REAL HYPERSURFACES OF COMPLEX SPACE FORMS WITH
SYMmetric RICCI *-TENSOR

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Abstract. Real hypersurfaces M’s in non-flat complex space forms such that
the symmetric part of the Ricci *-tensor of M is a constant multiple of the metric
are classified.

1. INTRODUCTION

This note is a continuation of our previous paper [1].

Let \((M, \phi, \xi, \eta, g)\) be an almost contact metric manifold with Ricci tensor \(S\). The
Ricci *-tensor \(S^*\) is defined by

\[ S^*(X, Y) = \frac{1}{2} \text{trace}(Z \mapsto R(X, \phi Y)\phi Z), \quad X, Y \in TM. \]

An almost contact metric manifold is said to be *-Einstein if \(S^*\) is a constant
multiple of the metric \(g\) on the holomorphic distribution \(T^\circ M\).

It should be remarked that Ricci *-tensor is not symmetric, in general. Thus the
condition “*-Einstein” automatically requires a symmetric property of the Ricci
*-tensor.

On real hypersurfaces in almost Hermitian manifolds, almost contact structures
are naturally induced from the almost Hermitian structure of the ambient space.
In our previous paper [1], the first named author investigated real hypersurfaces in
non-flat complex space forms in terms of Ricci *-tensor. In particular, he classi-
fied *-Einstein real hypersurfaces in non-flat complex space forms whose structure
vector fields are principal.

The purpose of present note is to generalize the classification result of [1]. We
shall weaken the assumption “*-Einstein” to “the symmetric part of \(S^*\) is a constant
multiple of \(g\) on \(T^\circ M\)”. More precisely, we shall prove the following two results.

Theorem 1.1. Let \(M\) be a connected real hypersurface of \(P_n(C)\) of constant holo-
monic sectional curvature \(4c > 0\). Assume that the symmetric part \(\text{Sym}S^*\) of

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Ricci $*$-tensor of $M$ is a constant multiple of the induced metric over the holomorphic distribution and the structure vector field $\xi$ is a principal curvature vector. Then $M$ is an open subset of one of the following:

(i) a geodesic hypersphere of radius $r$ $(0 < r < \pi/(2\sqrt{c}))$,
(ii) a tube over a totally geodesic complex projective space $P_k(\mathbb{C})$ of radius $\pi/(4\sqrt{c})$, where $0 < k < n - 1$,
(iii) a tube over a complex quadric $Q_{n-1}$ of radius $r$ $(0 < r < \pi/(4\sqrt{c}))$.

Theorem 1.2. Let $M$ be a connected real hypersurface of $H_n(\mathbb{C})$ of constant holomorphic sectional curvature $4c < 0$. Assume that the symmetric part $\text{Sym} S^*_{\text{Ricci}}$ of Ricci $*$-tensor of $M$ is a constant multiple of the induced metric over the holomorphic distribution and the structure vector field $\xi$ is a principal curvature vector. Then $M$ is an open subset of one of the following:

(i) a geodesic hypersphere of radius $r$ $(0 < r < \infty)$,
(ii) a tube over a totally geodesic complex hyperbolic hyperplane of radius $r$ $(0 < r < \infty)$,
(iii) a tube over a totally real hyperbolic space $H^n(\mathbb{R})$ of radius $r$ $(0 < r < \infty)$,
(iv) a horosphere.

2. Preliminaries

A complex $n$-dimensional Kähler manifold of constant holomorphic sectional curvature $4c < 0$ is called a complex space form, which is denoted by $\tilde{M}_n(4c)$. A complete and simply connected complex space form is a complex projective space $P_n(\mathbb{C})$, a complex Euclidean space $\mathbb{C}^n$ or a complex hyperbolic space $H_n(\mathbb{C})$, according as $c > 0$, $c = 0$ or $c < 0$. Let $M$ be a real hypersurface of a non-flat complex space form $\tilde{M}_n(4c)$.

Take a local unit normal vector field $N$ of $M$ in $\tilde{M}_n(4c)$. Then the Riemannian connections $\tilde{\nabla}$ of $\tilde{M}_n(4c)$ and $\nabla$ of $M$ are related by the following Gauss formula and Weingarten formula:

$$\tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \quad X, Y \in \mathfrak{X}(M),$$
$$\tilde{\nabla}_X N = -AX, \quad X \in \mathfrak{X}(M).$$

Here $g$ is the Riemannian metric of $M$ induced by the Kähler metric $G$ of the ambient space $\tilde{M}_n(4c)$. The $(1,1)$-tensor field $A$ is called the shape operator of $M$ derived from $N$.

An eigenvector $X$ of the shape operator $A$ is called a principal curvature vector. The corresponding eigenvalue $\lambda$ of $A$ is called a principal curvature. As is well known, the Kähler structure $(J, G)$ of the ambient space induces an almost contact metric structure $(\phi, \xi, \eta, g)$ on $M$. In fact, the structure vector field $\xi$ and its dual 1-form $\eta$ are defined by

$$\eta(X) = g(\xi, X) = G(JX, N), \quad X \in TM.$$

The $(1,1)$-tensor field $\phi$ is defined by

$$g(\phi X, Y) = G(JX, Y), \quad X, Y \in TM.$$
One can easily check that this structure \((\phi, \xi, \eta, g)\) is an almost contact structure on \(M\), that is, it satisfies
\[
\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0.
\]
It follows that
\[
\nabla_X\xi = \phi AX.
\]

Let \(\tilde{R}\) and \(R\) be the Riemannian curvature tensors of \(\tilde{M}_n(4c)\) and \(M\), respectively.

From the expression of the curvature tensor \(\tilde{R}\) of \(\tilde{M}_n(4c)\), we have the following equations of Gauss and Codazzi:
\[
\begin{align*}
R(X, Y)Z &= c(g(Y, Z)X - g(X, Z)Y \\
&\quad + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z) \\
&\quad + g(AY, Z)AX - g(AX, Z)AY, \\
(\nabla_X A)Y - (\nabla_Y A)X &= c(\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi).
\end{align*}
\]

To close this section, we recall the following two fundamental results (See e.g., [2]).

**Lemma 2.1.** If \(\xi\) is a principal curvature vector, then the corresponding principal curvature \(\alpha\) is locally constant.

**Lemma 2.2.** Assume that \(\xi\) is a principal curvature vector and the corresponding principal curvature is \(\alpha\). If \(AX = \lambda X\) for \(X \perp \xi\), then we have
\[
(2\lambda - \alpha)A\phi X = (\alpha \lambda + 2c)\phi X.
\]

We refer to the reader [2] about general theory of differential geometry of real hypersurfaces in complex space forms.

### 3. \(*\)-Einstein Real Hypersurfaces

Let us denote by \(S^*\) the Ricci \(*\)-tensor of a real hypersurface \(M\) which is defined by
\[
S^*(X, Y) = \frac{1}{2}\text{trace}(Z \mapsto R(X, \phi Y)\phi Z).
\]

Then the Gauss equation implies that
\[
S^*(X, Y) = 2cn(g(X, Y) - \eta(X)\eta(Y)) - g(\phi A\phi AX, Y),
\]
for all \(X, Y \in TM\).

The Ricci \(*\)-operator \(Q^*\) is the linear endomorphism field associated to \(S^*\):
\[
S^*(X, Y) = g(Q^*X, Y), \quad X, Y \in TM.
\]

The trace \(\rho^*\) of \(Q^*\) is called the \(*\)-scalar curvature of \(M\).

Let \(T^oM\) be a distribution defined by a subspace
\[
T^o_xM = \{X \in T_xM : X \perp \xi_x\}
\]
in the tangent space \(T_xM\). The formulas (1) imply that the distribution \(T^oM\) is invariant under \(\phi\). The distribution \(T^oM\) is called the holomorphic distribution of
If the Ricci $\ast$-tensor is a constant multiple of the Riemannian metric for the holomorphic distribution, i.e.

$$S^\ast(X, Y) = \frac{\rho^\ast}{2(n-1)}g(X, Y)$$

for $X, Y \in T^\circ M$ on $M$, then $M$ is said to be a $\ast$-Einstein real hypersurface.

The first author proved the following results in [1].

**Proposition 3.1.** Let $M$ be a connected $\ast$-Einstein real hypersurface of $P_n(C)$ of constant holomorphic sectional curvature $4c > 0$, whose structure vector field $\xi$ is a principal curvature vector. Then $M$ is an open subset of one of the following:

(i) a geodesic hypersphere of radius $r$ ($0 < r < \pi/(2\sqrt{c})$),
(ii) a tube over a totally geodesic complex projective space $P_k(C)$ of radius $\pi/(4\sqrt{c})$, where $0 < k < n - 1$,
(iii) a tube over a complex quadric $Q_{n-1}$ of radius $r$ ($0 < r < \pi/(4\sqrt{c})$).

**Proposition 3.2.** Let $M$ be a connected $\ast$-Einstein real hypersurface of $H_n(C)$ of constant holomorphic sectional curvature $4c < 0$, whose structure vector field $\xi$ is a principal curvature vector. Then $M$ is an open subset of one of the following:

(i) a geodesic hypersphere of radius $r$ ($0 < r < \infty$),
(ii) a tube over a totally geodesic complex hyperbolic hyperplane of radius $r$ ($0 < r < \infty$),
(iii) a tube over a totally real hyperbolic space $H^n(R)$ of radius $r$ ($0 < r < \infty$),
(iv) a horosphere.

Now we take the symmetric part $\text{Sym} S^\ast$ and the alternate part $\text{Alt} S^\ast$ of Ricci $\ast$-tensor $S^\ast$ of $M$;

$$\text{Sym} S^\ast(X, Y) = \frac{1}{2}(S^\ast(X, Y) + S^\ast(Y, X)),$$
$$\text{Alt} S^\ast(X, Y) = \frac{1}{2}(S^\ast(X, Y) - S^\ast(Y, X)),$$

for any $X, Y \in TM$.

Using (2), we see that

$$\text{Sym} S^\ast(X, Y) = 2cn(g(X, Y) - \eta(X)\eta(Y))$$
$$- \frac{1}{2}g((\phi A\phi A + A\phi A\phi)X, Y),$$
(3)
$$\text{Alt} S^\ast(X, Y) = \frac{1}{2}g((A\phi A\phi - \phi A\phi A)X, Y).$$
(4)

4. Proof of main theorems

To prove our theorems, we need the following lemma.

**Lemma 4.1.** Let $M$ be a real hypersurface of a non-flat complex space form $\tilde{M}_n(4c)$. If $\xi$ is a principal curvature vector, then the Ricci $\ast$-tensor of $M$ is symmetric, i.e. $\text{Alt} S^\ast = 0$. 
Proof. Let $X$ be a unit principal curvature vector orthogonal to $\xi$ with principal curvature $\lambda$. From Lemma 2.2, the tangent vector $\phi X$ is also a principal curvature vector. By calculating (4), we get $\text{Alt} S^∗(X, Y) = 0$, for any $X \in T^o M$ and $Y \in TM$.

On the other hand, by the assumption, we have $\phi A \phi A \xi = 0$ and (1) shows $A \phi A \phi \xi = 0$. Thus, we get $\text{Alt} S^∗(\xi, Y) = 0$ for any $Y \in TM$. □

Proof of theorems. Now let $M$ be a real hypersurface in $\widetilde{M}_n(4c)$ with $c \neq 0$ whose $\text{Sym} S^∗$ is a constant multiple of $g$ over $T^o M$. Assume that the structure vector field $\xi$ is principal. Then Lemma 4.1 implies that $S^∗(X, Y) = \text{Sym} S^∗(X, Y)$ for $X, Y \in TM$. Hence $M$ is $*$-Einstein. This fact, together with Propositions 3.1 and 3.2, yields the required results. □

References


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