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PUPPE EXACT SEQUENCE, UNIFORMITIES AND COMPLETENESS IN SOME FIBREWISE CATEGORIES —— A SURVEY ——

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Abstract. This paper is a summary of the author's doctoral dissertation. We study homotopy theory in MAP and prove that Puppe sequence is exact in the category MAP. We also give an application of Puppe exact sequence. Next, we discuss fibrewise uniformities. We develop the fibrewise covering uniformity theory corresponding the fibrewise (entourage) uniformity theory similar to James' [7] and the fibrewise generalized (resp. semi-) uniformity theory corresponding the generalized (resp. semi-) uniformity theory by Morita ([12]). Last we study, as applications, fibrewise Shanin compactification and characterizations of extendable fibrewise maps.

1. Introduction

The study of General Topology is concerned with the category TOP of topological spaces as objects, and continuous maps as morphisms. The concepts of space and map are equally important and one can even look at a space as a map from this space onto a one-point space and in this manner identify these two concepts. With this in mind, a branch of General Topology which has become known as General Topology of Continuous Maps, or Fibrewise General Topology, was initiated. Fibrewise General Topology is concerned most of all in extending the main notions and results concerning topological spaces to continuous maps.

From this point of view, we study, in the first part, homotopy theory in MAP and we prove that Puppe sequence is exact in MAP (for the definition of MAP, see section 2). We also give an application of Puppe exact sequence.

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A summary of doctoral thesis at Shimane University.

In the second part, we discuss fibrewise uniformities, their generalizations and applications in TOP_B (for the definition of TOP_B, see section 2).

Detailed construction of this paper is as follows: in section 2, we summarize notions and notations and terminologies used in this paper. In section 3, we study homotopy theory in MAP and prove that Puppe sequence is exact in MAP. We also give an application of Puppe exact sequence. From section 4 to 7, we study fibrewise uniformities and their generalizations. In the last section, we discuss extendability of fibrewise maps as applications of fibrewise uniformity theory.

Since this paper is a summary, we will omit all of the proofs.

2. Preliminaries

In this section, we refer to the notations used in the latter sections, further the notions and notations in Fibrewise Topology and Fibrewise Homotopy Theory.

In this paper, we assume that all spaces are topological spaces, all maps are continuous.

We will use the abbreviation *nbd(s)* for *neighborhood(s)*. We also use that for $b \in B$, $\mathcal{N}(b)$ is the set of all nbds of b and $N(b)$ is the set of all *open* nbds of b.

For a topological space X and $A \subset X$, $\mathrm{Cl}A$ or $\mathrm{Cl}_X A$ denote the closure of A in X.

Let (B, τ) be a fixed topological space with a fixed topology τ.

A topological space X with a map $p : X \to B$ is called a *fibrewise space over the base* B, p is called the *projection* and B is called the *base space*.

For each point $b \in B$, the *fibre* over b is the subset $X_b := p^{-1}(b)$ of X. Also for each subset B' of B, we denote $X_{B'} := p^{-1}B'$.

For fibrewise spaces $p : X \to B$ and $q : Y \to B$, a map $f : X \to Y$ is *fibrewise* if $p = q \circ f$.

The category TOP_B is a category which consists of fibrewise spaces are as objects and fibrewise maps as morphisms.

The objects of MAP are continuous maps from any topological space into any topological space. For two objects $p : X \to B$ and $p' : X' \to B'$, a morphism from p into p' is a pair (ϕ, α) of continuous maps $\phi : X \to X', \alpha : B \to B'$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\phi} & X' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow{\alpha} & B' \end{array}$$

is commutative. We note that this situation is a generalization of the category TOP_B since the category TOP_B is isomorphic to the particular case of MAP in which the spaces $B' = B$ and $\alpha = id_B$. We call an object $p : X \to B$ an $\mathbf{M}$-*fibrewise space* and denote (X, p, B). Also, for two $\mathbf{M}$-fibrewise spaces $(X, p, B), (X', p', B')$, we call the morphism (ϕ, α) from p into p' an $\mathbf{M}$-*fibrewise map*, and denote $(\phi, \alpha) : (X, p, B) \to (X', p', B')$.

Furthermore, in this paper we often consider the case that an $\mathbf{M}$-fibrewise space (X, p, B) has a section $s : B \to X$. We call it an $\mathbf{M}$-*fibrewise pointed space* and denote (X, p, B, s). For two $\mathbf{M}$-fibrewise pointed spaces (X, p, B, s), (X', p', B', s'), if an $\mathbf{M}$-fibrewise map $(\phi, \alpha) : (X, p, B) \to (X', p', B')$ satisfies $\phi s = s'\alpha$, we call it an $\mathbf{M}$-*fibrewise pointed map* and denote $(\phi, \alpha) : (X, p, B, s) \to (X', p', B', s')$.

3. Puppe exact sequence and its application in the fibrewise category MAP

We study Puppe exact sequence and its application in the fibrewise category MAP.

After we define $\mathbf{M}$-fibrewise pointed mapping cylinders, $\mathbf{M}$-fibrewise pointed mapping cones, $\mathbf{M}$-fibrewise pointed suspensions, $\mathbf{M}$-fibrewise pointed collapse and so on, we have next theorem. For the detail of definitions and notations, see [8]

Theorem 3.1. *For an* $\mathbf{M}$*-fibrewise pointed map* $(\phi, \alpha) : (X, p, B, s) \to (X', p', B', s')$ *where* α *is a bijection, the following sequence is exact.*

$$\begin{aligned} (X, p, B, s) &\xrightarrow{(\phi,\alpha)} (X', p', B', s') \\ &\xrightarrow{(\phi',\alpha')} \Gamma(\phi, \alpha) \\ &\xrightarrow{(\phi'',\alpha'')} \Sigma(X, p, B, s) \\ &\xrightarrow{(\phi''',\alpha''')} \Sigma(X', p', B', s') \\ &\xrightarrow{\qquad} \cdots \end{aligned}$$

Definition 3.2. An $\mathbf{M}$-fibrewise pointed space (X, p, B, s) is called $\mathbf{M}$-*fibrewise well-pointed* if $(s, id_B) : (B, id_B, B, id_B) \to (X, p, B, s)$ is an $\mathbf{M}$-fibrewise pointed cofibration and $s(B)$ is closed in X.

As an application, we can prove the generalized formula for the suspension of fibrewise product spaces.

Theorem 3.3. *Let* $\mathbf{M}$*-fibrewise pointed spaces* (X_i, p_i, B_i, s_i) $(i = 1, \ldots, n)$ *be* $\mathbf{M}$*-fibrewise well-pointed. Then the next formula holds.*

$$\sum \left\{ \prod_{i=1}^{n} (X_i, p_i, B_i, s_i) \right\} \cong^{\mathbf{M}}_{(\mathbf{P})} \bigvee_N{}^{\mathbf{M}} \sum (\bigwedge_{i \in N}{}^{\mathbf{M}} (X_i, p_i, B_i, s_i)),$$

where N *runs through all nonempty subsets of* $\{1, \ldots, n\}$.

Further, as an application of Theorem 3.3, introducing an intermediate fibrewise category TOP_B^H, we give an another short proof of the original formula in TOP_B, which is the following, using the concepts of TOP_B^H.

Proposition. ([7, Proposition 22.11]) Assume that fibrewise pointed spaces (X_i, p_i, B, s_i) $(i = 1, \ldots, n)$ are fibrewise non-degenerate spaces. Then the next formula holds in TOP_B.

$$\sum{}_B^B (X_1 \times_B \cdots \times_B X_n) \cong_B^B \bigvee_N{}_B \sum{}_B^B \bigwedge_{i \in N}{}_B X_i,$$

where N runs through all nonempty subsets of $\{1, \ldots, n\}$.

4. Fibrewise uniform spaces — entourages vs. coverings

In topological category, classically uniform spaces were investigated by using entourages and coverings ([3, Chapter 8]). Further, by generalizing the covering conditions of uniform spaces, semi-uniform spaces and generalized uniform spaces and their completions were investigated (see [12, Sections 1 and 2]).

On the other hand, in fibrewise category, fibrewise uniform spaces and their completions were investigated by using entourages (see [7]).

Our main theme is to develop these theories ([3, 7, 12]), that is:

(1) Developing the fibrewise covering uniformity theory corresponding the fibrewise (entourage) uniformity theory similar to James' [7].

(2) Developing the fibrewise generalized (resp. semi-) uniformity theory corresponding the generalized (resp. semi-) uniformity theory by Morita ([12]): (1) enables us to develop the fibrewise generalized (resp. semi-) uniformity theory, especially the fibrewise completion theory which is the extended one established in [12].

4.1. **Fibrewise uniform structures in the sense of James.**

We recall the definition of fibrewise uniform structures in [7].

Definition 4.1 (James [7])**.** Let X be a fibrewise set over B. Δ is the diagonal of $X \times X$. A *fibrewise uniform structure* on X is a filter Ω on $X \times X$ satisfying the following three conditions:

(J1) $\Delta \subset D$ for every $D \in \Omega$.

(J2) Let $D \in \Omega$. Then for each $b \in B$ there exist $W \in N(b)$ and $E \in \Omega$ such that $E \cap X_W^2 \subset D^{-1}$.

(J3) Let $D \in \Omega$. Then for each $b \in B$ there exist $W \in N(b)$ and $E \in \Omega$ such that

$$(E \cap X_W^2) \circ (E \cap X_W^2) \subset D.$$

In [7], it is proved that the family

$$\mathcal{N}(x) := \{D(x) \cap X_W | D \in \Omega, W \in N(p(x))\}$$

becomes a nbd system at x for every $x \in X$ and defines a fibrewise topology on X. We call this topology *fibrewise uniform topology*.

4.2. **Fibrewise entourage uniformities.**

Now we define a concept stronger than fibrewise uniform structures (Definition 4.1).

Definition 4.2. Let X be a fibrewise set over B. A *fibrewise entourage uniformity* on X is a filter Ω which satisfies (J1)–(J3) in Definition 4.1 and

(J4) If $E \subset X \times X$ satisfies that for each $b \in B$ there exist $W \in N(b)$ and $D \in \Omega$ such that $D \cap X_W^2 \subset E$, then $E \in \Omega$.

Fibrewise set with a fibrewise entourage uniformity is called *fibrewise entourage uniform space.*

Remark 4.3. Note that $D^{-1} \in \Omega$ for every $D \in \Omega$ in our theory. That is, our definition is symmetric.

For a fibrewise uniform structure Ω, the *fibrewise uniform topology* $\tau(\Omega)$ was defined in Section 4.1. We can similarly define $\tau(\Omega)$ for a fibrewise entourage uniformity.

4.3. **Fibrewise covering uniformities.** Let X be a fibrewise set over B and $W \in \tau$. Let μ_W be a non-empty family of coverings of X_W and $\{\mu_W\}_{W\in\tau}$ the system of $\mu_W, W \in \tau$. We say that $\{\mu_W\}_{W\in\tau}$ is a *system of coverings* of $\{X_W\}_{W\in\tau}$ (for this, we briefly use the notations $\{\mu_W\}$ and $\{X_W\}$). Let $\mathcal{U}$ and $\mathcal{V}$ be families of subsets of a set X. If $\mathcal{V}$ refines $\mathcal{U}$ in the usual sense, we denote $\mathcal{V} < \mathcal{U}$.

Definition 4.4. Let X be a fibrewise set over B, and $\mu = \{\mu_W\}$ be a system of coverings of $\{X_W\}$. We say that the system $\{\mu_W\}$ is a *fibrewise covering uniformity* (and a pair (X, μ) or $(X, \{\mu_W\})$ is a *fibrewise covering uniform space*) if the following conditions are satisfied:

(C1) Let $\mathcal{U}$ be a covering of X_W and for each $b \in W$ there exist $W' \in N(b)$ and $\mathcal{V} \in \mu_{W'}$ such that $W' \subset W$ and $\mathcal{V} < \mathcal{U}$. Then $\mathcal{U} \in \mu_W$.

(C2) For each $\mathcal{U}_i \in \mu_W, i = 1, 2$, there exists $\mathcal{U}_3 \in \mu_W$ such that $\mathcal{U}_3 < \mathcal{U}_i, i = 1, 2$.

(C3) For each $\mathcal{U} \in \mu_W$ and $b \in W$, there exist $W' \in N(b)$ and $\mathcal{V} \in \mu_{W'}$ such that $W' \subset W$ and $\mathcal{V}$ is a star refinement of $\mathcal{U}$.

(C4) For $W' \subset W$, $\mu_{W'} \supset \mu_W|_{X_{W'}}$, where

$$\mu_W|_{X_{W'}} = \{\mathcal{U}|_{X_{W'}} | \mathcal{U} \in \mu_W\} \ \text{ and } \ \mathcal{U}|_{X_{W'}} = \{U \cap X_{W'} | U \in \mathcal{U}\}.$$

For fibrewise covering uniform space (X, μ) we shall define the *fibrewise covering uniform topology* $\tau(\mu)$ as follows:

For every $x \in X$, $p(x) = b$, let $\mathcal{N}_x(\mu)$ be the family of all subsets which contains $\mathrm{st}(x, \mathcal{U})$ for some $\mathcal{U} \in \mu_W$ and $W \in N(b)$.

Then we can prove that $\{\mathcal{N}_x(\mu) | x \in X\}$ satisfies the axiom of nbd system and it defines the fibrewise covering uniform topology $\tau(\mu)$. That is,

Proposition 4.5. *For a fibrewise covering uniform space* (X, μ)*,* $\{\mathcal{N}_x(\mu) | x \in X\}$ *satisfies the axiom of nbd system.*

4.4. **Equivalence of fibrewise entourage uniformities and covering uniformities.** For the fibrewise entourage uniformity Ω on X, we can construct a system $\mu(\Omega) = \{\mu_W(\Omega)\}$ of coverings of $\{X_W\}$ as follows:

Construction 4.6. Let Ω be a fibrewise entourage uniformity on X. Then we shall construct a system $\mu(\Omega) = \{\mu_W(\Omega)\}$ of coverings of $\{X_W\}$ for every $W \in \tau$, as follows: For $D \in \Omega$ and $W \in \tau$, let

$$\mathcal{U}(D, W) := \{D(x) \cap X_W | x \in X_W\}.$$

Then it is easy to see that $\mathcal{U}(D, W)$ is a covering of X_W. Let $\mu_W(\Omega)$ be the family of coverings $\mathcal{U}$ of X_W satisfying that for each $b \in W$ there exist $W' \in N(b)$ and $D \in \Omega$ such that $W' \subset W$ and $\mathcal{U}(D, W') < \mathcal{U}$.

Conversely, for a fibrewise covering uniformity $\mu = \{\mu_W\}$ we shall construct a fibrewise entourage uniformity $\Omega(\mu)$ as follows:

Construction 4.7. Let $\mu = \{\mu_W\}$ be the fibrewise covering uniformity of a fibrewise covering uniform space (X, μ). For $\mathcal{U} \in \mu_W$, let

$$D(\mathcal{U}) := \cup\{U_\alpha \times U_\alpha | U_\alpha \in \mathcal{U}\}.$$

Let $\Omega(\mu)$ be the family of all subsets $D \subset X \times X$ satisfying a following condition:

$\Delta \subset D$, and for every $b \in B$ there exist $W \in N(b)$ and $\mathcal{U} \in \mu_W$ such that $D(\mathcal{U}) \subset D$.

Then we can show the following theorem.

Theorem 4.8. *Let B be a regular space and Ω be a fibrewise entourage uniformity. Then*

$$\Omega = \Omega(\mu(\Omega)).$$

Finally, we have the following proposition.

Proposition 4.9. (1) *For a fibrewise uniform space (X, Ω), it holds that $\tau(\Omega) = \tau(\mu(\Omega))$.*

(2) *Assume that B is regular. For a fibrewise covering uniform space (X, μ), where $\mu = \{\mu_W\}$, it holds that $\tau(\mu) = \tau(\Omega(\mu))$.*

5. Fibrewise generalized uniformities and completions

In the previous section, we considered fibrewise covering uniformities and fibrewise covering uniform spaces. In this section, by weakening the condition (C3) of fibrewise covering uniformity (Definition 4.4), we define fibrewise generalized uniformities (and fibrewise generalized uniform spaces). This concept is the fibrewise version of generalized uniformities in [12].

In this section, unless otherwise stated, we exclusively use that X is a fibrewise set over B and $\mu = \{\mu_W\}$ is a system of coverings of $\{X_W\}$. For undefined definitions, notations and terminology, see [10]

5.1. Fibrewise g-uniformities.

To weaken the condition (C3) of Definition 4.4, we define a following concept.

Let $\{\mu_W\}$ be a system of coverings of $\{X_W\}$. For an open set W of B and $Y \subset X$, let

$$\mathrm{Int}_{\mu_W} Y := \{x \in X_W | \exists W' \in N(p(x)), \exists \mathcal{U} \in \mu_{W'} \text{ such that } W' \subset W, \mathrm{st}(x, \mathcal{U}) \subset Y\}.$$

For a collection $\mathcal{U}$ of subsets of X, let

$$\mathrm{Int}_{\mu_W} \mathcal{U} := \{\mathrm{Int}_{\mu_W} U | U \in \mathcal{U}\}.$$

Note that, since X is not yet a topological space, $\mathrm{Int}_{\mu_W} Y$ is not the interior of Y in X_W.

Definition 5.1. Let $\mu = \{\mu_W\}$ be a system of coverings of $\{X_W\}$. Then $\mu = \{\mu_W\}$ is called a *fibrewise generalized uniformity* (we briefly say *fibrewise g-uniformity*) if it satisfies (C1), (C2) and (C4) of Definition 4.4 and

(FGU) For each $b \in B, W \in N(b)$ and $\mathcal{U} \in \mu_W$, there exist $W' \in N(b)$ and $\mathcal{V} \in \mu_{W'}$ such that $W' \subset W$ and $\mathrm{Int}_{\mu_W}\mathcal{U}$ is a covering of X_W and $\mathcal{V} < \mathrm{Int}_{\mu_W}\mathcal{U}$.

The pair (X, μ) or $(X, \{\mu_W\})$ is called a *fibrewise generalized uniform space* (we briefly say a *fibrewise g-uniform space*).

Proposition 5.2. *A fibrewise covering uniformity is a fibrewise g-uniformity.*

Corollary 5.3. *Let* $\{\mu_W\}$ *be a fibrewise g-uniformity and* $\mathcal{U} \in \mu_W$. *Then* $\mathrm{Int}_{\mu_W}\mathcal{U} \in \mu_W$ *and* $\mathrm{Int}_{\mu_W}\mathcal{U} < \mathcal{U}$.

Definition 5.4. (1) Let $\{\mu_W\}$ be a fibrewise g-uniformity and $\{\mu_W^0\}$ be a system of coverings of $\{X_W\}$ satisfying that $\mu_W^0 \subset \mu_W$ for all $W \in \tau$, and $\mu_{W'}^0 \supset \mu_W^0|_{X_{W'}}$ for every $W' \subset W$.

We say that $\{\mu_W^0\}$ is a *base* for $\{\mu_W\}$ if for each W and $\mathcal{U} \in \mu_W$ there exists $\mathcal{V} \in \mu_W^0$ such that $\mathcal{V} < \mathcal{U}$.

(2) Let $\{\mu_W^0\}$ be a system of coverings of $\{X_W\}$. We say that $\{\mu_W^0\}$ is a *fibrewise g-uniformity base* if $\{\mu_W^0\}$ satisfies (C2) and (C4) of Definition 4.4 and (FGU).

Unless otherwise stated, we use the notation $\{\mu_W^0\}$ for a fibrewise g-uniformity base.

Corollary 5.5. *Let* $\{\mu_W\}$ *be a fibrewise g-uniformity. Then there exists a base* $\{\mu_W^0\}$ *for* $\{\mu_W\}$ *such that every* μ_W *is consisted of open coverings of* X_W.

Considering the fibrewise g-uniformity base consisted of a system of *open* coverings, we can prove a characterisation of fibrewise R_0-ness.

Theorem 5.6. *Let* X *be a fibrewise space over* B.

(1) *If* X *admits a fibrewise g-uniformity compatible with the topology, then* X *is fibrewise* R_0.

(2) *Suppose that* X *is fibrewise* R_0. *For each open set* W *of* B, *let*

$$\mu_W^0 = \{\mathcal{U} \mid \mathcal{U} \textit{ is an open covering of } X_W\}.$$

Then $\{\mu_W^0\}$ *is a fibrewise g-uniformity base on* X *compatible with the topology.*

5.2. **Fibrewise completions of fibrewise g-uniform spaces.** We study fibrewise completeness of fibrewise g-uniform spaces and completions.

Definition 5.7. Let $\mathcal{F}$ be a b-filter base.

We say $\mathcal{F}$ is *Cauchy* if for each $W \in N(b)$ and $\mathcal{U} \in \mu_W$ there exist $F \in \mathcal{F}$ and $U \in \mathcal{U}$ such that $F \subset U$.

Definition 5.8. Let $\mathcal{F}$ be a Cauchy b-filter.

$\mathcal{F}$ is a *weak star b-filter* with respect to $\{\mu_W^0\}$ if for each $F \in \mathcal{F}$ there exist $W \in N(b)$ and $\mathcal{U} \in \mu_W^0$ such that $U \subset F$ for each $U \in \mathcal{U} \cap \mathcal{F}$, that is, $\cup(\mathcal{U} \cap \mathcal{F}) \subset F$.

Definition 5.9. $(X, \{\mu_W\})$ is said to be *fibrewise complete* if every weak star b-filter ($b \in B$) with respect to $\{\mu_W^0\}$ converges.

Definition 5.10. Let $(X, \{\mu_W\})$ and $(Y, \{\nu_W\})$ be fibrewise g-uniform spaces and $X \subset Y$. $(Y, \{\nu_W\})$ is a *fibrewise completion* of $(X, \{\mu_W\})$ if
(1) $(Y, \{\nu_W\})$ is fibrewise complete,
(2) $\{\nu_W|_X\} = \{\mu_W\}$,
(3) $(X, \tau(\{\mu_W\}))$ is dense in $(Y, \tau(\{\nu_W\}))$.

We can construct a fibrewise completion of a fibrewise g-uniform space $(X, \{\mu_W\})$.

Let Θ be the set of all weak star b-filters with respect to $\{\mu_W^0\}$ which do not converge and let $X^* := X \cup \Theta$. For $G \subset X$, we define

$$G^* := G \cup \{\mathcal{F} \in \Theta | G \in \mathcal{F}\}.$$

Let (X, μ) be a fibrewise g-uniform space with the fibrewise topology $\tau(\mu)$. We now define the projection $p^* : X^* \to B$ as follows:

$$p^*(y) = \begin{cases} p(y) & (y \in X) \\ b & (y = \mathcal{F} \in \Theta \text{ and } \mathcal{F} \text{ is a } b\text{-filter}). \end{cases}$$

Then noting $(X^*)_W = (p^*)^{-1}(W)$, the family

$$\{G^* \cap (X^*)_W | G \in \tau(\mu), W \in \tau\}$$

is a base for a topology on X^*. We denote the topology generated by this base by $\tau(\mu)^*$. Then since p^* is continuous, $\tau(\mu)^*$ is a fibrewise topology of X^*.

Let $\{\mu_W^0\}$ be a fibrewise g-uniformity base. For each $\mathcal{U} \in \mu_W^0$, let

$$\mathcal{U}^* := \{U^* \cap (X^*)_W | U \in \mathcal{U}\}.$$

Then $\mathcal{U}^*$ is a covering of $(X^*)_W$.

Put

$$(\mu_W)^* = \{\mathcal{U}^* | \mathcal{U} \in \mu_W\}.$$

Then, we have the following.

Theorem 5.11. *$(X^*, \{(\mu_W)^*\})$ is a fibrewise completion of $(X, \{\mu_W\})$.*

6. Fibrewise extensions of fibrewise spaces and a characterisation of completeness

We study relationships between fibrewise completions of fibrewise g-uniform spaces and fibrewise extensions of fibrewise spaces. Main purpose of this section is to show that the fibrewise completion of a fibrewise g-uniform space is characterized by a fibrewise extension of the fibrewise space and the converse. As an application of the fibrewise completion theory, we give fibrewise Shanin compactifications of fibrewise topological spaces.

6.1. **Fibrewise extensions.**

Lemma 6.1. *For an open set $G \subset X$*

$$G^* = X^* - \mathrm{Cl}(X - G).$$

where Cl *is the closure operator in X^*.*

Lemma 6.2. *Let B be a T_1-space. Then each point of $X^* - X$ is closed.*

Definition 6.3. Let X and Y be fibrewise spaces over B. Y is a *fibrewise extension* of X if X is a dense subspace of Y.

A fibrewise extension Y of X is called a *fibrewise* T_1*-extension* of X if $\{y\}$ is a closed set of Y for each $y \in Y - X$, and *strict* if $\{E_Y(G) | G \text{ is open in } X\}$ is a base for the open sets of Y, where

$$E_Y(G) := Y - \mathrm{Cl}_Y(X - G).$$

It is clear that if G is open then $G \subset E_Y(G)$.

For a collection $\mathcal{U}$ of subsets of X_W, we set

$$E_Y(\mathcal{U}) := \{E_Y(U) \cap Y_W | U \in \mathcal{U}\}.$$

We have next results.

Theorem 6.4. *Let $(X, \{\mu_W\})$ be a fibrewise g-uniform space and B be a T_1-space. Then the fibrewise completion $(X^*, \{(\mu_W)^*\})$ of $(X, \{\mu_W\})$ is characterised as a fibrewise g-uniform space $(Y, \{\nu_W\})$ satisfying following conditions:*

(1) *$(Y, \tau(\{\nu_W\}))$ is a strict fibrewise T_1-extension of $(X, \tau(\{\mu_W\}))$.*

(2) *For each open set W of B and each $\mathcal{U} \in \mu_W^0$, $E_Y(\mathcal{U})$ is an open covering of Y_W and the family*

$$\nu_W^0 := \{E_Y(\mathcal{U}) | \mathcal{U} \in \mu_W^0\},$$

is a base for $\{\nu_W\}$. Here $\{\mu_W^0\}$ is a base for $\{\mu_W\}$ which every μ_W^0 is consisted of open coverings of X_W.

(3) *$(Y, \{\nu_W\})$ is fibrewise complete.*

6.2. **Fibrewise Shanin compactification.** Let X be a fibrewise R_0-space and $\mathcal{G} = \cup \mathcal{G}_W$ be a base for the open sets of X such that $\mathcal{G}$ satisfies following four conditions:

(a) $X_W \in \mathcal{G}_W$ for all open set $W \subset B$.
(b) If $G, H \in \mathcal{G}_W$, then $G \cap H \in \mathcal{G}_W$.
(c) If $W' \subset W$, then $\mathcal{G}_{W'} \supset \{G \cap X_{W'} | G \in \mathcal{G}_W\}$.
(d) If $x \in G$ and $G \in \mathcal{G}_W$, then there exist $G_i \in \mathcal{G}_W, i = 1, \cdots, k$ such that $x \notin G_i$ and $G \cup (\cup G_i) = X_W$.

Let μ_W^0 be the collection of all finite open coverings of X_W consisting of sets of $\mathcal{G}_W$, then $\{\mu_W^0\}$ is a g-uniformity base compatible with the topology, because of Theorem 5.6 and (d) above.

Let $\{\mu_W\}$ be a fibrewise g-uniformity generated by $\{\mu_W^0\}$ and $(X^*, \{(\mu_W)^*\})$ be its fibrewise completion. We can show that X^* is fibrewise compact over B. Then, we have

$$\left[\cup_{i=1}^k G_i\right]^* = \cup_{i=1}^k G_i^*$$

for $G_i \in \mathcal{G}, i = 1, \ldots, k$.

Theorem 6.5. *Let X be a fibrewise R_0-space over a T_1-space B and $\mathcal{G} = \{\mathcal{G}_W | W \in \tau\}$ be a base for the open sets of X satisfying the conditions* (a)$-$(d) *above. Then there exists a fibrewise compact space Y satisfying following properties:*

(1) *Y is a fibrewise T_1-extension of X.*

(2) *$\{E_Y(G) | G \in \mathcal{G}\}$ is a base for the open sets of Y.*

(3) $E_Y(G_1 \cup \cdots \cup G_n) = E_Y(G_1) \cup \cdots \cup E_Y(G_n)$ *holds for any finite elements* $G_1, \cdots, G_n \in \mathcal{G}$.

Moreover, such a space Y *is essentially unique.*

We call this "*fibrewise Shanin compactification*".

7. Fibrewise semi-uniformities

We introduce the concept of fibrewise semi-uniformities and we have the following theorem: "Suppose that B is a regular space. A fibrewise space X admits a fibrewise semi-uniformity compatible with the original topology if and only if X is fibrewise regular".

7.1. **Fibrewise semi-uniformities.** Let $\{\mu_W\}$ be a system of coverings of $\{X_W\}$. For $b \in B$, $W, W' \in N(b)$ with $W' \subset W$, $\mathcal{U} \in \mu_W$ and $\mathcal{V} \in \mu_{W'}$, we define the following:

> $\mathcal{V}$ is a *fibrewise local star refinement* of $\mathcal{U}$ at b if for each $V \in \mathcal{V}$ there exist $\mathcal{W} \in \mu_{W'}$ and $U \in \mathcal{U}$ such that $\mathrm{st}(V, \mathcal{W}) \subset U$.

Definition 7.1. Let $\mu = \{\mu_W\}$ be a system of coverings of $\{X_W\}$. Then $\mu = \{\mu_W\}$ is a *fibrewise semi-uniformity* if it satisfies (C1), (C2) and (C4) of Definition 4.4 and

> **(FSU)** For each $b \in B, W \in N(b)$ and $\mathcal{U} \in \mu_W$, there exist $W' \in N(b)$ and $\mathcal{V} \in \mu_{W'}$ such that $W' \subset W$ and $\mathcal{V}$ is a fibrewise local star refinement of $\mathcal{U}$ at b.

The pair (X, μ) (or $(X, \{\mu_W\})$) is called a *fibrewise semi-uniform space.*

Proposition 7.2. *A fibrewise covering uniformity is fibrewise semi-uniformity and fibrewise semi-uniformity is fibrewise g-uniformity.*

Definition 7.3. (1) Let $\{\mu_W\}$ be a fibrewise semi-uniformity and $\{\mu^0_W\}$ be a system of coverings of $\{X_W\}$ satisfying that $\mu^0_W \subset \mu_W$ for all $W \in \tau$, and $\mu^0_{W'} \supset \mu^0_W|_{X_{W'}}$ for every $W' \subset W$.

We say that $\{\mu^0_W\}$ is a *base* for $\{\mu_W\}$ if for each W and $\mathcal{U} \in \mu_W$ there exists $\mathcal{V} \in \mu^0_W$ such that $\mathcal{V} < \mathcal{U}$.

(2) Let $\{\mu^0_W\}$ be a system of coverings of $\{X_W\}$. We say that $\{\mu^0_W\}$ is a *fibrewise semi-uniformity base* if $\{\mu^0_W\}$ satisfies (C2) and (C4) of Definition 4.4 and (FSU).

Unless otherwise stated, we use the notation $\{\mu^0_W\}$ for a fibrewise semi-uniformity base.

Theorem 7.4. *Let* $\{\mu^0_W\}$ *be a fibrewise semi-uniformity base. Then every Cauchy b-filter contains a weak star b-filter.*

Theorem 7.5. *The fibrewise completion of fibrewise semi-uniform space is also a fibrewise semi-uniform space.*

Finally, we have next theorem.

Theorem 7.6. *Suppose that* B *is regular. Let* X *be a fibrewise space over* B.

(1) *If X admits a fibrewise semi-uniformity compatible with the topology, then X is fibrewise regular.*
(2) *Let X be a fibrewise regular space and for every open set $W \subset B$ let $\{\mu_W^0\}$ be the family of open coverings of $\{X_W\}$. Then $\{\mu_W^0\}$ is a fibrewise semi-uniformity base compatible with the topology.*

8. Characterisations of extendable fibrewise maps

In this section, we study extendability of fibrewise maps from dense subspaces. That is, for a fibrewise space X, $A \subset X$ dense in X and a fibrewise map $f : A \to Y$, when f can be extended to whole space X?

In this section we assume that A is a dense subspace of a fibrewise space X and base space B is regular.

8.1. **Fibrewise extension of fibrewise maps.** Let G be an open set of the subspace A. We define an open set $E_X(G)$ of X with

$$E_X(G) := X - \mathrm{Cl}_X(A - G),$$

where Cl_X is the closure operator in X.

For a collection $\mathcal{G}$ of open subsets of A, put

$$E_X(\mathcal{G}) := \{E_X(G) | G \in \mathcal{G}\}.$$

Theorem 8.1. *Let Y be a fibrewise regular space and $f : A \to Y$ be a fibrewise map. Let $\nu = \{\nu_W\}$ be a fibrewise complete semi-uniformity on Y compatible with the topology of Y and $\nu_0 = \{\nu_W^0\}$ be a base for ν where every ν_W^0 consist of open coverings of Y_W. Let us put*

$$H(\nu_0) := \cup_{b \in B} \left[\cap \{ \cup E_X(f^{-1}(\mathcal{V})) | \mathcal{V} \in \nu_W^0, W \in N(b)\} \right].$$

Then there exists uniquely a fibrewise map $g : H(\nu_0) \to Y$ which is an extension of f. Moreover, if $\mathcal{V}'$ is a local star refinement of $\mathcal{V}$ at b, then

$$E_X(f^{-1}(\mathcal{V}')) \wedge H(\nu_0) < g^{-1}(\mathcal{V}).$$

The next theorem is the key result for extendability.

Theorem 8.2. *Let $f : A \to Y$ be a fibrewise map where Y is a fibrewise regular space. Let $\nu = \{\nu_W\}$ be a fibrewise complete semi-uniformity on Y compatible with the topology, and $\nu_0 = \{\nu_W^0\}$ a subbase for ν such that ν_W^0 consists of open coverings of Y_W for every $W \in \tau$. Let us put*

$$H(\nu_0) := \cup_{b \in B} \left[\cap \{ \cup E_X(f^{-1}(\mathcal{V})) | \mathcal{V} \in \nu_W^0, W \in N(b)\} \right].$$

Then the following hold:
(1) *f is extended to a fibrewise map $g : H(\nu_0) \to Y$.*
(2) *$H(\nu_0)$ is the largest subspace of X which contains A and over which f is extendable.*
(3) *$H(\nu_0) = \{x \in X | f(\mathcal{N}(x) \wedge A)$ converges to a point of $Y_{p(x)}\}$, where $\mathcal{N}(x)$ is the nbd filter of x in X.*

The following theorem is easily proved by Theorem 8.2 and Theorem 8.1.

Theorem 8.3. *Let (Y, ν) be a fibrewise complete semi-uniform fibrewise T_2 space, $f : A \to Y$ a fibrewise map, and $\nu_0 = \{\nu_W^0\}$ a subbase for ν such that ν_W^0 consists of open coverings of Y_W for every $W \in \tau$. Then f is extendable over X if and only if $\cup E_X(f^{-1}(\mathcal{V})) \supset X_b$ for every $b \in B, W \in N(b)$ and $\mathcal{V} \in \nu_W^0$.*

If the range space Y is fibrewise compact and fibrewise T_2, we can deduce more precise result.

Theorem 8.4. *Let Y be a fibrewise compact and fibrewise T_2 space. Then fibrewise map $f : A \to Y$ is extendable over X if and only if $\mathrm{Cl}_X f^{-1}(C) \cap \mathrm{Cl}_X f^{-1}(D) = \emptyset$ for any $W \in \tau$ and closed subsets C and D of Y_W with $C \cap D = \emptyset$.*

We have a dual form of Theorem 8.3.

Theorem 8.5. *Let (Y, ν) be a fibrewise complete semi-uniform fibrewise T_2 space, $f : A \to Y$ a fibrewise map, and $\nu_0 = \{\nu_W^0\}$ a subbase for ν such that ν_W^0 consists of open coverings of Y_W for every $W \in \tau$.*

Then fibrewise map $f : A \to Y$ is extendable over X if and only if for every $b \in B, W \in N(b)$ and $\mathcal{V} \in \nu_W^0$,

$$\left[\cap\{\mathrm{Cl}_X f^{-1}(Y - V) | V \in \mathcal{V}\}\right] \cap X_b = \emptyset.$$

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