OSCILLATION AND NONOSCILLATION THEOREMS
FOR SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS WITH \( p \)-LAPLACIAN

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Abstract. In this paper, we are concerned with the oscillation problem for the nonlinear differential equation
\[
(\phi_p(a(t)x'))' + b(t)g(x) = 0,
\]
where \( \phi_p(y) \) is the one-dimensional \( p \)-Laplacian operator, and \( g(x) \) satisfies \( xg(x) > 0 \) if \( x \neq 0 \).

1. Introduction

The purpose of this paper is to improve oscillation and nonoscillation theorems for the nonlinear differential equation
\[
(\phi_p(a(t)x'))' + b(t)g(x) = 0,
\]
where \( \phi_p(y) \) is a real-valued function defined by \( \phi_p(y) = |y|^{p-2}y \) with \( p > 1 \) a fixed real number, and \( a(t) \) and \( b(t) \) are positive and continuous on some half-line \((\alpha, \infty)\), and \( g(x) \) is a continuous function on \( \mathbb{R} \) satisfying the signum condition
\[
xg(x) > 0 \quad \text{if} \quad x \neq 0.
\]
We assume throughout this paper that
\[
\int_{\alpha}^{\infty} \frac{1}{a(t)} \, dt < \infty.
\]
A nontrivial solution \( x(t) \) of (1.1) is said to be oscillatory if there exists a sequence \( \{t_n\} \) tending to \( \infty \) such that \( x(t_n) = 0 \). Otherwise, it is said to be nonoscillatory.

When \( p = 2 \), equation (1.1) becomes the nonlinear self-adjoint differential equation
\[
(a(t)x')' + b(t)g(x) = 0,
\]
which has been devoted to the study of the oscillation problem by many authors (for example, see [3, 6, 7, 8, 9, 10]). Especially, using phase plain analysis of the Liénard system, Sugie et al. [8] discussed the oscillation problem for equation (1.4)

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whether the integral of the function $1/a(t)$ is divergent or convergent. In case (1.3), they gave the following pair of an oscillation theorem and a nonoscillation theorem.

**Theorem A** ([8, Theorem 3.1]). Let (1.2) and (1.3) hold. Suppose that $a(t)$ and $b(t)$ satisfy

$$a(t)b(t) \left( \int_t^\infty \frac{1}{a(\tau)} d\tau \right)^2 \geq 1$$

for $t$ sufficiently large, and that there exists a $\lambda$ with $\lambda > 1/16$ such that

$$\frac{g(x)}{x} \geq \frac{1}{4} + \frac{\lambda}{(\log |x|)^2}$$

for $|x|$ sufficiently small. Then all nontrivial solutions of (1.4) are oscillatory.

**Theorem B** ([8, Theorem 3.2]). Let (1.2) and (1.3) hold. Suppose that $a(t)$ and $b(t)$ satisfy

$$a(t)b(t) \left( \int_t^\infty \frac{1}{a(\tau)} d\tau \right)^2 \leq 1$$

for $t$ sufficiently large, and that $g(x)$ satisfies

$$G(x) = \int_0^x g(\chi) d\chi \leq \frac{1}{2} x^2 \quad \text{for} \quad x \in \mathbb{R}$$

and

$$\frac{g(x)}{x} \leq \frac{1}{4} + \frac{1}{16(\log |x|)^2}$$

for $x > 0$ or $x < 0$, $|x|$ sufficiently small. Then all nontrivial solutions of (1.4) are nonoscillatory.

Equation (1.1) also includes a half-linear differential equation. In the study of half-linear differential equations, its associated Riccati inequality plays an important role in the oscillation of solutions (see [1, 2, 4, 5]). Using Riccati technique, we can extend Theorems A and B to Theorems 1.1 and 1.2 below, respectively.

**Theorem 1.1.** Let (1.2) and (1.3) hold. Suppose that $a(t)$ and $b(t)$ satisfy

$$a(t)b(t) \left( \int_t^\infty \frac{1}{a(\tau)} d\tau \right)^p \geq 1$$

for $t$ sufficiently large, and that there exists a $\lambda$ with

$$\lambda > \frac{1}{2} \left( \frac{p-1}{p} \right)^{p+1}$$

such that

$$\frac{g(x)}{\phi_p(x)} \geq \left( \frac{p-1}{p} \right)^p + \frac{\lambda}{(\log |x|)^2}$$

for $|x|$ sufficiently small. Then all nontrivial solutions of (1.1) are oscillatory.
Theorem 1.2. Let (1.2) and (1.3) hold. Suppose that $a(t)$ and $b(t)$ satisfy

$$a(t)b(t)\left(\int_t^\infty \frac{1}{a(\tau)} d\tau\right)^p \leq 1$$

for $t$ sufficiently large, and that $g(x)$ satisfies

$$G(x) \leq \frac{p-1}{p} |x|^p \text{ for } x \in \mathbb{R}$$

and there exists a $\lambda$ with

$$0 < \lambda < \frac{1}{2} \left(\frac{p-1}{p}\right)^{p+1}$$

such that

$$\frac{g(x)}{\phi_p(x)} \leq \left(\frac{p-1}{p}\right)^p + \frac{\lambda}{\log|x|^2}$$

for $x > 0$ or $x < 0$, $|x|$ sufficiently small. Then all nontrivial solutions of (1.1) are nonoscillatory.

Remark. Since Theorem 1.1 coincides with Theorem A when $p = 2$, Theorem 1.1 is a complete generalization of Theorem A. Also Theorem 1.2 includes Theorem B if $\lambda \neq (\frac{p-1}{p})^{p+1}/2$. From Theorem B, we see that all nontrivial solutions of (1.1) are nonoscillatory in the case $\lambda = (\frac{p-1}{p})^{p+1}/2$ with $p = 2$. For this reason, we may conjecture that even if $\lambda = (\frac{p-1}{p})^{p+1}/2$ with $p \neq 2$, then all nontrivial solutions of (1.1) are nonoscillatory.

2. Transformation into a special case

In this section, we will find the canonical form of (1.1). For this purpose, we define

$$s = s(t) = \left(\int_t^\infty \frac{1}{a(\tau)} d\tau\right)^{-1}, \quad u(s) = x(t(s)),$$

where $t(s)$ is the inverse function of $s(t)$. Then we have

$$x'(t) = \frac{ds}{dt} \dot{u}(s) = \frac{s^2}{a(t)} \dot{u}(s),$$

$$\frac{d}{dt}\left(\phi_p(a(t)x'(t))\right)' = \frac{ds}{dt}\left(\frac{\phi_p(s^2 \dot{u}(s))'}{a(t)}\right) = \frac{s^2}{a(t)} \left(\phi_p(\dot{u}(s))' + \frac{2(p-1)}{s} \phi_p(\dot{u}(s))\right),$$

where $\dot{} = d/ds$, and therefore, equation (1.1) becomes the equation

$$\left(\phi_p(\dot{u})\right)' + \frac{2(p-1)}{s} \phi_p(\dot{u}) + \frac{a(t(s))b(t(s))}{s^{2p}} g(u) = 0.$$ 

Since $a(t)$ is positive for $t \geq \alpha$ and satisfies (1.3), the functions $s(t)$ and $t(s)$ are increasing and $s(t) \to \infty$ as $t \to \infty$. Hence, all nontrivial solutions of (1.1) are
Integrating both sides of this inequality from \( \varphi \) (2.2) \((\text{as an oscillation theorem and a nonoscillation theorem for equation})\) and therefore, \( t \) for \( t \geq T \), respectively.

Let \( c(s) = a(t(s))b(t(s))/s^{2p} \). Then conditions (1.5) and (1.8) coincide with \( s^{p}c(s) \geq 1 \) and \( s^{p}c(s) \leq 1 \), respectively. Thus, Theorems 1.1 and 1.2 are rewritten as an oscillation theorem and a nonoscillation theorem for equation

\[
(2.2) \quad (\phi_{p}(x'))' + \frac{2(p-1)}{t} \phi_{p}(x') + c(t)g(x) = 0,
\]

respectively.

**Proposition 2.1.** Assume (1.2) and suppose that \( c(t) \) satisfies

\[
(2.3) \quad t^{p}c(t) \geq 1
\]

for \( t \) sufficiently large, and that there exists a \( \lambda \) with (1.6) satisfying (1.7) for \( |x| \) sufficiently small. Then all nontrivial solutions of (2.2) are oscillatory.

**Proposition 2.2.** Let (1.2) and (1.9) hold. Suppose that \( c(t) \) satisfies

\[
(2.4) \quad t^{p}c(t) \leq 1
\]

for \( t \) sufficiently large, and that there exists a \( \lambda \) with (1.10) satisfying (1.11) for \( x > 0 \) or \( x < 0 \), \( |x| \) sufficiently small. Then all nontrivial solutions of (2.2) are nonoscillatory.

### 3. Proof of Proposition 2.1

To prove Proposition 2.1, we prepare some lemmas. We first consider the following lemma concerning properties of nonoscillatory solutions of (2.2).

**Lemma 3.1.** Assume (1.2) and suppose that \( c(t) \) satisfies (2.3) for \( t \) sufficiently large, and that equation (2.2) has a nonoscillatory solution. Then the solution tends to zero as \( t \to \infty \). Furthermore, if the solution is eventually positive, then its derivative is eventually negative.

**Proof.** Let \( x(t) \) be a nonoscillatory solution of (2.2). Then, without loss of generality, we may assume that there exists a \( T > 0 \) such that \( x(t) > 0 \) for \( t \geq T \).

To begin with, we will show that there exists a \( t_{1} \geq T \) such that \( x(t_{1}) < 0 \). By way of contradiction, we suppose that \( x'(t_{1}) > 0 \). Then \( x(t) \geq x(T) > 0 \) for \( t \geq T \). Hence, by (1.2) and (2.3), we have

\[
(t^{2(p-1)} \phi_{p}(x'(t)))' = -t^{2(p-1)} c(t) g(x(t)) \leq -t^{p-2} g(x(t)) < 0 \quad \text{for} \quad t \geq T.
\]

Integrating both sides of this inequality from \( T \) to \( t \), we get

\[
t^{2(p-1)} \phi_{p}(x'(t)) \leq T^{2(p-1)} \phi_{p}(x'(T)) \quad \text{for} \quad t \geq T,
\]

and therefore, \( x'(t) \leq T^{2} x'(T)/t^{2} \) for \( t \geq T \). Integrate this inequality to obtain

\[
x(t) \leq T^{2} x'(T) \left( \frac{1}{T} - \frac{1}{t} \right) + x(T) \leq T x'(T) + x(T) \quad \text{for} \quad t \geq T.
\]

Define \( m_{1} = \min \{ g(x) \mid x(T) \leq x \leq T x'(T) + x(T) \} \). Then we have

\[
(t^{2(p-1)} \phi_{p}(x'(t)))' \leq -t^{p-2} g(x(t)) \leq -t^{p-2} m_{1} \quad \text{for} \quad t \geq T.
\]
Integrating both sides of this inequality from $T$ to $t$, we get
\[ t^{2(p-1)} \phi_p(x'(t)) \leq -\frac{m_1}{p-1} (t^{p-1} - T^{p-1}) + T^{2(p-1)} \phi_p(x'(T)) \quad \text{for } t \geq T. \]

Hence, $x'(t)$ is negative for $t$ sufficiently large. This is a contradiction to the assumption that $x'(t) \geq 0$ for $t \geq T$. Thus, $x'(t_1) < 0$ for some $t_1 > T$.

Next, we will show that $x'(t) < 0$ for $t \geq t_1$. Suppose that there exists a $t_2 > t_1$ such that
\[ x'(t) < 0 \quad \text{for } t_1 \leq t < t_2 \quad \text{and} \quad x'(t_2) = 0. \]

Since $x(t)$ is a solution of (2.2), we have
\[ x''(t) = -\frac{2}{t} x'(t) - \frac{c(t)g(x(t))}{(p-1)|x'(t)|^{p-2}} = -x'(t) \left( \frac{2}{t} + \frac{c(t)g(x(t))}{(p-1)\phi_p(x'(t))} \right) \]
for $t_1 \leq t < t_2$. From (3.1) we see that
\[ \frac{2}{t} + \frac{c(t)g(x(t))}{(p-1)\phi_p(x'(t))} \to -\infty \quad \text{as } t \to t_2 - 0. \]

Hence, there exists a $\tau > 0$ such that $x''(t) < 0$ for $t_2 - \tau \leq t < t_2$, and therefore, $x'(t)$ is decreasing for $t_2 - \tau \leq t \leq t_2$. Thus, we obtain $x'(t) > x'(t_2)$ for $t_2 - \tau \leq t < t_2$, which is a contradiction to (3.1). We therefore conclude that $x(t)$ is decreasing for $t \geq t_1$.

Finally, we will show that $x(t)$ tends to zero as $t \to \infty$. Suppose that $x(t)$ does not tend to zero as $t \to \infty$. Since $x(t)$ is positive and decreasing for $t \geq t_1$, there exists a $\mu > 0$ such that $x(t) \to \mu$ as $t \to \infty$. Let $m_2 = \min\{g(x) \mid \mu \leq x \leq x(t_1)\}$.

Then we have
\[ (t^{2(p-1)} \phi_p(x'(t)))' \leq -t^{p-2} g(x(t)) \leq -t^{p-2} m_2 \quad \text{for } t \geq t_1. \]

Hence, integrating both sides of this inequality from $t_1$ to $t$, we get
\[ t^{2(p-1)} \phi_p(x'(t)) \leq -\frac{m_2}{p-1} (t^{p-1} - t_1^{p-1}) + t_1^{2(p-1)} \phi_p(x'(t_1)) \quad \text{for } t \geq t_1. \]

Let $L$ be a positive number satisfying $L^{p-1} < m_2/(p-1)$. Then there exists a $t_3 > t_1$ such that
\[ t^{2(p-1)} \phi_p(x'(t)) \leq -L^{p-1} t^{p-1} \quad \text{for } t \geq t_3, \]
and therefore, $x'(t) \leq -L/t$ for $t \geq t_3$. Thus, we see that
\[ x(t) \leq -L \log \frac{t}{t_3} + x(t_3) \quad \text{for } t \geq t_3. \]

This is a contradiction to the assumption that $x(t)$ is positive for $t \geq T$, thereby completing the proof. \[\square\]

We next consider some differential inequalities of the first order. For simplicity, we denote
\[ H(\xi) = (p-1) \left( (-\xi)^{p/(p-1)} + \xi + \frac{(p-1)^{p-1}}{p^p} \right) \]
for $\xi < 0$ and
\[
\gamma_p = \left(\frac{p-1}{p}\right)^{p-1}
\]
with $p > 1$ a fixed real number.

**Lemma 3.2.** Suppose that the differential inequality
\[
(3.2) \quad \dot{\xi} + H(\xi) \leq 0
\]
has a negative solution on $[s_0, \infty)$ with $s_0 > 0$. Then the solution tends to $-\gamma_p$ as $s \to \infty$.

**Proof.** Since
\[
H(-\gamma_p) = (p-1)\left\{\left(p-1\right)^p - \left(p-1\right)^{-1} + \frac{(p-1)^{p-1}}{p^p}\right\} = 0
\]
and
\[
\frac{d}{d\xi}H(\xi) = -p(-\xi)^{1/(p-1)} + p - 1,
\]
we see that $H(\xi) > 0$ if $\xi \neq -\gamma_p$. Let $\xi(s)$ be a negative solution of (3.2) on $[s_0, \infty)$. Then $\xi(s)$ satisfies
\[
(3.3) \quad \dot{\xi}(s) = -H(\xi(s)) \leq 0 \quad \text{for } s \geq s_0.
\]

Let $u(s)$ be the positive function defined by
\[
u(s) = \exp\left(-\int_{s_0}^s (-\xi(\sigma))^{1/(p-1)} d\sigma\right)
\]
for $s \geq s_0$. Differentiate $u(s)$ to obtain
\[
\dot{u}(s) = -u(s)(-\xi(s))^{1/(p-1)} < 0 \quad \text{for } s \geq s_0.
\]

Hence, we get
\[
(3.4) \quad \xi(s) = \frac{\phi_p(\dot{u}(s))}{\phi_p(u(s))} \quad \text{for } s \geq s_0.
\]

Differentiating both sides, we have
\[
\dot{\xi}(s) = \frac{(\phi_p(\dot{u}(s)))'}{\phi_p(u(s))} - (p-1) \left|\frac{\dot{u}(s)}{u(s)}\right|^p \quad \text{for } s \geq s_0.
\]

Hence, by (3.3) and (3.4), $u(s)$ satisfies
\[
(3.5) \quad \left(\phi_p(\dot{u}(s))\right)' + (p-1)\phi_p(\dot{u}(s)) + \left(\frac{p-1}{p}\right)^p \phi_p(u(s)) \leq 0 \quad \text{for } s \geq s_0.
\]

Put $v(s) = \dot{u}(s) + u(s)$. Then, by (3.5) we have
\[
\dot{v}(s) = \ddot{u}(s) + \dot{u}(s) \leq -\frac{(p-1)^{p-1}\phi_p(u(s))}{p^p|\dot{u}(s)|^{p-2}} < 0 \quad \text{for } s \geq s_0.
\]
Suppose that there exists an \( s_1 > s_0 \) such that \( v(s_1) < 0 \). Then we get \( v(s) \leq v(s_1) \) for \( s \geq s_1 \). Since \( u(s) > 0 \) for \( s \geq s_1 \), we obtain

\[
\dot{u}(s) = v(s) - u(s) < v(s_1) \quad \text{for } s \geq s_1.
\]

Therefore, we have

\[
u(s) = v(s_1)(s - s_1) + u(s_1) \to -\infty \quad \text{as } s \to \infty.
\]

This is a contradiction. Hence, we see that \( \dot{u}(s) + u(s) = v(s) \geq 0 \) for \( s \geq s_0 \). Thus, we get

\[
\xi(s) = \frac{\dot{u}(s)}{\phi_p(u(s))} \geq -1 \quad \text{for } s \geq s_0.
\]

Using (3.3) and (3.6), we can find a \( \mu \) such that \( -1 \leq \mu < 0 \) and \( \xi(s) \to \mu \) as \( s \to \infty \). If \( \mu \neq -\gamma_p \), there exists an \( s_2 \geq s_0 \) such that

\[
\dot{\xi}(s) \leq -H(\xi(s)) \leq -H((\mu - \gamma_p)/2) < 0
\]

for \( s \geq s_2 \). Then we obtain \( \xi(s) \to -\infty \) as \( s \to \infty \) which is a contradiction to (3.6). The proof is complete.

Lemma 3.3. Suppose that the differential inequality

(3.7)

\[
\dot{\eta} + \left( \eta + \frac{1}{2} \right)^2 + \frac{\delta}{s^2} \leq 0
\]

has a solution on \([s_0, \infty)\) with \( s_0 > 0 \) where \( \delta \) is a positive parameter. Then \( \delta \leq 1/4 \).

Proof. Let \( \eta(s) \) be a solution of (3.7) and define

(3.8)

\[
h(s) = -\dot{\eta}(s) - \left( \eta(s) + \frac{1}{2} \right)^2 \quad \text{for } s \geq s_0.
\]

Then we have

(3.9)

\[
h(s) \geq \frac{\delta}{s^2} \quad \text{for } s \geq s_0.
\]

Changing variable \( t = e^s \) and let \( u(s) \) be the positive function defined by

\[
x(t) = \exp \left( \int_{s_0}^{\log t} \eta(\sigma)d\sigma \right) \quad \text{for } t \geq e^{s_0}.
\]

Then, by (3.8) \( x(t) \) is a nonoscillatory solution of the linear differential equation

\[
x'' + \frac{2}{t} x' + \frac{1}{t^2} \left( \frac{1}{4} + h(t) \right) x = 0
\]

It follows from (3.9) and Sturm’s comparison theorem that all nontrivial solutions of

(3.10)

\[
x'' + \frac{2}{t} x' + \frac{1}{t^2} \left( \frac{1}{4} + \frac{\delta}{(\log t)^2} \right) x = 0
\]

are nonoscillatory. It is known that all nontrivial solutions of (3.10) are nonoscillatory if and only if \( \delta \leq 1/4 \) (for details, see [8]). The proof is now complete. \Box
We are now ready to prove Proposition 2.1.

Proof of Proposition 2.1. Let \( t_0 \) be a large number satisfying (2.3) for \( t \geq t_0 \) and let \( \varepsilon_0 \) be a small number satisfying (1.7) for \( 0 < |x| < \varepsilon_0 \). Since \( \varepsilon_0 \) is sufficiently small, by (1.6) we see that

\[
\frac{\gamma_p}{2}(1 + \varepsilon_0) \left( \frac{p - 1}{p} + \varepsilon_0 \right)^2 < \lambda.
\]

The proof is by contradiction. Suppose that equation (2.2) has a nonoscillatory solution \( x(t) \). Then, without loss of generality, we may assume that \( x(t) \) is eventually positive. By Lemma 3.1 there exists a \( t_1 > t_0 \) such that \( 0 < x(t_1) < \varepsilon_0 \) and \( x'(t) < 0 \) for \( t \geq t_1 \).

Making the change of variable \( t = e^s \), we can transform equation (2.2) into the equation

\[
(\phi_p(\dot{u}))' + (p - 1)\phi_p(\dot{u}) + e^{ps}c(e^s)g(u) = 0.
\]

Let \( u(s) \) be the solution of (3.12) corresponding to \( x(t) \). Then we have \( 0 < u(s) < \varepsilon_0 \) and \( \dot{u}(s) = tx'(t) < 0 \) for \( s \geq \log t_1 \). Define

\[
\xi(s) = \frac{\phi_p(\dot{u}(s))}{\phi_p(u(s))}
\]

which is negative for \( s \geq \log t_1 \). Differentiating \( \xi(s) \) and using (1.2), (1.7), (2.3) and (3.12), we have

\[
\dot{\xi}(s) = \frac{(\phi_p(\dot{u}(s)))'}{\phi_p(u(s))} - (p - 1)\left| \frac{\dot{u}(s)}{u(s)} \right|^p
\]

\[
= -(p - 1)\phi_p(\dot{u}(s)) - e^{ps}c(e^s)g(u(s)) - (p - 1)|\xi(s)|^{p/(p - 1)}
\]

\[
\leq - (p - 1)\xi(s) - \left( \frac{p - 1}{p} \right)^p - \frac{\lambda}{(\log u(s))^2} - (p - 1)|\xi(s)|^{p/(p - 1)}
\]

\[
(3.13) \quad = -H(\xi(s)) - \frac{\lambda}{(\log u(s))^2} \quad \text{for} \quad s \geq \log t_1.
\]

Hence, by Lemma 3.2 we see that

\[
(3.14) \quad \xi(s) \downarrow -\gamma_p \quad \text{as} \quad s \rightarrow \infty,
\]

and therefore, we have

\[
\frac{\dot{u}(s)}{u(s)} \geq -\frac{p - 1}{p} \quad \text{for} \quad s \geq \log t_1.
\]

Integrating both sides of this inequality from \( \log t_1 \) to \( s \), we obtain

\[
u(s) \geq u(\log t_1) \exp \left\{ -\frac{p - 1}{p} (s - \log t_1) \right\} \quad \text{for} \quad s \geq \log t_1.
\]

Hence, there exists an \( s_1 > \log t_1 \) such that

\[
u(s) \geq \exp \left\{ -\left( \frac{p - 1}{p} + \varepsilon_0 \right) s \right\} \quad \text{for} \quad s \geq s_1.
\]
Thus, together with (3.13), we get

\[
\dot{\xi}(s) \leq -H(\xi(s)) - \frac{\lambda}{((p-1)/p + \varepsilon_0)^2 s^2} \quad \text{for} \ s \geq s_1.
\]

From Taylor's expansion theorem, there exists a \(0 < K(\xi) < (p-1)/p\) such that

\[
H(\xi) = \frac{1}{2\gamma_p}(\xi + \gamma_p)^2 + \frac{p(p-2)}{6(p-1)^2}K(\xi)^{3-2p}(\xi + \gamma_p)^3 \quad \text{for} \ -\gamma_p \leq \xi < 0.
\]

Therefore, we can find an \(\varepsilon_1 > 0\) such that

\[
H(\xi) \geq \frac{1}{2\gamma_p(1 + \varepsilon_0)}(\xi + \gamma_p)^2 \quad \text{for} \ -\gamma_p \leq \xi \leq -\gamma_p + \varepsilon_1.
\]

By (3.14) there exists an \(s_2 \geq s_1\) such that \(-\gamma_p \leq \xi(s) \leq -\gamma_p + \varepsilon_1\) for \(s \geq s_1\). Let

\[
\eta(s) = \frac{\xi(s) - \varepsilon_0\gamma_p}{2\gamma_p(1 + \varepsilon_0)}.
\]

Then, from (3.17) we have

\[
H(\xi(s)) \geq \frac{1}{2\gamma_p(1 + \varepsilon_0)}(\xi(s) + \gamma_p)^2 = \frac{1}{2\gamma_p(1 + \varepsilon_0)}\{2\gamma_p(1 + \varepsilon_0)\eta(s) + \gamma_p(1 + \varepsilon_0)\}^2
\]

\[
= 2\gamma_p(1 + \varepsilon_0)\left(\eta(s) + \frac{1}{2}\right)^2 \quad \text{for} \ s \geq s_2.
\]

Hence, by (3.15) we obtain

\[
\dot{\eta}(s) = \frac{\dot{\xi}(s)}{2\gamma_p(1 + \varepsilon_0)} \leq \frac{1}{2\gamma_p(1 + \varepsilon_0)}\left\{-H(\xi(s)) - \frac{\lambda}{((p-1)/p + \varepsilon_0)^2 s^2}\right\}
\]

\[
\leq -\left(\eta(s) + \frac{1}{2}\right)^2 - \frac{\lambda}{2\gamma_p(1 + \varepsilon_0)((p-1)/p + \varepsilon_0)^2 s^2} \quad \text{for} \ s \geq s_2.
\]

Thus, from Lemma 3.3 we have

\[
\frac{\lambda}{2\gamma_p(1 + \varepsilon_0)((p-1)/p + \varepsilon_0)^2} \leq \frac{1}{4},
\]

which is a contradiction to (3.11). This completes the proof of Proposition 2.1. \(\square\)

4. PROOF OF PROPOSITION 2.2

Before proving Proposition 2.2, we will show that oscillatory solutions of (2.2) tend to zero.

**Lemma 4.1.** Let (1.2) and (1.9) hold. Suppose that \(c(t)\) satisfies (2.4) for \(t\) sufficiently large, and that equation (2.2) has a nontrivial oscillatory solution. Then the solution tends to zero as \(t \to \infty\).

**Proof.** Let \(x(t)\) be a nontrivial oscillatory solution of (2.2). Changing variable \(t = e^s\), we can transform equation (2.2) into the system

\[
\dot{u} = \phi_4(v),
\]

\[
\dot{v} = -(p-1)v - e^{ps}c(e^s)g(u),
\]

(4.1)
where \( q = p/(p-1) \). Let \((u(s), v(s))\) be the solution of (4.1) corresponding to \( x(t) \). Then there exists a sequence \( \{ s_n \} \) such that \( u(s_n) = 0 \). Consider the function

\[
U(u, v) = \begin{cases} 
\frac{1}{q} |v|^q + G(u) & \text{if } uv \leq 0, \\
G(u + \phi_q(v)) & \text{if } uv > 0.
\end{cases}
\]

Using (1.2) and (2.4), we have

\[
U(\Sigma_1)(u, v) = -\phi_q(v)(p-1)v + e^{p_\alpha c(e^*)}g(u) + g(u)\phi_q(v) \\
= -(p-1)|v|^q + (1 - e^{p_\alpha c(e^*)})g(u)\phi_q(v) \\
\leq -(p-1)|v|^q \leq 0 \quad \text{for } vu \leq 0,
\]

\[
\dot{U}(\Sigma_1)(u, v) = g(u + \phi_q(v))\{\phi_q(v) - (q-1)|v|^{q-2}(p-1)v - e^{p_\alpha c(e^*)}g(u)\} \\
= g(u + \phi_q(v))\{(1 - (p-1)(q-1))\phi_q(v) - (q-1)e^{p_\alpha c(e^*)}|v|^{q-2}g(u)\} \\
= -(q-1)e^{p_\alpha c(e^*)}|v|^{q-2}g(u)(u + \phi_q(v)) < 0 \quad \text{for } vu > 0.
\]

Hence, by (1.9) we obtain

\[
\lim_{s \to s_0 \pm 0} U(u(s), v(s)) = G(\phi_q(v(s_n))) \\
\leq \frac{p-1}{p}|v(s_n)|^{q-1} = \frac{1}{q}|v(s_n)|^q = \lim_{s \to s_0 \pm 0} U(u(s), v(s)).
\]

Put

\[
V(s) = \begin{cases} 
U(u(s), v(s)) & \text{if } s \neq s_n, \\
\lim_{s \to s_n} U(u(s), v(s)) & \text{if } s = s_n.
\end{cases}
\]

Then we conclude that the function \( V(s) \) is piecewise continuous and decreasing for \( s \geq s_1 \). Hence, \( V(s) \) is bounded, namely, there exists a \( B > 0 \) such that \( |v(s)| < B \) for \( s \geq s_1 \).

To complete the proof, it suffices to show that \( V(s) \) tends to zero as \( s \to \infty \).

Suppose that there exists a \( V_0 > 0 \) such that

\[
V(s) \searrow V_0 \quad \text{as } s \to \infty.
\]

Let \( S_{V_0} = \{(u, v) \mid U(u, v) < V_0 \} \). Then the solution \((u(s), v(s))\) does not enter \( S_{V_0} \) for \( s \geq s_1 \). The region \( S_{V_0} \) consists of two bounded and disjointed parts and encircles the origin.

We can find an \( \varepsilon_0 \) so small that

\[\{(u, v) \mid |u| < \varepsilon_0 \text{ and } |v| < \varepsilon_0\} \subset S_{V_0}\]

Since the positive orbit of (4.1) corresponding to \((u(s), v(s))\) rotates around the region \( S_{V_0} \) in a clockwise direction, there exists a sequence \( \{\sigma_n\} \) such that \( \sigma_n < s_n \), \( |u(\sigma_n)| = \varepsilon_0 \) and \( |v(\sigma_n)| > \varepsilon_0 \) for \( \sigma_n < s < s_n \). Hence, we have

\[
\varepsilon_0 = |u(s_n) - u(\sigma_n)| = \left| \int_{\sigma_n}^{s_n} \dot{u}(s) ds \right| = \left| \int_{\sigma_n}^{s_n} \phi_q(v(s)) ds \right| \leq \phi_q(B)(s_n - \sigma_n),
\]
and therefore,
\[ V_0 - V(s_1) \leq \lim_{s \to \infty} (U(u(s), v(s)) - U(u(\tau_1), v(\tau_1))) = \int_{s_1}^{\infty} \frac{d}{ds} U(u(s), v(s)) ds \]
\[ \leq -(p - 1) \sum_{n=1}^{\infty} |v(s)| ds \leq -(p - 1) \varepsilon_0^q \sum_{n=1}^{\infty} (s_n - \sigma_n) ds \]
\[ < -(p - 1) \varepsilon_0^q \sum_{n=1}^{\infty} \varepsilon_0 \phi_q(B) = -\infty, \]
which is a contradiction. The lemma is proved.

We are now able to prove Proposition 2.2.

**Proof of Proposition 2.2.** We prove only the case that condition (1.11) is satisfied for \( x > 0 \) sufficiently small, because the other case is carried out in the same manner.

By (1.10) and (3.16), there exist an \( \varepsilon_0 > 0 \) and an \( \varepsilon_1 > 0 \) such that
\[ H(\xi) \leq \frac{1 + \varepsilon_0}{2\gamma_p} (\xi + \gamma_p)^2 \quad \text{for} \quad -\gamma_p \leq \xi \leq -\left(\frac{p - 1}{p} - \varepsilon_1\right)^{p-1} \]
and
\[ (1 + \varepsilon_0)\lambda \leq \frac{1}{2\gamma_p} \left(\frac{p - 1}{p} - \varepsilon_1\right)^2. \]

Note that \( \varepsilon_1 \) depends on \( \varepsilon_0 \). Let
\[ \delta = \frac{(1 + \varepsilon_0)\lambda}{2\gamma_p ((p - 1)/p - \varepsilon_1)^2}. \]

Then \( 0 < \delta < 1/4 \). Define \( \eta(s) = -1/2 + z/s \), where \( z = (1 + \sqrt{1 - 4\delta})/2 > 0 \).
Then we see that \( \eta(s) \) satisfies equation
\[ \dot{\eta} = - \left(\eta + \frac{1}{2}\right)^2 - \frac{\delta}{s^2}. \]
We also see that there exists a \( \tau > 0 \) such that
\[ \eta(\tau) = -\frac{1}{2} + \frac{(1 + \varepsilon_0)}{2\gamma_p} \left(-\left(\frac{p - 1}{p} - \varepsilon_1\right)^{p-1} + \gamma_p\right) \]
and
\[ \eta(s) > -\frac{1}{2} \quad \text{for} \quad s \geq \tau. \]

Let \( \varepsilon_2 \) be a positive number satisfying
\[ \log \varepsilon_2 < -\left(\frac{p - 1}{p} - \varepsilon_1\right) \tau. \]
and (1.11) for \( 0 < x < \varepsilon_2 \).

The proof is by contradiction. Suppose that (2.2) has an oscillatory solution \( x(t) \). Then, from Lemma 4.1 we see that \( x(t) \) tends to zero as \( t \to \infty \). Let \( u(s) \)
be the solution of (3.12) corresponding to $x(t)$. Since $u(s)$ is also oscillatory and tends to zero as $s \to \infty$, there exist an $s_1$ and an $s_2$ such that

$$u(s_1) = u(s_2) = 0, \quad \dot{u}(s_1) > 0, \quad \dot{u}(s_2) < 0$$

and

$$0 < u(s) < \varepsilon_2 \quad \text{for} \quad s_1 < s < s_2.$$  

Note that we may assume that

$$e^{ps} c(e^s) \leq 1 \quad \text{for} \quad s_1 < s < s_2$$

by (2.4). Let

$$\xi(s) = \frac{\phi_p(\dot{u}(s))}{\phi_p(u(s))} \quad \text{for} \quad s_1 < s < s_2.$$  

Then, by (4.8) we have

$$\lim_{s \to s_1+0} \xi(s) = \infty \quad \text{and} \quad \lim_{s \to s_2-0} \xi(s) = -\infty.$$  

Since $\xi(s)$ is continuous on the bounded open interval $(s_1, s_2)$, there exist an $s_*$ and an $s^*$ such that $s_1 < s_* < s^* < s_2$,

$$\xi(s_*) = -\left(\frac{p-1}{p} - \varepsilon_1\right)^{p-1}, \quad \xi(s^*) = -\gamma_p$$

and

$$-\gamma_p \leq \xi(s) \leq -\left(\frac{p-1}{p} - \varepsilon_1\right)^{p-1} \quad \text{for} \quad s_* \leq s \leq s^*.$$  

Hence, we have

$$\frac{\dot{u}(s)}{u(s)} = \xi(s)^{1/(p-1)} \leq -\left(\frac{p-1}{p} - \varepsilon_1\right) \quad \text{for} \quad s_* \leq s \leq s^*.$$  

Integrating both sides of this inequality from $s_*$ to $s \leq s^*$, and using (4.7) and (4.9), we obtain

$$\begin{align*}
\log u(s) &\leq -\left(\frac{p-1}{p} - \varepsilon_1\right) (s - s_*) + \log u(s_*) \\
&< -\left(\frac{p-1}{p} - \varepsilon_1\right) (s - s_*) + \log \varepsilon_2 \\
&< -\left(\frac{p-1}{p} - \varepsilon_1\right) (s - s_*) - \left(\frac{p-1}{p} - \varepsilon_1\right) \tau \\
&= -\left(\frac{p-1}{p} - \varepsilon_1\right) (s - s_* + \tau) \quad \text{for} \quad s_* \leq s \leq s^*.
\end{align*}$$
Hence, together with (1.11), (3.12), (4.2), (4.10) and (4.12), we have
\[
\dot{\xi}(s) = \left(\frac{\phi_p(u(s))}{\phi_p(u(s))}\right) - (p-1) \left|\frac{\dot{u}(s)}{u(s)}\right|^p
\]
\[
= -\left((p-1)\frac{\phi_p(u(s)) - e^{\rho_0}c(e^s)g(u(s))}{\phi_p(u(s))}\right) - (p-1) \left|\frac{\dot{u}(s)}{u(s)}\right|^p
\]
\[
\geq -\left((p-1)\frac{|\xi(s)|^{p/(p-1)} + \xi(s) + \frac{(p-1)^{p-1}}{p^p}}{(p-1)/p - \epsilon_1)^2(\log u(s))^2} - \frac{\lambda}{(\log u(s))^2}
\]
\[
\geq -\frac{1 + \epsilon_0}{2\gamma_p} \left((\xi(s) + \gamma_p)^2 - \frac{\lambda}{(p-1)/p - \epsilon_1)^2(\log u(s))^2}
\]
\[
\geq -\frac{1 + \epsilon_0}{2\gamma_p} \left((\xi(s) + \gamma_p)^2 - \frac{\lambda}{(p-1)/p - \epsilon_1)^2(\log u(s))^2}
\]
for \(s_* \leq s \leq s^\ast\).

Put
\[
\zeta(s) = -\frac{1}{2} + \frac{1 + \epsilon_0}{2\gamma_p} \left((\xi(s) + s_\ast - \tau) + \gamma_p\right) \quad \text{for} \quad \tau \leq s \leq s^\ast - s_\ast + \tau.
\]
Then, from (4.3) we have
\[
\dot{\zeta}(s) = \frac{1 + \epsilon_0}{2\gamma_p} \left((\xi(s) + s_\ast - \tau)\right)
\]
\[
> -\left(\frac{1 + \epsilon_0}{2\gamma_p} \right)^2 \left((\xi(s) + s_\ast - \tau) + \gamma_p\right)^2 - \frac{1 + \epsilon_0}{2\gamma_p} \frac{\lambda}{((p-1)/p - \epsilon_1)^2 s^2}
\]
\[
= -\left(\zeta(s) + \frac{1}{2}\right)^2 - \frac{\delta s^2}{s^2} \quad \text{for} \quad \tau \leq s \leq s^\ast - s_\ast + \tau.
\]
By (4.5) and (4.11), we have
\[
\eta(\tau) = -\frac{1}{2} + \frac{1 + \epsilon_0}{2\gamma_p} \left((\xi(s_\ast) + \gamma_p\right) = \zeta(\tau).
\]
Comparing equation (4.4) and inequality (4.14), we see that
\[
\eta(s) \leq \zeta(s) \quad \text{for} \quad \tau \leq s \leq s^\ast - s_\ast + \tau.
\]
However, by (4.6), (4.11) and (4.13), we have
\[
\eta(s^\ast - s_\ast + \tau) > -\frac{1}{2} \quad \text{and} \quad \zeta(s^\ast - s_\ast + \tau) = -\frac{1}{2}.
\]
This is a contradiction to (4.15) with \(s = s^\ast - s_\ast + \tau\). The proof is now complete. □

REFERENCES


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