beta^{p-1}(1-\alpha)^{p-1}}\right)^{l(q-1)}. \tag{19}$$

Suppose that

$$\frac{(1-\alpha)^{p-1} + \alpha^{p-1}(1-\beta)^{p-1}}{(1-\beta)^{p-1} + \beta^{p-1}(1-\alpha)^{p-1}} = 1. \tag{20}$$

Then we have

$$\frac{(1-\alpha)^{p-1}}{1-\alpha^{p-1}} = \frac{(1-\beta)^{p-1}}{1-\beta^{p-1}}.$$

The function $(1-t)^{p-1}/(1-t^{p-1})$ is strictly increasing for $0<t<1$ if $1<p<2$; that is strictly decreasing if $p>2$. Since $p\neq 2$ and $\alpha\neq\beta$, it follows that (20) never holds. Therefore the right hand side of (19) diverges when $l\to\infty$ or $l\to-\infty$. $\square$

Proof of Theorem 7 (iib). Let $\mathcal{T}=(V,E,r)$ be a tree which has infinitely many $x\in V$ such that $\deg(x)\geq 3$. If there is a subtree such that either

$$\deg(x_0)\geq 3, \deg(x_1)=2,\dots,\deg(x_l)=2, \deg(x_{l+1})=1$$

for some $l\geq 0$, or

$$\deg(x_0)\geq 3, \deg(x_1)=2,\dots,\deg(x_l)=2,\dots,$$

then we may remove it from $\mathcal{T}$ since we can make it a parabolic end.

Let $\{\{z_0^i, z_1^i,\dots,z_{m_i-1}^i, z_{m_i}^i\}\}_i$ be all of paths such that

$$\deg(z_0^i)\geq 3, \deg(z_1^i)=2,\dots,\deg(z_{m_i-1}^i)=2, \deg(z_{m_i}^i)\geq 3$$

for some $m_i\geq 2$. Let $V'=V\setminus\{z_j^i\}_{i,j}$ and $E'=E\cup\{(z_0^i,z_{l_i}^i)\}_i\setminus\{(z_{j-1}^i,z_j^i)\}_{i,j}$. Then $\deg^{(V',E')}(x)\geq 3$ for all $x\in V'$. We choose $\{x_l\}_{l\in\mathbb{Z}}\subset V'$ be a two-sided infinite path, i.e., $\cdots\sim x_{-2}\sim x_{-1}\sim x_0\sim x_1\cdots$.

Let $\mathcal{T}''=(V'',E'',r'')$ be a tree such that

$$V=\{x_l, y_{l,k}\}_{l\in\mathbb{Z},k\in\mathbb{N}},$$
$$E=\{(x_{l-1},x_l),(x_l,y_{l,1}),(y_{l,k},y_{l,k+1})\}_{l\in\mathbb{Z},k\in\mathbb{N}},$$
$$r(x_{l-1},x_l)=a^l,\qquad r(x_l,y_{l,1})=a^lb,\qquad r(y_{l,k},y_{l,k+1})=a^lb\gamma^k,$$

where a, b and γ are as in Lemma 18. Then that lemma shows that

$$\sup_{l\in\mathbb{Z}} H^{\mathcal{T}''}(x_0,x_l)=\infty.$$

Let $\mathcal{S}_l'=\mathcal{S}((V',E'),\{(x_{l-1},x_l),(x_l,x_{l+1})\},x_l)$. We choose a resistance r_l' on $E(\mathcal{S}_l')$ such that

$$g_{x_l}^{\mathcal{S}_l'}(x_l)=g_{x_l}^{\mathcal{S}_l''}(x_l),$$

where $\mathcal{S}_l''=\mathcal{S}(\mathcal{T}'',\{(x_{l-1},x_l),(x_l,x_{l+1})\},x_l)$. Then Lemma 17 shows that there exists a resistance r' on E' such that

$$H^{\mathcal{T}'}(x_0,x_l)=H^{\mathcal{T}''}(x_0,x_l),$$

and therefore

$$\sup_{l\in\mathbb{Z}} H^{\mathcal{T}'}(x_0,x_l)=\infty.$$

Next we choose a resistance r on E such that $r=r'$ on $E\cap E'$ and

$$r'(z_0^i,z_{m_i}^i)=\sum_{j=1}^{m_i} r(z_{j-1}^i,z_j^i)\qquad\text{for each } i.$$

Then a similar argument to Proof of Theorem 7 (iia) implies

$$H^{\mathcal{T}}(x_0,x_l)=H^{\mathcal{T}'}(x_0,x_l),$$

and therefore

$$\sup_{l\in\mathbb{Z}} H^{\mathcal{T}}(x_0,x_l)=\infty.$$

This completes the proof. □

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Hisayasu Kurata: Yonago National College of Technology, Yonago, 683-8502 JAPAN

E-mail address: kurata@yonago-k.ac.jp