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BIHARMONIC-EXTENSION SPACE

IBTESAM BAJUNAID

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Abstract. Defining biharmonic spaces in a general way in the axiomatic potential theory of Brelot, we obtain some extension properties of biharmonic functions defined outside a compact set, and use them to study the removable singularity of bounded biharmonic functions.

1. Introduction

Bounded biharmonic functions defined outside a compact set in $\mathbb{R}^n$ have some distinguishing limiting properties at infinity ([8]) such as (1) if $n \geq 4$, $\lim_{|x|\to\infty} b(x)$ exists; (2) if $n \geq 2$, $\lim_{|x|\to\infty} \Delta b(x) = 0$; and conversely (3) if $n = 2$ or $n \geq 5$ and if h is a harmonic function defined outside a compact set in $\mathbb{R}^n$ and tends to 0 at infinity, there exists a bounded biharmonic function b near infinity such that $\Delta b = h$.

In view of the Almansi representation of biharmonic functions in $\mathbb{R}^n$, it is of some interest therefore to know that given biharmonic function $u(x)$ outside a compact set in $\mathbb{R}^n$ whether there exists a biharmonic function $B(x)$ in the whole space $\mathbb{R}^n$ such that $u(x) - B(x)$ is bounded near infinity. Some aspects of a similar problem in a Riemannian manifold were considered in [7]. In this note we discuss this question in the axiomatic case of Brelot harmonic spaces.

Defining a biharmonic space $(\Omega, H, H^*, \lambda)$ in an expanded way, we prove some necessary and sufficient conditions for the above mentioned biharmonic extension to be valid in Ω. Then we give some sufficient condition for this extension to take place, here we introduce a notion of flux in a harmonic space with potentials > 0. (M. Nakai, as given in [14], defined a notion of flux in a harmonic space without potentials > 0 in the context of a similar extension with harmonic functions in the axiomatic case).

Using this extension, we make some remarks concerning the removable singularity for bounded biharmonic function in a biharmonic space. The last section is about the boundary-value problem (with Wiener boundaries) for biharmonic functions defined in relatively compact domains.

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2. Preliminaries

Let Ω be a connected, locally compact but not compact space. Let H and H^* be two harmonic sheaves defined on Ω satisfying the three axioms of M. Brelot [9]; the constants are assumed to be harmonic in both sheaves, we suppose that the harmonic space (Ω, H^*) has potentials > 0 in Ω.

As for the harmonic space (Ω, H), there may or may not be any potential $>$ 0 in Ω, but we suppose that it satisfies the local axiom of proportionality and possesses locally the property A^* of analyticity (A. de La Pradelle [13, p. 391, p. 397]) with respect to the adjoint sheaf of harmonic functions associated to (Ω, H). In particular, these assumptions on (Ω, H) are verified if the harmonic space (Ω, H) is self-adjoint in the sense of F. Y. Maeda [12, p. 63], satisfies the local axiom of proportionality and has the property that a harmonic function h defined in a domain $\omega \subset \Omega$ is identically 0 if it is 0 in a neighborhood of a point in ω.

Finally, we fix a Radon measure λ in Ω such that every superharmonic function in Ω is locally λ-integrable. For example, if Ω_n is a regular exhaustion of Ω containing a point z, that is, $z \in \Omega_n \subset \overline{\Omega}_n \subset \Omega_{n+1}$ and $\Omega = \cup \Omega_n$, and if $\rho_z^{\Omega_n}$ is the harmonic measure on $\partial\Omega_n$ and if $e_n \geq 0$ is a sequence of numbers, take $\lambda = \sum e_n \rho_z^{\Omega_n}$. We call $(\Omega, H, H^*, \lambda)$ a biharmonic space.

Examples of biharmonic spaces

(1) The Euclidean space $\mathbb{R}^n$, $n \geq 3$, with $H = H^* =$ the sheaf of classical harmonic functions defined as the solutions of the Laplacian Δ, is a biharmonic space; here we take λ as the Lebesgue measure.

(2) A bounded domain Ω in $\mathbb{R}^n$, $n \geq 2$, with the C^2-solutions H and H^* of a second order elliptic differential operator $Lu = \sum a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum b_i \frac{\partial u}{\partial x_i}$ with locally Lipschitz coefficients as given in Mme. Hervé [11, pp. 560-563] is a biharmonic space. Here we take λ as the Lebesgue measure.

(3) A hyperbolic Riemannian manifold (resp. a hyperbolic Riemann surface) Ω with $H = H^* =$ the harmonic functions in the usual sense and λ is the volume measure (resp. the surface measure).

(4) $\Omega = (0, \infty)$ with the locally affine functions as harmonic is a biharmonic space. Here we take λ as the linear Lebesgue measure.

Lemma 2.1 ([4])**.** *Let $(\Omega, H, H^*, \lambda)$ be a biharmonic space. Then given any locally λ-integrable function f on an open set ω in Ω, there exists an (Ω, H) δ-superharmonic function u in ω such that locally u has a Riesz representation with associated signed measure $f d\lambda$. We denote this relation by $Lu = -f$ in ω.*

We call a function b in an open set ω in Ω biharmonic if there exists an (Ω, H^*)-harmonic function h^* in ω such that $Lb = -h^*$ in ω.

If u is a superharmonic function in Ω, the harmonic support of u is the complement, in Ω, of the largest open set where u is harmonic.

3. Biharmonic-Extension Spaces

Definition 3.1. A biharmonic space $(\Omega, H, H^*, \lambda)$ is said to be a biharmonic-extension space if for any biharmonic function b defined outside a compact set in

Ω, there exists a biharmonic function B in Ω such that $(b - B)$ is bounded near infinity.

Terminology: The term "near infinity" is used to denote a set that is the complement of a compact set in Ω.

For an outerregular compact set k (that is, if ω is a relatively compact open set containing k, every point of ∂k is regular for the Dirichlet solution in $\omega \setminus k$), let $B_k f$ stands for the Dirichlet solution in $\Omega \setminus k$ with boundary values f on ∂k and 0 at infinity.

Theorem 3.2. *Let $(\Omega, H, H^*, \lambda)$ be a biharmonic space, where (Ω, H) is a harmonic space with potentials > 0. The following are equivalent:*

1. *$(\Omega, H, H^*, \lambda)$ is a biharmonic-extension space.*
2. *If p^* is a locally λ-integrable (Ω, H^*) potential with compact harmonic support and if $Lq = -p^*$, there exists a biharmonic function B in Ω such that $(q - B)$ is bounded near infinity.*
3. *If $h^* = B_k h^*$ in $\Omega \setminus k$, where k is an outerregular compact set and if $Lb = -h^*$, there exists a biharmonic function B in Ω such that $(b - B)$ is bounded near infinity.*

Proof. (1) $\Rightarrow$ (2): It is clear.

(2) $\Rightarrow$ (3): Let $h^* = B_k h^*$ in $\Omega \setminus k$ and $Lb = -h^*$. Since h^* is harmonic near infinity, $h^* = p_1^* - p_2^* + u^*$ near infinity, where p_1^* and p_2^* are finite continuous (Ω, H^*) potentials with compact harmonic support and u^* is harmonic in Ω ([7]). Since $p_i^* = B_k p_i^*$ near infinity for $i = 1, 2$ and since $h^* = B_k h^*$, we have $u^* = B_k u^*$ near infinity; u^* is harmonic. Therefore we deduce that $u^* \equiv 0$. Let $Lq_i = -p_i^*$. Then by (2), there exist biharmonic functions B_i in Ω such that $(q_i - B_i)$ is bounded near infinity. Consequently, since $b - (q_1 - q_2)$ is an (Ω, H) harmonic function v near infinity, $b - v - (B_1 - B_2)$ is bounded near infinity. However since v is harmonic outside a compact set in the harmonic space with potentials > 0, there exists an (Ω, H) harmonic function u in Ω such that $(u - v)$ is bounded near infinity ([1]). Now set $B = u + (B_1 - B_2)$ to conclude that $(b - B)$ is bounded near infinity.

(3) $\Rightarrow$ (1): Let b be a biharmonic function defined outside a compact set. Let $Lb = -g^*$. Since (Ω, H^*) is a harmonic space with potentials > 0, $g^* = u^* + h^*$ near infinity, where u^* is an (Ω, H^*) harmonic function in Ω, and $h^* = B_k h^*$ in $\Omega \setminus k$ for an outerregular compact set k. Let B_1 be a biharmonic function in Ω such that $LB_1 = -u^*$. Then $L(b - B_1) = -h^*$ and consequently by (3) there exists a biharmonic function B in Ω such that $b - B_1 - B$ is bounded near infinity.

This completes the proof of the theorem. □

$\mathbb{R}^3$ and $\mathbb{R}^4$ are not biharmonic-extension spaces since $|x|$ in $\mathbb{R}^3$ and $\log|x|$ in $\mathbb{R}^4$ are biharmonic near infinity which cannot be extended biharmonically in the above sense. However $\mathbb{R}^n$, $n \geq 5$, is a biharmonic-extension space, the reason for which being that there is a positive solution Q for $\Delta^2 Q = \delta$ that is bounded near infinity. Generalizing this situation we say as in [7] that a biharmonic space $(\Omega, H, H^*, \lambda)$ is *tapered* if G^* is an (Ω, H^*) potential, there exists an (Ω, H) potential Q in Ω

bounded near infinity such that $LQ = -G^*$. This definition does not depend on the choice of G^*. Apart from $\mathbb{R}^n$, $n \geq 5$, and any relatively compact domain ω in $\mathbb{R}^n$, $n \geq 2$, is an example of tapered spaces including any Poincaré ball in $\mathbb{R}^n$, $n \geq 2$, in which a bounded quasiharmonic function u (that is, $\Delta u = 1$) exists ([15]).

We have some results in [7] proved in a tapered Riemannian manifold; if the proof of any of them carries over to the present axiomatic case without many changes, we assume that this result is proved in the axiomatic case of the biharmonic space $(\Omega, H, H^*, \lambda)$ also. In the sequel we fix a finite continuous (Ω, H^*) potential G^* with compact harmonic support in Ω.

Theorem 3.3. *In a biharmonic space $(\Omega, H, H^*, \lambda)$, the following are equivalent:*

1. *$(\Omega, H, H^*, \lambda)$ is a tapered space.*
2. *For any given biharmonic function b outside a compact set in Ω, there exists a biharmonic function B in Ω such that $(b - B)$ is bounded near infinity and $|L(b - B)| \leq p^*$ outside a compact set in Ω, where p^* is an (Ω, H^*) potential.*

Proof. (1) $\Rightarrow$ (2): Let b be a biharmonic function defined outside a compact set and $Lb = -h^*$. Since h^* is harmonic outside a compact set, there exist finite continuous (Ω, H^*) potentials p_1^* and p_2^* with compact harmonic support and an (Ω, H^*) harmonic function u^* in Ω such that $h^* = p_1^* - p_2^* + u^*$ near infinity. If $Ls_1 = -p_1^*, Ls_2 = -p_2^*$ and $LB_1 = -u^*$, then $b = s_1 - s_2 + B_1 + v$ near infinity, where v is an (Ω, H) harmonic function near infinity. Write $v = q_1 - q_2 + v_1$, where q_1 and q_2 are potentials with compact harmonic support and v_1 is (Ω, H) harmonic in Ω, and let $B = B_1 + v_1$. Then B is biharmonic in Ω and $b = s_1 - s_2 + q_1 - q_2 + B$. Since $(\Omega, H, H^*, \lambda)$ is tapered by assumption, $s_i\,(i = 1, 2)$ can be considered as (Ω, H) potential bounded near infinity ([7, Theorem 10]). Since q_i is a potential with compact harmonic support, it is also bounded near infinity.

Hence $(b - B)$ is bounded near infinity and $|L(b - B)| \leq p_1^* + p_2^*$ outside a compact set.

(2) $\Rightarrow$ (1): We have to show that for the fixed potential G^*, there exists a function Q bounded near infinity such that $LQ = -G^*$. Let q be some superharmonic function in (Ω, H) such that $Lq = -G^*$. Since q is biharmonic outside a compact set, by the assumption (2), there exists a biharmonic function B in Ω such that $(q - B)$ is bounded outside a compact set and $|L(q - B)| \leq p^*$ near infinity.

This implies that the harmonic function LB in (Ω, H^*) is bounded by the potential $p^* + G^*$ outside a compact set. Hence $LB \equiv 0$, that is, B is harmonic in (Ω, H). Now we write $Q = q - B$ to obtain $LQ = -G^*$. $\square$

Remark 3.4. The condition that the biharmonic space $(\Omega, H, H^*, \lambda)$ is tapered is sufficient to conclude that $(\Omega, H, H^*, \lambda)$ is a biharmonic-extension space. However it is not necessary. Consider, for example, $\Omega = (0, \infty)$ with the local affine functions forming the harmonic sheaf $H = H^*$ and λ the linear Lebesgue measure. Then a biharmonic function u outside the compact set $[X, Y]$ is of the form $ax^3 + bx^2 + cx + d$ in (Y, ∞) and $\alpha x^3 + \beta x^2 + \gamma x + \delta$ in $(0, X)$. If we take $B(x) = ax^3 + bx^2 + cx + d$ in

Ω, clearly $B(x)$ is biharmonic in Ω and $(u - B)$ is bounded near infinity. However $(\Omega, H, H^*, \lambda)$ is not tapered.

Recall that for a given nonempty set A in Ω, R_1^A stands for the infimum of all superharmonic functions $s > 0$ in Ω such that $s \geq 1$ on A; and $\widehat{R_1^A}$ is its lower semicontinuous regularization, namely $\widehat{R_1^A}(x) = \liminf_{y \to x} R_1^A(y)$ for each $x \in \Omega$.

Corollary 3.5. *Let $(\Omega, H, H^*, \lambda)$ be a tapered space in which there is an (Ω, H^*) potential p^* tending to 0 at infinity. Then, if h^* is an (Ω, H^*) harmonic function tending to 0 at infinity, there exists a bounded biharmonic function b near infinity such that $Lb = -h^*$.*

Proof. Since there is a potential p^* tending to 0 at infinity, any (Ω, H^*) potential p_1^* with compact harmonic support in Ω tends to 0 at infinity. For, if k is an (Ω, H^*) outerregular compact set containing the harmonic support of p_1^* in its interior, $p_1^* = B_k p_1^*$ in $\Omega \setminus k$; hence $p_1^* \leq \alpha p^*$ in $\Omega \setminus k$ for some α; hence p_1^* tends to 0 at infinity.

Also, if h^* tends to 0 at infinity, for some (Ω, H^*) outerregular compact set k, $h^* = B_k h^*$ in $\Omega \setminus k$ and hence if $\sup_{\partial k} |h^*| = \beta$ and if $p_2^* = \beta \left(\widehat{R_1^k}\right)$ in (Ω, H^*), p_2^* is an (Ω, H^*) potential such that $|h^*| \leq p_2^*$ in $\Omega \setminus k$.

Now, let b_1 be a biharmonic function such that $Lb_1 = -h^*$. Then by Theorem 3.3, there exists a biharmonic function B in Ω such that $b = b_1 - B$ is bounded near infinity and $|L(b_1 - B)| \leq p_3^*$; this implies (since $|Lb_1| = |h^*| \leq p_2^*$ near infinity) that $|LB| \leq p_3^* + p_2^*$ near infinity. Hence $LB \equiv 0$ and consequently $Lb = Lb_1 = -h^*$. □

In the converse direction, we prove the following:

Proposition 3.6. *In a tapered space $(\Omega, H, H^*, \lambda)$ the following are equivalent:*

1. *If b is a bounded biharmonic function near infinity, then Lb tends to 0 at infinity.*
2. *There exists an (Ω, H^*) potential tending to 0 at infinity and every bounded biharmonic function in Ω is (Ω, H) harmonic.*

Proof. (1) $\Rightarrow$ (2): Since $(\Omega, H, H^*, \lambda)$ is tapered. If $LQ = -G^*, Q$ is bounded near infinity. Hence by (1), G^* tends to 0 at infinity. Secondly, if B is a bounded biharmonic function in Ω, LB should tend to 0 at infinity by (1) since LB is (Ω, H^*) harmonic and $LB \equiv 0$, that is, B is (Ω, H) harmonic.

(2) $\Rightarrow$ (1): Suppose b is a bounded biharmonic function near infinity. Then by Theorem 3.3, there exists a biharmonic function B in Ω such that $(b - B)$ is bounded near infinity, and $|L(b - B)| \leq p^*$, where p^* is an (Ω, H^*) potential which by constructions is finite continuous and also has compact harmonic support; hence p^* tends to 0 at infinity. Since B should be bounded biharmonic in Ω, B is harmonic by (2). Thus $|L(b)| = |L(b - B)| \leq p^*$ outside a compact set and consequently Lb tends to 0 at infinity. □

Remark 3.7. In any harmonic space (Ω, H), for a given nonpolar compact set k, a relatively compact component ω of $\Omega \setminus k$ is said to be a P-domain (resp. an

S-domain) if $R_1^k < 1$ (resp. $R_1^k \equiv 1$) in ω. In the context of statement (2) of Proposition 3.6, we remark that if (Ω, H) is a harmonic space with potentials > 0 and if for a nonpolar compact set k, $\Omega \setminus k$ contains an S-domain ω, then no potential p in Ω tends to 0 at infinity. Thus, let $\beta = \inf_k p$; then $p \geq \beta R_1^k$ in $\Omega \setminus k$ and consequently $\limsup_{x\in\omega, x\to\infty} p(x) \geq \beta$.

We conclude this section with a different characterization of bounded biharmonic functions defined outside a compact set in $\mathbb{R}^n$, $n \geq 5$, which is a tapered space.

Theorem 3.8. *Let h be a harmonic function defined outside a compact set in $\mathbb{R}^n, n \geq 5$. Then $h \in L^p(\omega)$ for some $1 \leq p < \infty$ and some neighborhood ω of infinity if and only if there exists a bounded biharmonic function b near infinity such that $\Delta b = h$.*

To prove this theorem, we need the following lemmas.

Lemma 3.9. *If s is a subharmonic function in $L^p(\mathbb{R}^n)$, $1 \leq p < \infty$ and $n \geq 2$, then $s \leq 0$ in $\mathbb{R}^n$.*

Proof. For $x_0 \in \mathbb{R}^n$, let $S_n = \{x : |x - x_0| = 1\}$ and σ_n be the surface area of S_n. Since $t = s^+ \geq 0$ and t^p is subharmonic, we have $t^p(x_0) \leq \frac{1}{\sigma_n}\int_{S_n} t^p(r,\omega)\, d\omega$, where $x = (r, \omega)$ is the polar coordinate with $|x - x_0| = r$.

Since $t \in L^p(\mathbb{R}^n)$ by hypothesis,

$$\infty > \int_0^\infty \int_{S_n} t^p(r,\omega)\, r^{n-1} dr d\omega \geq \int_0^\infty \sigma_n t^p(x_0)\, r^{n-1} dr.$$

This is possible if and only if $t^p(x_0) = 0$. Since x_0 is arbitrary, $t^p \equiv 0$ in $\mathbb{R}^n$ and hence $s \leq 0$ in $\mathbb{R}^n$. □

For the statement of the following lemma, we shall say that a subharmonic function f defined outside a compact set in $\mathbb{R}^n$ extends subharmonically in $\mathbb{R}^n$, if there exists a subharmonic function g in $\mathbb{R}^n$ such that g is not majorized by a harmonic function in $\mathbb{R}^n$ and $f = g$ outside a compact set.

Lemma 3.10. *Let u be an L^p-subharmonic function, $1 \leq p < \infty$, defined outside a compact set in $\mathbb{R}^n$, $n \geq 2$. Then u cannot be extended subharmonically in $\mathbb{R}^n$.*

Proof. Suppose there exists a subharmonic function v not majorized by a harmonic function in $\mathbb{R}^n$ such that $u = v$ outside a compact set. Then, for large r, the function s defined as u in $|x| \geq r$ and $D_r u$ in $|x| < r$ is subharmonic in $\mathbb{R}^n$ and $s \geq v$, where $D_r u$ is the Dirichlet solution in $|x| < r$ with boundary values u.

If $u(x) \in L^p$ in $|x| \geq r, s(x)$ is in the harmonic Hardy class in $|x| < r$ (Axler [6, p. 103]) and hence there exists a harmonic function $H(x)$ in $|x| < r$ such that $|s^p| < H$. Then $\int_{|x|<r} |s(x)|^p dx \leq c_n H(0)$ for a constant c_n. That is, s belongs to L^p in $|x| < r$, which implies that $s \in L^p(\mathbb{R}^n)$ since $s(x) = u(x)$ in $|x| \geq r$. Then, by Lemma 3.9, $s \leq 0$ and hence $v \leq 0$ in $\mathbb{R}^n$, this is a contradiction. □

Lemma 3.11. *Let u be a subharmonic function in an open set ω containing $|x| \geq r$ in $\mathbb{R}^n$, $n \geq 2$. Suppose $u \in L^p(\omega)$ for some $1 \leq p < \infty$. Then u is upper bounded in $|x| \geq r$.*

Proof. By hypothesis, $u^+(x)$ is an L^p-subharmonic function in the open set ω containing $|x| \geq r$.

(1) In $\mathbb{R}^2$, if u^+ is not upper bounded in $|x| \geq r$, it can be extended subharmonically in $\mathbb{R}^2$ ([12, Corollary 1]). This is a contradiction (Lemma 3.10) since $u^+ \in L^p$ in $|x| \geq r$. This means that u^+ and hence u is upper bounded in $|x| \geq r$.

(2) In $\mathbb{R}^n$ with $n \geq 3$, there exists a subharmonic function $s(x)$ in $\mathbb{R}^n$ and some $\alpha \leq 0$ such that $u^+(x) = s(x) - \alpha|x|^{2-n}$ in $|x| \geq r$ ([3, Theorem 1']). Hence $s(x) \geq \alpha|x|^{2-n}$.

Denoting by $M(R, s)$ the mean-value of $s(x)$ on $|x| = R$, suppose $\lim_{R\to\infty} M(R, s) = \infty$. Then $\lim_{R\to\infty} M(R, u^+) = \infty$. Hence u^+ can be extended subharmonically in $\mathbb{R}^n$ ([3, Theorem 2']). This is a contradiction (Lemma 3.10); thus $\lim_{R\to\infty} M(R, s) < \infty$, in which case s has a harmonic majorant h in $\mathbb{R}^n$. Since h is lower bounded, it is a constant c and $c \geq 0$. Hence u^+ is bounded in $|x| \geq r$ and u is upper bounded by $c - \alpha|x|^{2-n}$ in $|x| \geq r$.

Thus, for all $n \geq 2$, u is upper bounded in $|x| \geq r$ in $\mathbb{R}^n$. □

Let ω be a bounded open set in $\mathbb{R}^2$ and Ω be an open set containing ω. Let f be a C^2-function on Ω. If $\frac{\partial g}{\partial n^+}(s)$ denotes the outer normal derivative at a point s on $\partial\omega$, then $\int_{\partial\omega} \frac{\partial g}{\partial n^+}(s)\, ds$ is defined as the outward flux of g on ω. As a particular case of the Green's Formula, we see that $\iint_{\omega} \Delta g(x) = \int_{\partial\omega} \frac{\partial g}{\partial n^+}(s)\, ds$.

Suppose $h(z)$ is a harmonic function defined on $|x| > R$. Let a and b be two positive numbers larger than R. Since $\Delta h(z) = 0$ when $|x| > R$, we obtain $\int_{|s|=a} \frac{\partial h}{\partial n^-}(s)\, ds + \int_{|s|=b} \frac{\partial h}{\partial n^+}(s)\, ds = 0$ from the Green's Formula on the annulus $\omega = \{z : a < |x| < b\}$. This implies that $\int_{|s|=a} \frac{\partial h}{\partial n^+}(s)\, ds = \int_{|s|=b} \frac{\partial h}{\partial n^+}(s)\, ds$. Since a and b are arbitrary, the constant $\alpha = \int_{|s|=r} \frac{\partial h}{\partial n^+}(s)\, ds$ is independent of $r\,(> R)$. We define α as the flux at infinity of h.

Lemma 3.12. *Let h be a harmonic function defined outside a compact set in $\mathbb{R}^n$, $n \geq 2$. Then h tends to 0 at infinity if and only if $h \in L^p(\omega)$ for some $1 \leq p < \infty$, and some neighborhood ω of infinity.*

Proof. Suppose $h \in L^p(\omega)$. Then by Lemma 3.11, h is bounded near infinity and hence tends to a limit l at infinity; l should be 0 since $h \in L^p(\omega)$.

Conversely, suppose h tends to 0 at infinity.

(1) In $\mathbb{R}^n$, $n \geq 3$, write $h = p_1 - p_2 + H$ near infinity, where $p_i\,(i = 1, 2)$ is a finite continuous potential with compact harmonic support in $\mathbb{R}^n$ and H is harmonic in $\mathbb{R}^n$. Since p_i and h tend to 0 at infinity, $H \equiv 0$. Now, for sufficiently large r, if $k = \{x : |x| \leq r\}$, $B_r p_i(x) = B_k p_i(x) = p_i(x)$ for $|x| > r$. Consequently $|p_i(x)| \leq \alpha_i |x|^{n-2}$ in $|x| > r$ where $\alpha_i = \max_{|x|=r} p_i(x)$ and hence if ω is the open set $|x| > r, p_i \in L^p(\omega)$ for $p > n/(n-2)$. Hence $h \in L^p(\omega)$.

(2) In $\mathbb{R}^2$, write $h = s_1 - s_2 + H$ near infinity, where $s_i\,(i = 1, 2)$ is a finite continuous logarithmic potential with compact harmonic support in $\mathbb{R}^2$ (that is, $s_i(x) - \alpha_i \log|x| \to 0$ as $|x| \to \infty$, where α_i is the flux of s_i at infinity)

and H is harmonic in $\mathbb{R}^2$. Then $\alpha_1 = \alpha_2 = \alpha$ since flux h and flux H at infinity are 0; also since h and $s_i - \alpha \log|x|$ tend to 0 at infinity, $H \equiv 0$. Now for sufficiently large r, $|s_i - \alpha \log|x|| \leq \frac{M}{|x|}$ when $|x| > r$. Hence $|h(x)| \leq \frac{2M}{|x|}$ when $|x| > r$; consequently $h \in L^p$ in $|x| > r$, if $p > 2$.

□

Proof of Theorem 3.8:

(1) Suppose $h \in L^p(\omega)$ for some $p \geq 1$. Then by Lemma 3.12, h tends to 0 at infinity and consequently, Corollary 3.5 can be used to assist the existence of a bounded biharmonic function b near infinity such that $\Delta b = h$.
(2) Conversely, let h be harmonic outside a compact set such that $\Delta b = h$ for some bounded biharmonic function b. Then by Proposition 3.6, h tends to 0 at infinity. Consequently by Lemma 3.12, $h \in L^p(\omega)$ for any $p > n/(n-2)$ and some neighborhood ω of the point at infinity.

□

4. Flux Condition

Let $(\Omega, H, H^*, \lambda)$ be a biharmonic space, where (Ω, H) has potentials > 0. A reformulation of Theorem 3.2 states that $(\Omega, H, H^*, \lambda)$ is a biharmonic-extension space if and only if the following two conditions are satisfied:

(1) Condition F: If p^* is a finite continuous (Ω, H^*) potential with compact harmonic support and if $Lq = -p^*$, there exist a biharmonic function B in Ω and a constant α such that $(q - \alpha Q - B)$ is bounded near infinity. (Recall that $LQ = -G^*$, where G^* is a fixed (Ω, H^*) potential, finite continuous with compact harmonic support).
(2) Condition E: There exists a biharmonic function V in Ω such that $Q + V$ is bounded near infinity.

Definition 4.1. Let $(\Omega, H, H^*, \lambda)$ be a biharmonic space, where (Ω, H) has potentials > 0. We say that $(\Omega, H, H^*, \lambda)$ satisfies the flux condition if it satisfies the condition F but not the condition E.

Remark 4.2.

(1) Let $(\Omega, H, H^*, \lambda)$ be a biharmonic space satisfying the flux condition. Then, for a given finite continuous (Ω, H^*) potential p^* with compact harmonic support, the constant α in the flux condition is uniquely determined. For, suppose we have another set of solutions $Lq_1 = -p^*$ and $LQ_1 = -G^*$ such that $(q_1 - \alpha_1 Q_1 - B_1)$ is bounded near infinity; now $(q - q_1)$ and $(Q - Q_1)$ are harmonic in (Ω, H), then this implies that there exists a biharmonic function B_2 in Ω such that $(\alpha - \alpha_1) Q + B_2$ is bounded near infinity. If $\alpha \neq \alpha_1$ this means that $Q + \frac{1}{\alpha - \alpha_1} B_2$ is bounded near infinity; that is the condition E is satisfied. This is a contradiction. We call this α the flux of p^* at infinity.
(2) $\mathbb{R}^3$ and $\mathbb{R}^4$ are nice examples of biharmonic spaces satisfying the flux condition. For, in $\mathbb{R}^3$, take $G^*(x) = \frac{1}{|x|}$ and $Q(x) = -\frac{1}{2}|x|$. Then if p^* is any

potential with its associated measure μ having compact harmonic support, since $(|x| * \mu - \mu(\mathbb{R}^3)|x|)$ is bounded near infinity ([7]), it follows that the condition F is satisfied in $\mathbb{R}^3$. To show that $\mathbb{R}^3$ does not satisfy the condition E, we can make use of the Almansi representation of biharmonic function (see Proposition 4.5 below). It is also easy to prove that in this case α is the flux of p^* at infinity according to the above definition is actually proportional to the classical definition of flux at infinity of p^* in $\mathbb{R}^3$.

(3) Let h^* be an (Ω, H^*) harmonic function defined outside a compact set in a biharmonic space satisfying the flux condition. Write $h^* = p_1^* - p_2^* + u^*$ near infinity, where p_1^* and p_2^* are finite continuous (Ω, H^*) potentials with compact harmonic support and u^* is harmonic uniquely determined in Ω. We define *the flux at infinity of* h^* as $(\operatorname{flux} p_1^* - \operatorname{flux} p_2^*)$ at infinity . It can be checked that the flux at infinity of h^* is well-determined.

Lemma 4.3. *Let $(\Omega, H, H^*, \lambda)$ be a biharmonic space satisfying the flux condition. If b is a biharmonic function defined outside a compact set in Ω, it is of the form $b(x) = \alpha Q(x) + B(x) + u(x)$ near infinity, where $LQ = -G^*$ in Ω, $B(x)$ is biharmonic in $\Omega, u(x)$ is bounded biharmonic near infinity, and $\alpha = -\operatorname{flux} Lb$ is uniquely determined.*

Proof. Let α be the flux at infinity of $h^* = -Lb$. That is, if $h^* = p_1^* - p_2^* + u^*$ near infinity and if $\operatorname{flux} p_i^*$ is α_i, then $\alpha = \alpha_1 - \alpha_2$. Let $Ls_i = -p_i^*$. Then $(s_i - \alpha_i Q - B_i)$ is bounded near infinity. Let $LB_0 = -u^*$ in Ω.

Hence $b = s_1 - s_2 + B_0 + v_1$ near infinity, where v_1 is an (Ω, H) harmonic function and hence $v_1 = v +$ (a bounded harmonic function v_2) near infinity, where v is (Ω, H) harmonic in Ω. This leads to the representation $b = (\alpha_1 - \alpha_2)Q + (B_1 - B_2 + B + v) +$ (a bounded biharmonic function) near infinity, as given in the lemma.

For the uniqueness of α in the representation $b = \alpha Q + B + u$, notice that if $b = \alpha_1 Q + B_1 + u_1$ is another representation with $\alpha \neq \alpha_1$, we should have $Q + \frac{1}{\alpha - \alpha_1}(B - B_1)$ bounded near infinity. This is a contradiction. □

Theorem 4.4. *Let $(\Omega, H, H^*, \lambda)$ be a biharmonic space satisfying the flux condition. For a given biharmonic function b outside a compact set, there exists a biharmonic function B in Ω such that $(b - B)$ is bounded near infinity if and only if* $\operatorname{flux} Lb$ *is* 0.

Proof. Write $b = \alpha Q + B + u$ near infinity as in Lemma 4.3. Suppose $-\alpha = \operatorname{flux} Lb$ is 0, then $(b - B)$ is bounded near infinity. Conversely, suppose there exists a biharmonic function B_1 in Ω such that $(b - B_1)$ is bounded near infinity. Then if $\alpha \neq 0$, we should have $Q + \frac{1}{\alpha}(B - B_1)$ bounded near infinity, this is a contradiction. □

We conclude this section with a simple property of s-harmonic functions (that is, $\Delta^s u = 0$) in $\mathbb{R}^n$ which is related to the condition E.

Proposition 4.5. *For $s \in \mathbb{N}$ and $3 \leq n \leq 2s$, let $\Delta^{s-1} Q = -|x|^{2-n}$ in $\mathbb{R}^n$. Then, for any m-harmonic function $v(x)$ in $\mathbb{R}^n, Q(x) + v(x)$ is unbounded near infinity.*

Proof. Since $\Delta^s Q = c\delta$ and $n \leq 2s$, up to and additive s-harmonic function, $Q(x)$ is of the form $A|x|^{2s-n}$ if n is odd and of the form $|x|^{2s-n}(B + A\log|x|)$ if n is even. Suppose $Q(x) + v(x)$ is bounded near infinity for some m-harmonic function $v(x)$. Let $l = \max(m, s)$.

Using the Almansi representation, we can express $Q + v$ in the form $A|x|^{2s-n} + |x|^{2l-2}h_1(x) + \cdots + h_l(x)$ if n is odd and $|x|^{2s-n}(B + A\log|x|) + |x|^{2l-2}h_1(x) + \cdots + h_l(x)$ if n is even, where h_i is harmonic in $\mathbb{R}^n$. For a fixed point x_0 in $\mathbb{R}^n$, let $\rho^a_{x_0}$ denote the harmonic measure on $|x| = a$ (defined by the Poisson kernel). By assumption, $\int (Q(x) + v(x))\, d\rho^a_{x_0}$ is bounded for large a.

That is, $Aa^{2s-n} + a^{2l-2}h_1(x_0) + \cdots + h_l(x_0)$ is bounded for large a, if n is odd; and $a^{2s-n}(B + A\log a) + a^{2l-2}h_1(x_0) + \cdots + h_l(x_0)$ is bounded for large a, if n is even. However, by allowing $a \to \infty$, it is easy to check that this is not possible (notice that in both cases $A \neq 0$).

Hence the proof of the proposition is completed. □

5. Biharmonic Extensions in parabolic spaces

It can be shown, for example as in [7], that if $b(x)$ is a biharmonic function defined outside a compact set in $\mathbb{R}^2$, then there exist two uniquely determined constants α and β and a biharmonic function $B(x)$ in $\mathbb{R}^2$ unique up to an additive constant and a bounded biharmonic function $u(x)$ outside a compact set in $\mathbb{R}^2$ such that $b(x) = \alpha\log|x| + \beta|x|^2\log|x| + B(x) + u(x)$ near infinity. We consider briefly this result in Brelot harmonic spaces without positive potentials; parabolic Riemann surfaces and parabolic Riemannian manifolds are examples of such spaces.

Let H and H^* be two Brelot harmonic sheaves defined on Ω, both without positive potentials defined on Ω, (Ω, H) satisfies locally the axiom of proportionality and the axiom A^*, λ is a fixed Radon measure in Ω with respect to which the biharmonic functions are defined as before.

Let k be an outerregular compact set in (Ω, H) and let $F \geq 0$ be a fixed unbounded continuous function on Ω, equal to 0 on k and harmonic in $\Omega \setminus k$. (F is like $\log|x|$ in $|x| > 1$). Let $F^* \geq 0$ be a similarly fixed function in (Ω, H^*). Let Q be a function on Ω such that $LQ = -F^*$. Let $\operatorname{flux} F$ (and $fluxF^*$) at infinity be 1. We recall that any (Ω, H) harmonic function h defined outside a compact set in Ω is of the form $h = \alpha F + f + u$ near infinity, where f is harmonic in Ω and u is bounded harmonic near infinity.

Theorem 5.1. *Let $(\Omega, H, H^*, \lambda)$ be a biharmonic space as above without positive potentials. Then the following are equivalent:*

(1) *If h^* is a bounded harmonic function defined outside a compact set in (Ω, H^*) and if $Lb = -h^*$, then there exist a constant α and a biharmonic function $B(x)$ in Ω such that $(b - \alpha F - B)$ is bounded near infinity.*
(2) *If s^* is a locally λ-integrable superharmonic function in (Ω, H^*) with compact harmonic support and if $Lq = -s^*$, then there exist constant α and β and a biharmonic function B in Ω such that $q(x) = \alpha F(x) + \beta Q(x) + B(x) + u(x)$ near infinity, where $u(x)$ is bounded biharmonic function near infinity for which* $\operatorname{flux} Lu$ *at infinity is* 0 *and* $\beta = \operatorname{flux} s^*$.

(3) *Any biharmonic function b defined outside a compact set is of the form $b(x) = \alpha F(x) + \beta Q(x) + B(x) + u(x)$ near infinity, where $B(x)$ is biharmonic in Ω, $u(x)$ is bounded biharmonic function near infinity, $-\beta = \operatorname{flux} Lb$ and $0 = \operatorname{flux} Lu$.*

Proof. (1) $\Rightarrow$ (2): Let $Lq = -s^*$, where s^* is a locally λ-integrable (Ω, H^*) superharmonic function with compact harmonic support in Ω. Then s^* is of the form $\beta F^* + f^* + h^*$ near infinity, where f^* is (Ω, H^*) harmonic in Ω and h^* is bounded harmonic near infinity. Let $LB_1 = -f^*$ and $Lb = -h^*$. Then $L(q - \beta Q - B_1) = -h^*$ near infinity and hence by (1), there exist a constant α and a biharmonic function B in Ω such that $q - \beta Q - B_1 = \alpha F + B + u$ near infinity, where u is bounded biharmonic function near infinity and $\operatorname{flux} Lu = -\operatorname{flux} h^* = 0$, since h^* is bounded near infinity. This proves (2).

(2) $\Rightarrow$ (3): Let b be biharmonic function defined outside a compact set. Let $Lb = -u^*$. Since u^* can be written as $s_1^* - s_2^*$ near infinity, where s_1^* and s_2^* are finite continuous superharmonic functions with compact harmonic support in (Ω, H^*) and consequently since $\operatorname{flux} u^* = \operatorname{flux} s_1^* - \operatorname{flux} s_2^*$, we obtain (3).

(3) $\Rightarrow$ (1): Let h^* be a bounded (Ω, H^*) harmonic function defined outside a compact set and let $Lb = -h^*$. Then by (3), $b(x) = \alpha F(x) + \beta Q(x) + B(x) + u(x)$ near infinity, where $u(x)$ is bounded biharmonic function near infinity. Since h^* is bounded, $\operatorname{flux} h^*$ is 0 and consequently, $-\beta = \operatorname{flux} Lb = -\operatorname{flux} h^* = 0$. Hence we have (1).

Hence the proof of Theorem 5.1 is completed. □

6. Removable singularity for bounded biharmonic functions

Let $(\Omega, H, H^*, \lambda)$ be a biharmonic space and k be a compact set in Ω. Suppose for an open set $\omega \supset k$, there exists a bounded biharmonic function in $\omega \setminus k$ which does not extend biharmonically in ω. Is it then possible to find in any open set $\omega_0 \supset k$, a bounded biharmonic function in $\omega_0 \setminus k$ which does not extend as a biharmonic function in ω_0? We leave out the simple case where the answer is yes if $\omega \supset \omega_0 \supset k$. It is proved in this section that the answer is yes even if $\omega_0 \supset \omega \supset k$, provided $(\Omega, H, H^*, \lambda)$ is a biharmonic-extension space.

First we prove a similar result for bounded harmonic functions in a harmonic space (Ω, H) with or without potentials > 0. If there is no potential > 0 in Ω, we fix as in the previous section an unbounded continuous function $F \geq 0$ in $\Omega, 0$ on an outerregular compact set k, and harmonic in $\Omega \setminus k$. Then a combination of Lemma 1 in [2] and Theorem 2.2 in [5] permits us to prove the following:

Lemma 6.1. *Let (Ω, H) be a harmonic space with or without potentials > 0 in Ω. Let ω be an open set and k a compact set in Ω such that $k \subset \omega$. Let h be a harmonic function in $\omega \setminus k$. Then there exist a harmonic function s in $\Omega \setminus k$ and a harmonic function t in ω such that $h = s - t$ in $\omega \setminus k$. Moreover, s can be taken as bounded near infinity if there are potentials > 0 in Ω, and $(s - \alpha F)$ can be taken as bounded near infinity for some constant α if there are no potentials > 0 in Ω.*

Lemma 6.2. *Let $(\Omega, H, H^*, \lambda)$ be a biharmonic space. Let k be a compact set and ω an open set in Ω such that $k \subset \omega$. Let b be a biharmonic function in $\omega \setminus k$. Then there exist a biharmonic function u in $\Omega \setminus k$ and a biharmonic function v in ω such that $b = u - v$ in $\omega \setminus k$.*

Proof. Let $Lb = -h^*$ in $\omega \setminus k$. Write $h^* = s^* - t^*$ as in Lemma 6.1, where s^* is harmonic in $\Omega \setminus k$ and t^* is harmonic in ω. Let u_1 and v_1 be such that $Lu_1 = -s^*$ and $Lv_1 = -t^*$. Then u_1 is biharmonic in $\Omega \setminus k$ and v_1 is biharmonic in ω such that $b = u_1 - v_1 +$ (a harmonic function h_1) in $\omega \setminus k$.

Again by Lemma 6.1, $h_1 = s_1 - t_1$ in $\omega \setminus k$, where s_1 is (Ω, H) harmonic in $\Omega \setminus k$ and t_1 is (Ω, H) harmonic in ω. Writing $u = u_1 + s_1$ and $v = v_1 + t_1$, we obtain the decomposition $b = u - v$ in $\omega \setminus k$ as stated in the lemma. □

Theorem 6.3. *Let $(\Omega, H, H^*, \lambda)$ be a biharmonic space in which every biharmonic function is infinite continuous. Let k be a compact set and ω an open set such that $k \subset \omega \subset \Omega$. Suppose there exists a bounded biharmonic function b in $\omega \setminus k$ which does not extend biharmonically in ω. Then in any domain $\Omega_0 \supset \omega$, there exists a bounded biharmonic function B in $\Omega_0 \setminus k$ which does not extend biharmonically in Ω_0, provided $(\Omega_0, H, H^*, \lambda)$ is a biharmonic-extension space.*

Proof. By Lemma 6.2, $b = u - v$ in $\omega \setminus k$, where u is biharmonic in $\Omega \setminus k$ and v is biharmonic in ω. Suppose $(\Omega_0, H, H^*, \lambda)$ is a biharmonic-extension domain and $k \subset \omega \subset \Omega_0$. Then we can find a biharmonic function b_1 in Ω_0 such that $(u - b_1)$ is bounded near infinity in Ω_0. Define $B = u - b_1$ in $\Omega_0 \setminus k$.

Let ω_1 be a relatively compact open set such that $k \subset \omega_1 \subset \overline{\omega}_1 \subset \omega$. Since every biharmonic function is assumed to be finitely continuous, v is bounded on $\overline{\omega}_1$ and hence u is bounded in $\omega_1 \setminus k$; b_1 being biharmonic in Ω_0, it is also bounded in $\omega_1 \setminus k$. Thus B is bounded in $\omega_1 \setminus k$ as well as near infinity in Ω_0. Consequently, B is a bounded biharmonic function in $\Omega_0 \setminus k$. However B cannot extend biharmonically in Ω_0 for $B = b + v - b_1$ in $\omega_1 \setminus k$, where $v - b_1$ is biharmonic in ω. Hence if B extends biharmonically in Ω_0, b should extend biharmonically in ω. It contradicts the hypothesis. □

Corollary 6.4. *Let $(\Omega, H, H^*, \lambda)$ be a biharmonic-extension space in which every biharmonic function is finite continuous. Suppose k is a compact set and ω is an open set such that $k \subset \omega$ and that every bounded biharmonic function in $\omega \setminus k$ extends bihamronically in ω. Then for any open set $\omega_0 \supset k$ if b is bounded biharmonic in $\omega_0 \setminus k$, b extends biharmonically in ω_0.*

Proof. In view of the above theorem, it is enough to prove that if Ω_0 is any domain in Ω, $(\Omega_0, H, H^*, \lambda)$ is also a biharmonic-extension space.

Let p_0^* be a locally λ-integrable (Ω_0, H^*) potential with compact harmonic support A in Ω_0 and let $Lq_0 = -p_0^*$ in Ω_0. Let p_1^* be an (Ω, H^*) potential in Ω with the same support A such that $p_1^* = p_0^*$ in a neighborhood of A. Then p_1^* is locally λ-integrable in Ω and let $Lq_1 = -p_1^*$ in Ω. Since Ω is a biharmonic-extension space, there exists a biharmonic function B_1 in Ω such that $(q_1 - B_1)$ is bounded outside a compact set. Since $(q_1 - B_1)$ is biharmonic outside A and since a biharmonic function is finite continuous, if X is a relatively compact open set such that $A \subset X \subset \overline{X} \subset \Omega_0, (q_1 - B_1)$ is bounded in $\Omega \setminus X$.

Let $B_0 = q_0 - q_1 + B_1$ in Ω_0. Since $p_1^* - p_0^*$ is harmonic in Ω_0, B_0 is biharmonic in Ω_0 and $(q_0 - B_0)$ is bounded outside X in Ω_0. Hence Ω_0 is a biharmonic-extension space. □

7. Boundary-value problem for biharmonic functions

Let R be a Riemannian manifold and ω a relatively compact domain in R. Let $\overline{\omega}$ be the Wiener compactification of ω and let $\partial\omega = \overline{\omega} - \omega$. Suppose f and g are two finite continuous functions on $\partial\omega$. If H_f^ω denotes the Dirichlet solution in ω with boundary value f. Since $\partial\omega$ is resolutive, $h = H_f^\omega$ is a bounded harmonic function in ω. If $G(x,y)$ is the Green function in ω, $G \in L^1(\omega)$ and hence if $u(x) = \int_\omega G(x,y)\, h(y)\, dy, u(x)$ is bounded and $\Delta u = -h$; since h is in $C^\infty(\omega)$, we can assume u also is in $C^\infty(\omega)$.

Recall that if μ is a measure in ω with $||\mu||$ finite, then $\int G(x,y)\, d\mu(y)$ is a potential in ω. Hence

$$u(x) = \int_\omega G(x,y)\, h^+(y)\, dy - \int_\omega G(x,y)\, h^-(y)\, dy$$

is the difference of two potentials and hence harmonizable. Thus, u is a bounded Wiener function in ω and consequently u extends as a continuous function on the Wiener compactification of $\overline{\omega}$.

Let $v = H_{g-u}^\omega$. Then v is a bounded harmonic function in ω and if $b = u + v$, then b is a bounded biharmonic function in ω, $\Delta b = -h, b$ tends to g and Δb tends to f at the regular points of $\partial\omega$. (see [15, Section 4, Chapter VII]).

Proceeding in the same way, we prove the following theorem in the axiomatic case where the theory of Wiener compactification is due to Constntinecu and Cornea [10] and that of self-adjoint harmonic spaces is due of F. Y. Maeda [12].

Theorem 7.1. *Let $(\Omega, H, H^*, \lambda)$ be a biharmonic space, where (Ω, H) is a self-adjoint harmonic space. Let ω be a relatively compact domain in Ω. Let Γ and Γ^* be the Wiener harmonic boundaries of ω in (Ω, H) and (Ω, H^*) respectively. Assume that a biharmonic function in an open set in Ω is finite continuous. Then if g and f^* are finite continuous on Γ and Γ^* respectively, there exists a unique biharmonic function b in ω such that b and Lb are bounded, b tends to g on Γ and Lb tends to f^* on Γ^*.*

Proof. Extend f^* and g as finite continuous functions on the Wiener boundaries $\partial\omega^*$ and $\partial\omega$ respectively. Let $h^* = H_{-f^*}^{\omega^*}$ in ω^* and $Lv = -h^*$. Let ω_1 be an (Ω, H) regular domain containing ω_c, the closure of ω in Ω. By the assumption on the continuity of the biharmonic functions, there exists a finite continuous function u in ω_1 such that $Lu = -1$, that is, if $G_{\omega_1}(x,y)$ is the symmetric (Ω, H) Green kernel in ω_1, $u(x) = \int_{\omega_1} G_{\omega_1}(x,y)\, d\lambda(y)$ is finite continuous in ω_1. Since u is finite continuous on $\omega_1 \supset \omega_c$, u is bounded on ω_c.

Hence, for $x \in \omega$,

$$\int_{\omega} G(x,y)\, d\lambda(y) \le \int_{\omega} G_{\omega_1}(x,y)\, d\lambda(y) \le \int_{\omega_1} G_{\omega_1}(x,y)\, d\lambda(y) = u(x) < \infty.$$

Consequently, since h^* is bounded in ω and since $G_\omega(x,y)$ is symmetric by hypothesis, $\int_\omega G_\omega(x,y)\, {h^*}^+(y)\, d\lambda(y)$ and $\int_\omega G_\omega(x,y)\, {h^*}^-(y)\, d\lambda(y)$ are well-defined bounded potentials in ω; hence v is the difference of two bounded potentials and hence harmonizable; also v being biharmonic, is continuous. Thus v is a bounded Wiener function in ω and hence v extends continuously on the Wiener compactification of $\overline{\omega}$.

Since g is a finite continuous function on $\partial\omega$, there exists a bounded (Ω, H) harmonic function h_1 in ω tending to $g - v$ on Γ. Let $b = v + h_1$ in ω. Then b is a bounded biharmonic function in ω such that b tends to g on Γ, and $Lb = Lv = -h^*$ tends to f^* on Γ^*.

For the uniqueness of b, notice that if u is bounded biharmonic in ω and if Lu is bounded in ω^* such that u and Lu tend to 0 on Γ and Γ^* respectively, then $u \equiv 0$. For, Lu is bounded (Ω, H^*) harmonic in ω and tends to 0 on Γ^* and hence $Lu \equiv 0$; this means that u is bounded (Ω, H) harmonic in ω and tends to 0 on Γ and hence $u \equiv 0$. $\square$

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P. O. Box 60618, Riyadh, 11555, Saudi Arabia
E-mail address: ibajunaid@hotmail.com