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MONOTONE MULTITERMINAL RESISTORS AND LARGE NETWORKS

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Abstract. We show how the set of strongly monotone and Lipschitz multiterminal resistors is closed under forming networks and choosing terminals. We show that an infinite network of multiterminal resistors can serve as an approximation for all large finite subnetworks. We show that we may obtain the hierarchy of infinite networks of A. Zemanian from the set of multiterminal resistors. The techniques include new notation, nodal analysis and loop space analysis.

1. Introduction

Ohtsuki and colleagues showed in [18] and [19] that a network H has a unique solution if constructed from multiport resistors with hybrid characteristic functions which are continuous and strongly monotone. Here we show first that if we assume (for simplicity) all resistors are Lipschitz, and choose a set of ports from the nodes of H, we obtain a multiport resistor of the same type, Lipschitz and strongly monotone, both voltage controlled and current controlled.

This paper is self contained, hence is accessible to mathematicians.

Much of the value in this work consists in it being a rethinking of multiterminal nonlinear resistors, with notation that allows us to deduce what we will obtain if we connect not just several multiterminal devices together, but many, and take limits. We show, using a nodal analysis, that if the multiterminal resistors are voltage controlled, with continuous, strictly monotone and coercive conductance functions, then a finite circuit fabricated from these elements is of the same type. These conditions, in the one variable case, mean that the conductance function is an increasing homeomorphism. Anderson et al. [1] studied the interconnection of two nonlinear n-ports, with each conductance function being the subdifferential of a convex function, without all the previous conditions. Their paper is an extension to nonlinear networks of the results of [2]. This refers back to [3] to describe the types of permitted interconnections of the two n-ports. Thus [1] is related to this

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paper, but here we do not focus on this question, but pass on to consider large networks and their limiting behaviour.

Our study forms part of the body of work that deals with DC operating points of circuits, as circuit components are resistors when the terminal voltages and currents are constant in time. In addition, it deals with the voltage-charge relationship for a capacitor network [12], and there are commercial multiterminal components to which the monotone operator theory applies. In [16], page 303, the authors report that the charge on a MOS transistor is a strongly monotonic function of the voltage, once parasitics are taken into account. Thus this paper could be called "monotone multiterminal capacitors and large networks," but traditionally the basic results are presented for resistors.

We especially consider large networks, that is, we study their limiting behaviour as the network becomes very large. Such a network is given by granular material, formed by a large number of bodies of conducting material, packed together and touching neighbouring bodies. We construct a theory that is relevant to those large finite networks that approximate some infinite networks, in that their voltage-current characteristics are all much the same. To do this, we study infinite networks of multiterminal resistors.

Following on from the seminal works [14], which gave a unique current flow in an infinite network of two terminal resistors, and [26], which analysed extremal problems on an infinite network, the hierarchy of "transfinite" resistive networks has been defined by A. Zemanian, [27], [28], [29] and [30]. The idea of a 1-network, the basic transfinite graph, is that one forms infinite loops from infinite paths, for example by shorting together some ends of the graph. These 1-networks have been used in a study of the classical equations of traffic flow in an infinite network [7] to give existence of equilibrium flows.

The motivation of this paper was to give a theory of transfinite resistive networks which was much more accessible, rather less general, and more powerful, in that the theory was rich and required few definitions. The conclusion is that we may construct transfinite resistive networks of all ranks as multiterminal resistors whose current-voltage characteristics are quite well understood.

We develop our approach to infinite networks of multiterminal resistors, using a current space analysis. The conclusion is simply that when suitable bounds are assumed, an infinite network of multiterminal resistors with Lipschitz and strongly monotone resistance functions, with a choice of some nodes as terminals, is a multiterminal resistor of the same type. Now, with infinite networks at our disposal, we are able to return to discuss finite networks, and show that we can describe the limiting behaviour of their voltage-current characteristics.

Existence and solution algorithms for general resistive networks, often containing only two terminal devices, received considerable attention around the 1960s and 70s, sometimes using degree theory [18], [21] and sometimes assuming mappings to be monotone [12], [21], [1]. Around the 1990s, there has been continuing interest in this sort of problem, using dissipativity [17], homotopy methods [24], [15], and the simplex method [25], as well as a survey [23] . Multiterminal resistors have been included in studies of network dynamics, for example [17]. The paper [8] shows that multiterminal representations can speed up calculations in SPICE.

The articles [5] and [6] and the book [30] give the most accessible account of some topics in the theory of 1-networks of two terminal resistors. The volume [22] presents a sound study of infinite networks.

Large networks of resistors have been measured experimentally, see [11].

2. Definitions

2.1. **Multiterminal resistors.** We will consider a multiterminal resistor or element to be our basic building block for constructing resistive circuits as explained in [9]. Rather than giving node to datum voltages, and modelling a multiterminal resistor as a digraph with n nodes and $n-1$ branches, one node having degree $n-1$ and the others degree one, together with an allowable set of branch currents and voltages, or equivalently as an $n-1$ port with half the terminals shorted, we use a more symmetric representation of terminal voltages as elements of a quotient vector space.

Given any set T we let $\mathbb{R}^T$ denote the functions $f : T \to \mathbb{R}$, and write 1_T for the element of $\mathbb{R}^T$ taking the value 1 at all $t \in T$. Write $\langle a \rangle$ for the linear span of $\{a\}$ for $a \in \mathbb{R}^T$. We suppose T is finite. Then the quotient space, $\mathbb{R}^T/\langle 1_T \rangle$, and $\langle 1_T \rangle^\perp = \{y \in \mathbb{R}^T : \sum y_t = 0\}$ are in duality under the pairing $(y, v + \langle 1_T \rangle) = \sum_{t \in T} v_t y_t$.

Definition 1. *We define a multiterminal resistor, or resistor for short, d, to be a nonempty finite set $T(d)$ of "terminals", together with a set*

$$G(d) \subset (\mathbb{R}^{T(d)}/\langle 1_{T(d)} \rangle) \times \langle 1_{T(d)} \rangle^\perp$$

of allowable terminal "voltages" and "currents."

The notation G_d is avoided because we put superscripts and subscripts on the d, which would give $G_{d_B^F}$. We use the convention [10] that a current flowing into a resistor through a terminal is positive. We call d voltage controlled if $G(d)$ is the graph of a function with domain $\mathbb{R}^{T(d)}/\langle 1_T(d) \rangle$, denoted $G(d)$ too, called the conductance function.

2.2. **Networks.** We wish to construct a network from some resistors by joining some of their terminals together. The first theorem in this paper gives simple conditions on the conductance functions which ensure that the network itself becomes a resistor of the same type if we choose some nodes as terminals.

Definition 2. *Let D denote a set of resistors, and let G denote the function taking each $d \in D$ to $G(d)$. Let N be a nonempty set of "nodes." Suppose that for all d, there is a function $\varphi_d : T(d) \to N$, giving $\varphi : D \to \prod_{d \in D} N^{T(d)}$, the incidence function. We call (D, G, N, φ) a network of of resistors, or multiterminal resistors if we wish to emphasise they need not be two terminal resistors.*

The network (D, G, N, φ) is called finite if D and N are finite. Given (D, G, N, φ), we say $d \in D$ is incident to $n \in N$ if there is $t \in T(d)$ such that $\varphi_d(t) = n$, and we say also that n is incident to d. We say nodes a and b are adjacent if they are both incident to the same resistor in D. We say resistors d and e are adjacent if they are both incident to the same node in N.

With loss of flexibility, but giving the same results, we could equate nodes with elements of a partition of $\bigcup_{d\in D} T(d)$, i.e. assume φ to be surjective. If φ is surjective we obtain a hypergraph on N, that is, [4], relaxing the finiteness assumed there, a family $\{E_m\}_{m\in M}$ of nonempty subsets of N, whose union is N, by taking E_d for $d \in D$, to be the nodes to which d is incident. This enables us to say (D, G, N, φ) is connected to mean φ is surjective and the hypergraph is connected [4], that is, the intersection graph of the edges is connected. This means that given any two nodes a and b, there is a finite sequence $x_0, \ldots, x_n$ of nodes with x_i and x_{i-1} adjacent for $i = 1$ to n, $x_0 = a$ and $x_n = b$.

A subnetwork of (D, G, N, φ) is a network $(D^*, G^*, N^*, \varphi^*)$ such that $D^* \subset D$, $N^* \subset N$, $\varphi_d(T_d) \subset N^*$ for all $d \in D^*$, and φ^* is the restriction of φ to D^*. A component is a maximal connected subnetwork. We say a network is locally finite if for each node there are only finitely many resistors incident to it. Given a network $H = (D, G, N, \varphi)$ we write $E(H)$ for its resistor set D, and $V(H)$ for its node set N. Given a network $H = (D, G, N, \varphi)$ and a set $E \subset D$ we write $\langle E\rangle$ for the subnetwork with resistor set E, incidence function $\varphi|_E$, and node set $\bigcup_{d\in E} \varphi_d(T(d))$. If $V \subset N$, $\langle V\rangle$ denotes the subnetwork with node set V, resistor set $\{d \in D : \varphi_d(T(d)) \subset V\}$ and incidence function the restriction of φ to its resistor set.

2.3. **Kirchhoff's Laws.**

Definition 3. *Let $H = (D, G, N, \varphi)$ be given, and suppose that i maps each $d \in D$ to $i_d \in \langle 1_{T(d)}\rangle^\perp$. For $n \in N$, the flow of i into n, $I(i, n) := -\sum_{d\in D}\sum_{\varphi_d(t)=n} i_d(t)$. We say i satisfies Kirchhoff's current law (KCL) on $A \subset N$ if for all $n \in A$, $I(i, n)$ is zero. If there is ambiguity we write $\{i_d\}_{d\in D}$ for i. We suppose that v maps each $d \in D$ to $v_d + \langle 1_{T(d)}\rangle \in \mathbb{R}^{T(d)}/\langle 1_{T(d)}\rangle$. We say v satisfies Kirchhoff's voltage law (KVL) on H if there exists $p : N \to \mathbb{R}$ such that for $d \in D$, $p \circ \varphi_d + \langle 1_{T(d)}\rangle = v_d + \langle 1_{T(d)}\rangle$. We say p is a potential for v.*

3. Finite networks of monotone resistors

Suppose X is a Banach space and X' is its dual. We say $M : X \to X'$ is strongly monotone [13], page 18, with constant $K > 0$ to mean that for x and y in X, $(M(x) - M(y), x - y) \geq K\|x - y\|^2$. We say $M : X \to X'$ is Lipschitz with constant K to mean that for x and y in X, $\|M(x) - M(y)\| \leq K\|x - y\|$. We write $\|M\|_{Lip}$ for the least such K. We say $M : X \to X'$ is strictly monotone [13] to mean that for x and y in X, $x \neq y$ implies $(M(x) - M(y), x - y) > 0$. We say $M : X \to X'$ is coercive [13], page 23, to mean that $(M(x), x)/\|x\| \to \infty$ as $\|x\| \to \infty$.

Our first result shows that by forming a finite network from voltage controlled resistors whose conductance functions are strictly monotone, coercive and continuous, we obtain, by using some of the nodes as terminals, a resistor whose conductance function is strictly monotone, coercive and continuous. We note that a hybrid resistor is really both current and voltage controlled if its characteristic function, (giving some port voltages and other port currents in terms of port currents and voltages on appropriate ports), is strictly monotone, coercive and continuous.

This is related to two terminal case studied in [12], where they allowed weaker conditions, depending on whether a resistor was a a tree branch or not. Although Dolezal [13] considers monotone multiports, there is no real overlap with this work. The paper [1] is strongly related, and yet quite different in approach. The closest result to this paper is Theorem 5 of [19], which was clarified in Remark 16 of [18]. The point of assuming conductances to be monotone is to give elegant, simple and positive results for what is a rather undeveloped theory. The theory does not cover transistors as resistors, because circuits with transistors can have multiple operating points, see [23].

Then in the next section we move on to an infinite network, and, having to control bounds, we show that by forming an infinite network from voltage controlled resistors whose conductance functions are strongly monotone and Lipschitz, we obtain, by using some of the nodes as terminals, a resistor whose conductance function is strongly monotone and Lipschitz. The point of using strongly monotone and Lipschitz maps is to obtain an elegant theory, and control the parameters when taking limits.

The non-Lipschitz theory uses weaker assumptions, and it is an open question as to what estimates would allow one to pass from finite to infinite networks in this setting. We solve the finite and non-Lipschitz case by a nodal analysis, but approach the infinite and Lipschitz case by a loop space analysis. We do the finite network theory partly to give a result without the difficulties of infinite networks, partly to show how a nodal analysis can appear, partly to include current sources, and partly because we use it in the subsection on approximation of finite subnetworks.

3.1. **Non-Lipschitz Case. Problem A** Let (D, G, N, φ) be a connected finite network of resistors. Let $B \subset N$ be given and let $e_B + \langle 1_B \rangle \in \mathbb{R}^B/\langle 1_B \rangle$, and $i^*_{N\setminus B} \in \mathbb{R}^{N\setminus B}$, with $i^*_{N\setminus B} \in \langle 1_N \rangle^\perp$ if B is empty. (We assume that just $i^*_{N\setminus B}$ is given if B is empty and just $e_B + \langle 1_B \rangle$ is given if $B = N$.) Does there exist $e + \langle 1_N \rangle \in \mathbb{R}^N/\langle 1_N \rangle$, with e extending e_B, and $i^* \in \langle 1_N \rangle^\perp$, extending $i^*_{N\setminus B}$, such that for all $d \in D$, there exists $i_d \in \langle 1_{T(d)} \rangle^\perp$, such that $(e \circ \varphi_d + \langle 1_{T(d)} \rangle), i_d) \in G(d)$, and such that for all $n \in N$,

$$\begin{aligned} I(\{i_d\}_{d\in D}, n) + i^*(n) &= 0, \text{ that is,} \\ \sum_{d\in D} \sum_{\varphi_d(t)=n} i_d(t) &= i^*(n)? \end{aligned} \tag{1}$$

A solution will be $(e + \langle 1_N \rangle, i) \in (\mathbb{R}^N/\langle 1_N \rangle, \prod_{d\in D} \langle 1_{T(d)} \rangle^\perp)$, or, if all $d \in D$ are voltage controlled, simply $e + \langle 1_N \rangle \in \mathbb{R}^N/\langle 1_N \rangle$. We note the contrast with the presentation in terms of current sources between nodes and voltage sources connected in series with network branches, used for example in [19]. The following result essentially appears in [19].

Theorem 1. *Let $H = (D, G, N, \varphi)$ be a connected finite network of resistors. Let $B \subset N$ be given and let $e_B \in \mathbb{R}^B$, and $i^*_{N\backslash B} \in \mathbb{R}^{N\backslash B}$, with $i^*_{N\backslash B} \in \langle 1_N \rangle^\perp$ if B is empty. Suppose each $d \in D$ is a voltage controlled resistor whose conductance function is strictly monotone, coercive and continuous. Then Problem A has a unique solution $e + \langle 1_N \rangle$.*

Proof. (a) Suppose $B = N$. Then (1) gives i^* and summing over n shows $i^* \in \langle 1_N \rangle$.

(b) Suppose $B = \emptyset$. Define $G : \prod_{d\in D} \mathbb{R}^{T(d)}/\langle 1_{T(d)} \rangle \to \prod_{d\in D} \langle 1_{T(d)} \rangle^\perp$ by, for each $d \in D$,

$$(G(\{e_d + \langle 1_{T(d)} \rangle\}_{d\in D}))_d = G(d)(e_d + \langle 1_{T(d)} \rangle).$$

Define $A : \mathbb{R}^N/\langle 1_N \rangle \to \prod_{d\in D} \mathbb{R}^{T(d)}/\langle 1_{T(d)} \rangle$ to be the linear operator mapping $e + \langle 1_N \rangle$ to the point in the product with dth component $e \circ \varphi_d + \langle 1_{T(d)} \rangle$. We have $\prod_{d\in D} \mathbb{R}^{T(d)}/\langle 1_{T(d)} \rangle$ with dual $\prod_{d\in D} \langle 1_{T(d)} \rangle^\perp$ under the pairing

$$(\{e_d + \langle 1_{T(d)} \rangle\}_{d\in D}, \{i_d\}_{d\in D}) = \sum_{d\in D} (e_d, i_d).$$

The dual $A' : \prod_{d\in D} \langle 1_{T(d)} \rangle^\perp \to \langle 1_N \rangle^\perp$ is given by $A'(\{i_d\}_{d\in D}) = i \in \langle 1_N \rangle^\perp$, where for all $n \in N$,

$$i(n) = \sum_{d\in D} \sum_{\varphi_d(t)=n} i_d(t).$$

We claim A is injective. Suppose $A(e + \langle 1_N \rangle) = 0$. Then for each d, $e \circ \varphi_d$ is constant on all terminals of d, and since H is connected, e is constant on N, proving the claim.

Let $i^* \in \langle 1_N \rangle^\perp$ be given. Now $e + \langle 1_N \rangle$ is a solution iff

$$A'GA(e + \langle 1_N \rangle) = i^*. \tag{2}$$

Now $A'GA$ is continuous, strictly monotone by the injectivity of A, and coercive, again because of the injectivity of A. By e.g. [20], Chapter III, Section 2.8, $A'GA$ is surjective, and injective by strict monotonicity, giving a unique solution to (2).

(c) Now we suppose B is equal to neither N nor $\emptyset$. Consider $\mathbb{R}^N$ as the direct sum $\mathbb{R}_B \oplus \mathbb{R}_{N\backslash B}$. Suppose e_B and $i^*_{N\backslash B}$ are given, and we want $e_{N\backslash B}$ and i^*_B with $(i^*_B, i^*_{N\backslash B}) \in \langle 1_N \rangle^\perp$, and

$$A'GA((e_B, e_{N\backslash B}) + \langle 1_N \rangle) = (i^*_B, i^*_{N\backslash B}). \tag{3}$$

Define $Q : \mathbb{R}^{N\backslash B} \to \mathbb{R}^N/\langle 1_N \rangle$ by $Q(x_{N\backslash B}) = (0, x_{N\backslash B}) + \langle 1_N \rangle$. Then the dual $Q' : \langle 1_N \rangle^\perp \to \mathbb{R}^{N\backslash B}$ is given by $Q'(x)(n) = x(n)$ for all $n \in N \setminus B$. Note Q is injective. Now (3) is equivalent to

$$Q'A'GA(Q(e_{N\backslash B}) + ((e_B, 0) + \langle 1_N \rangle)) = i^*_{N\backslash B}. \tag{4}$$

The map taking $e_{N\backslash B}$ to the left hand side is monotone, indeed strictly monotone, using injectivity of Q and A. It is likewise coercive, as well as continuous. Hence this map is a bijection, and (4) has a unique solution, solving Problem A. □

In the next result we require only that B contain at least one node, but of course we need it to contain at least two to have a nontrivial case.

Theorem 2. *Suppose $H = (D, G, N, \varphi)$ is a connected finite network, with each resistor being voltage controlled, with coercive, strictly monotone and continuous conductance function. Let $B \subset N$ be nonempty. Let us form a resistor d_B with terminal set B and the set $G(d_B)$ of allowable terminal voltages and currents being $(e_B + \langle 1_B\rangle, i^*_B)$, where $e_B + \langle 1_B\rangle$ is arbitrary and i^*_B is given by Problem A corresponding to $e_B + \langle 1_B\rangle$ and $i^*_{N\setminus B} = 0$. Then d_B is voltage controlled, and its conductance function $G(d_B)$, is continuous, strictly monotone and coercive.*

Proof. By definition, d_B is a resistor, and by Theorem 1 it is voltage controlled. We suppose that $B \neq N$, the case $B = N$ being altogether similar. Now $G(d_B)(e^*_B + \langle 1_B\rangle) = i^*_B$ iff there is $e^*_{N\setminus B}$ such that

$$A'GA((e^*_B, e^*_{N\setminus B}) + \langle 1_N\rangle) = (i^*_B, 0_{N\setminus B}). \tag{5}$$

Let us show monotonicity of $G(d_B)$. Suppose (5) holds, and likewise

$$A'GA((e^{'}_B, e^{'}_{N\setminus B}) + \langle 1_N\rangle) = (i^{'}_B, 0_{N\setminus B}). \tag{6}$$

Then

$$\begin{aligned}(i^{'}_B - i^*_B, e^{'}_B - e^*_B + \langle 1_B\rangle) &= (A'GA((e^{'}_B, e^{'}_{N\setminus B}) + \langle 1_N\rangle) - A'GA((e^*_B, e^*_{N\setminus B}) + \langle 1_N\rangle),\\ &\quad (e^{'}_B, e^{'}_{N\setminus B}) - (e^*_B, e^*_{N\setminus B}) + \langle 1_N\rangle)\\ &= \sum_{d\in D}(G(d)(e^{'} \circ \varphi_d + \langle 1_{T(d)}\rangle) - G(d)(e^* \circ \varphi_d + \langle 1_{T(d)}\rangle),\\ &\quad e^{'} \circ \varphi_d - e^* \circ \varphi_d + \langle 1_{T(d)}\rangle)\\ &\geq 0,\end{aligned}$$

since each $G(d)$ is monotone. Thus $G(d_B)$ is monotone. We claim it is strictly monotone. Suppose the left hand side is zero, then for all $d \in D$,

$$(G(d)(e^{'} \circ \varphi_d + \langle 1_{T(d)}\rangle) - G(d)(e^* \circ \varphi_d + \langle 1_{T(d)}\rangle), e^{'} \circ \varphi_d - e^* \circ \varphi_d + \langle 1_{T(d)}\rangle) = 0. \tag{7}$$

Hence, using the strict monotonicity of $G(d)$, for all nodes n incident to d, $e^{'}(n)$ and $e^*(n)$ differ by a constant. Since H is connected, $e^{'}$ and e^* differ by a constant, and therefore $G(d_B)$ is strictly monotone.

We claim $G(d_B)$ is coercive.

$$\begin{aligned}(i^*_B, e_B + \langle 1_B\rangle) &= (A'GA((e_B, e_{N\setminus B}) + \langle 1_N\rangle), (e_B, e_{N\setminus B}) + \langle 1_N\rangle)\\ &= \sum_{d\in D}(G(d)(e \circ \varphi_d + \langle 1_{T(d)}\rangle), e \circ \varphi_d + \langle 1_{T(d)}\rangle)\\ &\geq \sum_{d\in D} c_d(\|e \circ \varphi_d + \langle 1_{T(d)}\rangle\|)(\|e \circ \varphi_d + \langle 1_{T(d)}\rangle\|),\end{aligned}$$

where any norms may be used, since the spaces are finite dimensional, and for each d, $c_d : (0, \infty) \to (0, \infty)$ is increasing, and $c_d(x)$ diverges to ∞ as $x \to \infty$. Let

$A(d)$ denote the operator A followed by projection on $\mathbb{R}^{T(d)}/\langle 1_{T(d)}\rangle$. There exists $c \in (0,\infty)$ such that for all $e \in \mathbb{R}^N$,

$$\max_{d\in D} \|A(d)(e+\langle 1_N\rangle)\| \geq c\|e+\langle 1_N\rangle\|,$$

since A is injective and $\mathbb{R}^N/\langle 1_N\rangle$ is finite dimensional. Hence

$$\begin{aligned} (i_B^*, e_B + \langle 1_B\rangle,) &\geq \max_{d\in D} c_d(\|A(d)(e+\langle 1_N\rangle)\|)\|A(d)(e+\langle 1_N\rangle)\| \\ &\geq \max_{d\in D} c_d(c\|e+\langle 1_N\rangle\|)c\|e+\langle 1_N\rangle\| \\ &\geq \max_{d} c_d((c\|e_B+\langle 1_B\rangle\|)c\|e_B+\langle 1_B\rangle\|, \end{aligned}$$

assuming that we use the quotient norm from the ℓ^∞ norm on $\mathbb{R}^{T(d)}/\langle 1_{T(d)}\rangle$, and therefore $(i_B^*, e_B+\langle 1_B\rangle,)/\|e_B+\langle 1_B\rangle\| \to \infty$ as $\|e_B+\langle 1_B\rangle\| \to \infty$.

Now we claim that $G(d_B)$ is continuous. Suppose we have a convergent sequence, $e_B^n+\langle 1_B\rangle \to e_B^0+\langle 1_B\rangle$ as $n\to\infty$. Suppose that for all $n \in \{0,1,\ldots\}$, $e_{N\setminus B}$ satisfies

$$A'GA((e_B^n, e_{N\setminus B}^n)+\langle 1_N\rangle) = (i_B^n, 0) \in \langle 1_N\rangle^\perp.$$

Now $\{i_B^n\}_{n\in\mathbb{N}}$ is bounded since a monotone operator is locally bounded at points in the interior of its domain by [20], page 104. We want to show if a subsequence $i_B^{n(m)} \to i_B^*$ then $i_B^* = G(d_B)(e_B^0+\langle 1_B\rangle)$. We claim $\{e_N^n+\langle 1_N\rangle\}$ is bounded. Now

$$\begin{aligned} (i_B^n, e_B^n+\langle 1_B\rangle) &= ((i_B^n,0),(e_B^n,e_{N\setminus B}^n)+\langle 1_N\rangle) \\ &= \sum_{d\in D}(G(d)(e^n\circ\varphi_d+\langle 1_{T(d)}\rangle), e^n\circ\varphi_d+\langle 1_{T(d)}\rangle), \end{aligned}$$

and coercivity gives $\{e^n\circ\varphi_d+\langle 1_{T(d)}\rangle\}$ bounded for each $d\in D$, so that $\{e_N^n+\langle 1_N\rangle\}$ is bounded. Taking a subsequence $n(m(k))$, we have i_B^* and $e_{N\setminus B}^*$ such that $(e_B^{n(m(k))}, e_{N\setminus B}^{n(m(k))})+\langle 1_N\rangle \to (e_B^0, e_{N\setminus B}^*)+\langle 1_N\rangle$, and $i_B^{n(m(k))} \to i_B^*$. Then

$$A'GA((e_B^0, e_{N\setminus B}^*)+\langle 1_N\rangle) = (i_B^*, 0)$$

since G is continuous, giving

$$i_B^* = G(d_B)(e_B^0+\langle 1_B\rangle).$$

□

4. 1-networks of monotone resistors

In this section we define a 1-network in terms of ends, rather than more general terminals as in [6], to keep it as simple as we can. We note G plays no role in this definition, and in some others.

Definition 4. *We define an end in a connected, infinite, locally finite network $H = (D,G,N^0,\varphi)$ to be a function e mapping each finite subset F of D to an infinite component $e(F)$ of $(D\setminus F, G|_{D\setminus F}, N^0, \varphi|_{D\setminus F})$ such that if $E\subset F$ then $V(e(F))\subset V(e(E))$.*

Assumption L We will assume $H = (D, G, N^0, \varphi)$ to be an infinite locally finite network with a finite number of components, all infinite, and a finite number of ends in each.

Definition 5. *A 1-node is a nonempty set of ends. We assume every end is in exactly one 1-node, and denote the set of 1-nodes by N^1. We will call $H^1 = (D, G, N^0, \varphi, N^1)$ a 1-network.*

Slightly more elaborate 1-networks are useful, and are constructed by including a node $n \in N^0$ in any 1-node, and we shall do this in the proof of the next theorem.

Suppose i satisfies KCL on N^0, and $W \subset N^0$ is given. We define the flow $I(i, W)$ of i into W to be $\sum_{d\in D} \sum_{\varphi_d(t)\notin W} i_d(t)$, if this converges absolutely. We write $I(i, n)$ for $I(i, \{n\})$, if $n \in N^0$; see Definition 3.

Suppose i satisfies KCL on N^0 and e is an end in a component of H. Suppose that F is a finite subset of D, such that if $f \neq e$ an end, then $e(F) \neq f(F)$. Suppose $E \supset F$. Then $I(i, V(e(E))) = I(i, V(e(F)))$. We define the flow of i into e, $I(i, e)$, to be $I(i, V(e(E)))$. We define the flow of i into a 1-node n to be $I(i, n) := \sum_{e\in n} I(i, e)$. We say that KCL holds at $n \in N^1$ to mean the flow of i into n is 0.

4.1. **Lipschitz Case.** We remark that a Lipschitz strongly monotone function from a reflexive Banach space to its dual is bijective, and its inverse is a Lipschitz strongly monotone function too. The next theorem is expressed in terms of resistance functions.

Assumption M Let $H^1 = (D, G, N^0, \varphi, N^1)$ be a 1-network. Suppose L_1, L_2, and L_3 are positive reals. Suppose that for each d, $G(d)$ is strongly monotone and Lipschitz, with inverse $R(d)$, called the resistance function. We use the ℓ^1 norm on $\langle 1_{T(d)}\rangle^\perp$, and the quotient norm from the ℓ^∞ norm on $\mathbb{R}^{T(d)}/\langle 1_{T(d)}\rangle$. We assume the following bounds. Suppose that for each $d \in D$, there is a number $r_d > 0$, and the following hold. For all x and x^*, and all $d \in D$,

$$\|R(d)(x) - R(d)(x^*)\| \le L_1 r_d \|x - x^*\| \tag{8}$$

$$(R(d)(x) - R(d)(x^*), x - x^*) \ge L_2 r_d \|x - x^*\|^2 \tag{9}$$

$$\|R(d)(0)\| \le L_3 r_d \tag{10}$$

$$\sum_{d\in D} r_d < \infty. \tag{11}$$

4.2. **A 1-network is a resistor.** In this subsection we give our main existence results for 1-networks. They are perhaps easier to think of if we take $B \subset N^1$.

Theorem 3. *Suppose $H^1 = (D, G, N^0, \varphi, N^1)$ is a 1-network satisfying Assumptions L and M. Let $B \subset N := N^1 \cup N^0$ be nonempty and finite. Let $e_B + \langle 1_B\rangle \in \mathbb{R}^B/\langle 1_B\rangle$ be given. There exists a unique $x \in \prod_{d\in D}\langle 1_{T(d)}\rangle^\perp$, such that :*

1) $\sum_{d\in D} \|x_d\|^2 r_d < \infty$ (finite power),

2) x satisfies KCL at all $n \in N \setminus B$, *3) For all $i \in \prod_{d \in D} \langle 1_{T(d)} \rangle^{\perp}$ satisfying the preceding conditions 1) and 2) with x replaced by i,*

$$\sum_{d \in D} (R(d)x_d, i_d) + \sum_{b \in B} I(i, b)e_b = 0. \tag{12}$$

(Tellegen's equation).

Proof. We form the multiterminal resistor $d(B)$ with terminal set $T(d(B))$ indexed by B, $T(d(B)) = \{t_b : b \in B\}$, and resistance function $R(d(B))$ being the constant function $e_B + \langle 1_B \rangle$. We form the augmented 1-network H^* with resistor set $D^* = D \cup \{d(B)\}$, with incidence function the extension φ^* of φ given by $\varphi^*_{d(B)}(t_b) = b$ for all $b \in B$. With respect to H^*, the flow of the current i into the node $b \in B \cap N^1$ is now $I(i, b) := \sum_{e \in b} I(i, e) - i_{d(B)}(t_b)$, which will be zero for KCL to hold.

We let K consist of functions $i \in \prod_{d \in D^*} \langle 1_{T(d)} \rangle^{\perp}$ satisfying KCL on N^0 and on N^1, and such that

$$\|i\|^2 := \sum_{d \in D} r_d \|i_d\|^2 < \infty.$$

One checks that this gives a norm, for if $\|i\| = 0$, then for all $d \in D$, $i_d = 0$, so for $b \in B$, the flow of $i|_D$ into b is zero, and hence $i_{d(B)}(t_b) = 0$. One checks that K is complete, thus a Banach space. Note that we have an equivalent norm $\|i\|_{D^*}$ by setting $r_{d(B)} = 1$, and

$$\|i\|^2_{D^*} := \sum_{d \in D^*} r_d \|i_d\|^2.$$

Note K is reflexive, being a closed subspace of the space consisting of $i \in \prod_{d \in D^*} \langle 1_{T(d)} \rangle^{\perp}$ such that

$$\|i\|^2_{D^*} := \sum_{d \in D^*} r_d \|i_d\|^2 < \infty. \tag{13}$$

We claim that we may define $R : K \to K'$, the dual space, by: for i and i^* in K,

$$(Ri, i^*) = \sum_{d \in D} (R(d)i_d, i^*_d).$$

Let us define $\varrho_d := r_d^{-1} R(d)$.

$$\begin{aligned} (Ri, i^*) &= \sum_{d \in D} r_d (\varrho_d i_d, i^*_d) \\ &\leq \sum_{d \in D} r_d \|\varrho_d i_d\| \|i^*_d\| \\ &\leq \sqrt{\sum_{d \in D} r_d \|\varrho_d(i_d)\|^2} \|i^*\|. \end{aligned}$$

By (8) and (10),

$$\|\varrho_d i_d\| \leq L_3 + L_1 \|i_d\|. \tag{14}$$

Hence

$$(Ri, i^*) \leq (L_3\sqrt{\sum r_d} + L_1\|i\|)\|i^*\|. \tag{15}$$

This shows, using (11), that R maps K to K'. We claim R is strongly monotone. Let i and i^* be in K. Then by (9),

$$\begin{aligned}(Ri - Ri^*, i - i^*) &= \sum_{d\in D} r_d(\varrho_d i_d - \varrho_d i^*_d, i_d - i^*_d)\\ &\geq \sum r_d L_2 \|i_d - i^*_d\|_1^2\\ &= L_2\|i - i^*\|^2.\end{aligned}$$

We claim R is Lipschitz continuous. Let i and i^* be in K. Then

$$\begin{aligned}\|Ri - Ri^*\| &= \sup_{\|j\|\leq 1} \sum r_d(\varrho_d i_d - \varrho_d i^*_d, j_d)\\ &\leq \sup_{\|j\|\leq 1} \sum r_d L_1\|i_d - i^*_d\|\|j_d\|\\ &\leq L_1 \sup_{\|j\|\leq 1} \sqrt{\sum r_d\|i_d - i^*_d\|^2}\sqrt{\sum r_d\|j_d\|^2}\\ &\leq L_1\|i - i^*\|.\end{aligned}$$

Hence $R : K \to K'$ is bijective [20], page 121. Now $e_B + \langle 1_B\rangle$ gives an element f of K' by $(f, i) = (e_B + \langle 1_B\rangle, i_{d(B)})$. Therefore, there is a unique $x \in K$ such that for all $i \in K$,

$$(Rx, i) + \sum_{b\in B} e_b i_{d(B)}(t_b) = 0. \tag{16}$$

We have $i_{d(B)}(t_b) = I(i|_D, b)$ for all $b \in B$, giving (12).

□

Before we show in our next theorem that these resistors that we obtain from a 1-network have a strongly monotone and Lipschitz conductance function, we need to say what we mean by a 1-network being 1-connected.

Definition 6. *We say that the 1-network H^1 is 1-connected if for any ends e and f there is a finite sequence $\{G_1, \ldots, G_K\}$ of components of (D, G, N^0, φ) with e an end in G_1, f an end in G_K, and, if $K > 1$, for $j = 1$ to $K - 1$, there are ends in G_j and in G_{j+1}, both in the same 1-node.*

Theorem 4. *In Theorem 3, assuming H^1 is 1-connected, we obtain a a voltage controlled resistor d_B say, with terminal set B, and with conductance function $G(d_B)$ which is Lipschitz and strongly monotone, defined by $G(d_B)(e_B + \langle 1_B\rangle)(b) = -I(x, b)$ for $b \in B$, x being given by Theorem 3.*

Proof. The map $G(d_B)$ is defined for all terminal voltages by Theorem 3, and we have only to check that it is Lipschitz and strongly monotone. Suppose $e_B + \langle 1_B\rangle$

and $e^*_B+\langle 1_B\rangle$ are given. Let x and x^* be the corresponding currents. For all $i\in K$,

$$\sum_{d\in D}(R(d)x_d - R(d)x^*_d, i_d) + \sum_{b\in B}(e_b - e^*_b)i_{d(B)}(b) = 0.$$

Note that for all ends e, the linear map $x\mapsto I(x,e)$ is bounded, since $I(x,e) = I(x,V(e(E))$ for any large E, and this is a finite sum of terms $i_d(t)$. Put $i = x - x^*$ to give

$$\begin{aligned}
\|x_{d(B)} - x^*_{d(B)}\|\ \|e_B - e^*_B + \langle 1_B\rangle\| &\geq -\sum_{b\in B}(e_b - e^*_b)(x_{d(B)}(b) - x^*_{d(B)}(b)) \\
&= \sum_{d\in D}(R(d)x_d - R(d)x^*_d, x_d - x^*_d) \\
&\geq \sum_{d\in D} L_2 r_d\|x_d - x^*_d\|^2 \\
&\geq L\|x_{d(B)} - x^*_{d(B)}\|^2 \qquad (17)
\end{aligned}$$

for some L independent of x and x^* since $x\mapsto x_{d(B)}$ is bounded. Hence $G(d_B)$ has Lipschitz constant L^{-1}.

There is some scope for giving the following definitions in different ways.

Given $H^1 = (D, G, N^0, \varphi, N^1)$, and H^* as in Theorem 3,we refer to $d:\mathbb{Z}\to D$ such that for all i and j, d_i is adjacent to d_j iff $|i-j| = 1$, as an endless path of resistors (in H^1, if there is ambiguity). We say it is from end e to end f, if for all finite subsets F of D, there exists $k\in\mathbb{N}$ such that d_i is in $E(f(F))$ for $i>k$ and in $E(e(F))$ for $i<-k$. We say it is from 1-node n to 1-node m, if it is from an end $e\in n$ to an end $f\in m$.

An endless path of resistors and terminals from 1-node n to 1-node m is an endless path of resistors together with sequences of terminals $s_i\in T(d_i)$ and $t_i\in T(d_i)$ with $\varphi_{d_i}(t_i) = \varphi_{d_{i+1}}(s_{i+1})$ for all $i\in\mathbb{Z}$. Analogously we define a one-ended path of resistors and terminals from node n to 1-node m using sequences defined on $\mathbb{N}$ in place of $\mathbb{Z}$. We define a one-ended path of resistors and terminals from 1-node n to node m using sequences defined on $-\mathbb{N}$ in place of $\mathbb{Z}$. Analogously we define a path of resistors and terminals from node n to node m using finite sequences.

A loop of resistors and terminals in H^* is a finite sequence $d_i : i\in\mathbb{Z}_K$, defined on the additive group $\mathbb{Z}_K$, for some $K\geq 2$, with d_i and d_j adjacent iff $i-j=\pm 1$, together with sequences of terminals $s_i\in T(d_i)$ and $t_i\in T(d_i)$ with $\varphi_{d_i}(t_i) = \varphi_{d_{i+1}}(s_{i+1})$ for all $i\in\mathbb{Z}_K$.

A 1-loop in H^* is a finite sequence $d^k : k\in\mathbb{Z}_K$ of paths, one ended paths and endless paths of resistors and terminals, for some $K\geq 2$, such that there is an injective sequence $n^k : k\in\mathbb{Z}_K$ of nodes and 1-nodes, with d^i from n^{i-1} to n^i for all $i\in\mathbb{Z}_K$, and d^k_i and d^h_j adjacent iff $k=h$ and $|i-j|=1$. A 1-path in H^* is defined similarly, using a finite sequence $\{d^k : k = 0\ldots, K-1\}$, where $K\geq 1$, and nodes or 1-nodes n^0 to n^K, so we do not require a path, one-ended path or endless path from n^K to n^0. A loop current of value $Z\in\mathbb{R}$ is defined to be a function $z\in\prod_{d\in D^*}\mathbb{R}^{T(d)}$ such that there is a loop of resistors and terminals and for each s_i and t_i in $T(d_i)$ (with $\varphi_{d_i}(t_i) = \varphi_{d_{i+1}}(s_{i+1})$), we have $z_{d_i}(s_i) = Z$, $z_{d_i}(t_i) = -Z$, and

$z_d(t) = 0$ otherwise. Analogously, a 1-loop current of value $Z \in \mathbb{R}$ is a function $z \in \prod_{d\in D^*} \mathbb{R}^{T(d)}$ such that there is a 1-loop of resistors and terminals and for each s_i^k and t_i^k in $T(d_i^k)$ with $\varphi_{d_i^k}(t_i^k) = \varphi_{d_{i+1}^k}(s_{i+1}^k)$, we have $z_{d_i^k}(s_i^k) = Z$, $z_{d_i^k}(t_i^k) = -Z$, and $z_d(t) = 0$ otherwise.

Now we use 1-loop currents in H^* to show that for some $L \in \mathbb{R}$,

$$\|e_B - e_B^* + \langle 1_B\rangle\| \le L\|x - x^*\|. \tag{18}$$

Now

$$\|e_B - e_B^* + \langle 1_B\rangle\| = \frac{1}{2}|(e_B - e_B^*)(n) - (e_B - e_B^*)(m)| \tag{19}$$

for some m and n in B. Since H^1 is 1-connected, there exists a 1-path P of resistors and terminals from n to m, in H^1. From the path of resistors and terminals in H^* from m to n, given by $d(B)$, t_m and t_n, and from P, we form a loop or 1-loop, P^* say. Let i be the 1-loop or loop current of value 1 in P^*. Now $i \in K$, and by (12),

$$\sum_{d\in P}(R(d)x_d - R(d)x_d^*, i_d) + \sum_{b\in B} I(i,b)(e_b - e_b^*) = 0. \tag{20}$$

Therefore

$$\begin{aligned}
|(e_B - e_B^*)(n) - (e_B - e_B^*)(m)| &\le 2\sum_{d\in P}\|R(d)x_d - R(d)x_d^*\| \\
&\le 2\sum_{d\in D} r_d L_1\|x_d - x_d^*\| \\
&\le 2(\sum r_d)^{1/2}\|x - x^*\|.
\end{aligned}$$

This gives (18). By this and (17),

$$\begin{aligned}
\|e_B - e_B^* + \langle 1_B\rangle\|^2 &\le -\sum_{b\in B}(e_b - e_b^*)(x_{d(B)}(b) - x_{d(B)}^*(b)) \\
&= (G(d_B)e_B - G(d_B)e_B^*, e_B - e_B^*),
\end{aligned}$$

and $G(d_B)$ is strongly monotone. Note that $G(d_B)$ gives the currents into each terminal $b \in B$ of d_B, which is minus the current into the terminal t_b of $d(B)$. □

4.3. **Approximation of Finite Subnetworks.** We show that if we approximate a 1-network by finite subnetworks formed by shorting together all nodes near each 1-node, their conductance functions approximate that of the 1-network. Thus, under Assumption F, a 1-network serves as a metaphor, or approximation, for any finite network which is large enough but of unknown size. For clarity we restrict ourselves by making assumptions about connectedness.

Assumption F In the following, we let H^1 be a 1-connected 1-network satisfying Assumptions L and M, and let B be a nonempty finite subset of N. Let F denote a finite subset of D. We say F satisfies Assumption F if for any component G of H^1, $\langle F \cap E(G)\rangle$ is connected, there is no finite component of $\langle E(G) \setminus F\rangle$, for each end e in G there is only one end in $e(F)$, and F contains all resistors adjacent to any node $n \in N^0$ in the given terminal set B.

Given $e_B+\langle 1_B\rangle$, and F finite, with $F \subset D$, and supposing F satisfies Assumption F, we consider the following finite network with resistor set F. For all $a \in N^1$, we identify all $n \in V(e(F)) \cap V\langle F\rangle$ such that $e \in a$, to form a node a^F. Then our finite network has resistor set F, conductance function $G|_F$, node set the equivalence classes of $V\langle F\rangle$ given by this identification, and corresponding incidence function. Note it is connected. We have a resistor d_B^F say, given by this network with terminal set $B^F := (B \cap N^0) \cup \{b^F : b \in B \cap N^1\}$, and conductance function $G(d_B^F)$, say. We have an identification of B and B^F.

Given e_B on B, this identification induces e_B on B^F, and denote by $(p^F + \langle 1_{V\langle F\rangle}\rangle, x^F)$ the solution given by Theorem 1, so p^F gives the nodal voltages extending e_B, corresponding to a current x^F. Thus $I(x^F, b^F) = -G(d_B^F)(e_B + \langle 1_B\rangle)(b^F)$. The next result shows that for all large F, these conductances are approximately equal.

Theorem 5. *Let $H^1 = (D, G, N^0, \varphi, N^1)$ be a 1-connected 1-network satisfying Assumptions L and M, and let B be a nonempty finite subset of N. Given $M > 0$ and $\epsilon > 0$, there exists a finite $F_0 \subset D$ such that for F satisfying Assumption F, with $F_0 \subset F \subset D$, and $\|e_B + \langle 1_B\rangle\| \le M$, for all $b \in B$,*

$$|I(x^F, b^F) - I(x, b)| \le \epsilon.$$

Proof. (a) We bound $\|x^F\| := \sqrt{\sum_{d\in F} r_d \|x_d^F\|^2}$ independently of F.

$$\begin{aligned}
\|x^F\|^2 &= \sum_{d\in F} r_d \|x_d^F\|^2 \\
&\le L_2^{-1}\Big(\sum_{d\in F} (R(d)x_d^F, x_d^F) + L_3 r_d \|x_d^F\|\Big) \\
&\le L_2^{-1}\Big(\Big|\sum_{b\in B} I(x,b)e_b\Big| + L_3 \sum_{d\in F} r_d \|x_d^F\|\Big) \text{ by (9)} \\
&\le L_2^{-1}\Big(\sum_{b\in B} |I(x,b)|\,\|e_B + \langle 1_B\rangle\| + L_3 \sum_{d\in F} r_d \|x_d^F\|.\Big) \qquad (21)
\end{aligned}$$

Let F_0 satisfy Assumption F. Now

$$I(x^F, b) = \sum_{e\in b} I(x^F, V(e(F_0)))$$

for $b \in N^1$. Hence, there exists $K = K(F_0) > 0$ such that

$$\sum_{b\in B} |I(x^F, b)| \le K \sqrt{\sum_{d\in F_0} r_d \|x_d^F\|^2} \le K\|x^F\|. \qquad (22)$$

Hence, by (21) and (22),

$$\|x^F\| \le L_2^{-1}\Big(K\|e_B + \langle 1_B\rangle\| + L_3 \sqrt{\sum r_d}\ \Big). \qquad (23)$$

(b) Suppose $b \in N^1 \cap B$ is given. Let F as above be given and let $e \in b \in B$ be given. Let $m \in V(e(F_0)) \cap V\langle F_0\rangle$ be given. Take a path P in $e(F_0)$ from m to a node $n \in V(e(F)) \cap V\langle F\rangle$. We may assume all resistors of P are in F, by taking the first such n.

$$\begin{aligned}
|p^F(m) - e_B(b)| &\leq 2\sum_{d\in P} \|R(d)x_d^F\| \\
&\leq 2\sum_{d\in P} r_d(L_3 + L_1\|x_d^F\|) \\
&\leq 2\sum_{d\notin F_0} r_d L_3 + 2\sum_{d\in F\setminus F_0} r_d L_1\|x_d^F\| \\
&\leq 2L_3\sum_{d\notin F_0} r_d + 2L_1\sqrt{\sum_{d\notin F_0} r_d}\ \|x^F\| \\
&\leq 2L_3\sum_{d\notin F_0} r_d + 2\frac{L_1}{L_2}\sqrt{\sum_{d\notin F_0} r_d}(K\|e_B + \langle 1_B\rangle\| + L_3\sqrt{\sum r_d}\ ), \text{ by (23).}
\end{aligned}$$

(c) Similarly, in Theorem 4, x being the current given by e_B, and $v = \{R(d)x_d\}_{d\in D}$, v satisfies KVL on (D, G, N^0, φ) by Tellegen's equation, using loop currents. Thus we find there exists a potential $p : N^0 \to \mathbb{R}$, unique up to an additive constant on each component, for v, and satisfying $p(n) = e_B(n)$ for all $n \in B\cap N^0$. We say that a sequence n_k of nodes converges to $n^1 \in N^1$ when for all finite $F \subset D$, for all large k, there is $e \in n^1$ such that $n_k \in V(e(F))$. We find, using 1-loop currents, that there exists a unique $p : N \to \mathbb{R}$ which is continuous and such that $p(n) = e_B(n)$ for all $n \in B$, and the restriction of p to N^0 is a potential for v. We have, with m as in (b), and P a one ended path in $e(F^0)$ from m to e,

$$\begin{aligned}
|p(m) - e_B(b)| &\leq 2\sum_{d\in P} \|R(d)x_d\| \\
&\leq 2\sum_{d\in P} r_d(L_3 + L_1\|x_d\|) \\
&\leq 2L_3\sum_{d\notin F_0} r_d + 2L_1(\sum_{d\in P} r_d\|x_d\|^2)^{1/2}(\sum_{d\notin F_0} r_d)^{1/2}.
\end{aligned}$$

By Assumption F, since all $d \in F \setminus F_0$ are in some $e(F_0)$,

$$\begin{aligned}
|I(x^F, b^F) - I(x, b)| \leq\ & 2\|G(F_0, B)\|_{Lip}(2\frac{L_1}{L_2}\sqrt{\sum_{d\notin F_0} r_d}(K\|e_B + \langle 1_B\rangle\| + L_3\sqrt{\sum r_d}\ ) \\
&+4L_3\sum_{d\notin F_0} r_d + 2L_1(\sum_{d\in P} r_d\|x_d\|^2)^{1/2}(\sum_{d\notin F_0} r_d)^{1/2}).
\end{aligned}$$

The result follows. □

5. Transfinite Networks

The first type of transfinite network is the 1-network, and the key point of view is that it may be considered as a multiterminal resistor by taking the terminal set to be N^1. We do not go over the details of the general definition of transfinite networks here, but by forming a 1-network of 1-networks we obtain a 2-network, as conceived of in [27], that is, we consider infinite paths of infinite paths. But under our conditions, this is a 1-network of multiterminal resistors, and hence just another multiterminal resistor. One may think either of a) starting with infinitely many 1-networks and joining them together, or b) starting with a 1-network of resistors and substituting each resistor with a 1-network of resistors.

A 3-network is a 1-network of 2-networks, and hence a resistor again, and so on. This gives a simple construction of some of the transfinite networks considered in Chapters 5 and 6 of [28], and Chapters 4 and 5 of [30]. We do not generalize the definitions of Zemanian, rather, we particularize, in order to obtain a manageable structure. The existence and uniqueness result of [30], Theorem 6.8-1 and Th 6.9-1, will follow from this paper under the assumption of Condition M, rather than Condition 6.1-1 of [30]. The no-gain property gave a-priori bounds in the case of two terminal devices, which are not available if general multiterminal devices are considered.

We have focussed on working with particular conductance functions that allowed us to give the conductance function of a 1-network in terms of the conductance functions of its components, the former conductance function having the same properties as these latter, so that the 1-network could be used as a component too. We were able to deal with this analytical problem in case the conductances are all Lipschitz and strongly monotone. It is important to note that the 1-network enjoys the same properties that we required of its components, because then we may repeat the process, forming new networks from these 1-networks, and so on, inductively.

6. Conclusion

In this paper the accent has been on understanding what happens when we join together a number of multiterminal resistors, especially how the resulting circuit gives a multiterminal resistor again. Section 3 studied finite networks, Section 4 studied 1-networks, and Section 5 sketched the application to the hierarchy of transfinite networks.

We saw that a finite network, and then a 1-network, together with a terminal set, B, gives a resistor whose conductance can be calculated, and then we saw how 1-networks approximate large finite networks.

One implication of our theory is that 1-networks, 2-networks and so on may be largely understood as multiterminal resistors with conductance functions which are Lipschitz continuous and strongly monotone. We remark that the figures in [30], for example the one on the cover, show that transfinite networks may be visualized to be much like multiterminal resistors.

The other main conclusion is that (classes of) finite networks of multiterminal resistors have a limiting behaviour as the number of resistors becomes large.

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