CIRCLES AND TOTALLY GEODESIC KÄHLER SUBMANIFOLDS

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ABSTRACT. The purpose of this paper is to characterize all totally geodesic Kähler submanifolds by some circles.

1. INTRODUCTION.

This paper is a part of a joint work with Professors T. Adachi and S. Maeda [SMA]. We first review the definition of circles. Let \( \gamma(s) \) be a regular curve parametrized by its arclength \( s \) in a Riemannian manifold \( M \). Then the curve \( \gamma \) is called a circle if there exist a field \( V = V(s) \) of unit vectors along \( \gamma \) and a constant \( k (\geq 0) \) satisfying

\[
\begin{align*}
\nabla_\gamma \dot{\gamma} &= kV, \\
\nabla_\gamma V &= -k \dot{\gamma},
\end{align*}
\]

where \( \nabla_\gamma \) denotes the covariant differentiation along \( \gamma \) with respect to the Riemannian connection \( \nabla \) of \( M \). The constant \( k \) is called the curvature of the circle. A circle of curvature zero is nothing but a geodesic. For each point \( x \in M \), each orthonormal pair \( (u, v) \) of vectors at \( x \) and each positive constant \( k \), there exists locally a unique circle \( \gamma = \gamma(s) \) on \( M \) with initial condition that \( \gamma(0) = x, \dot{\gamma}(0) = u \) and \( \nabla_\gamma \dot{\gamma}(0) = kv \). For details, see [NY].

We here recall the following two parallel surfaces in Euclidean space. Let \( f_1 \) be a totally umbilic imbedding of a 2-dimensional standard sphere \( S^2(c) \) of curvature \( c \) into Euclidean space \( \mathbb{R}^5 \), and let \( f_2 = \iota \circ f \) be an isometric parallel immersion of \( S^2(c) \) into \( \mathbb{R}^5 \). Here \( f \) is the second standard minimal immersion of \( S^2(c) \) into \( S^4(3c) \) and \( \iota \) is a totally umbilic imbedding of \( S^4(3c) \) into \( \mathbb{R}^5 \). We know that for each great circle \( \gamma \) on \( S^2(c) \), both of the curves \( f_1 \circ \gamma \) and \( f_2 \circ \gamma \) are circles in the ambient space \( \mathbb{R}^5 \). This implies that we cannot distinguish \( f_1 \) from \( f_2 \) by the extrinsic shape of \( S^2(c) \) in \( \mathbb{R}^5 \). However we emphasize that we can distinguish these two isometric immersions \( f_1 \) and \( f_2 \) by the extrinsic shape of

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(small) circles of $S^2(c)$ in $\mathbb{R}^5$. In fact, for each small circle $\gamma$ on $S^2(c)$, the curve $f_1 \circ \gamma$ is also a circle in $\mathbb{R}^5$ but the curve $f_2 \circ \gamma$ is a helix of proper order 4 in the ambient $\mathbb{R}^5$ (for details, see [S]).

In this context we are interested in the extrinsic shape of circles of the submanifold.

We recall a Kähler circle in a Kähler manifold $(M, J)$ with complex structure $J$.

Let $\gamma$ be a circle in a Kähler manifold $M$. Then we see from (1) that $\langle \dot{\gamma}, JV \rangle$ is constant along $\gamma$. Therefore it makes sense to define a Kähler circle as a circle $\gamma$ satisfying the condition that $\dot{\gamma}$ and $Y$ span a holomorphic plane, that is, $Y = J\dot{\gamma}$ or $Y = -J\dot{\gamma}$. Note that if $\gamma$ is a Kähler circle, then (1) reduces to $\nabla \dot{\gamma} \dot{\gamma} = kJ\dot{\gamma}$ or $\nabla \dot{\gamma} \dot{\gamma} = -kJ\dot{\gamma}$.

Let $M$ be a complex $n$-dimensional complex space form $M_n(c)$, which is locally either a complex projective space $\mathbb{C}P^n(c)$ of holomorphic sectional curvature $c (> 0)$, a complex Euclidean space $\mathbb{C}^n$ or a complex hyperbolic space $\mathbb{C}H^n(c)$ of holomorphic sectional curvature $c (< 0)$. Kähler circles $\gamma$ of curvature $k$ on $M$ are the following plane curves:

| $\gamma$ circle $\subset \mathbb{C}P^1(c)$ | Kähler | $\subset \mathbb{C}P^n(c)$, |
| $\gamma$ circle $\subset \mathbb{C}^1$ | Kähler | $\subset \mathbb{C}^n$, |
| $\gamma$ circle $\subset \mathbb{C}H^1(c)$ | Kähler | $\subset \mathbb{C}H^n(c)$. |

In this paper we pay particular attention to Kähler circles. Nomizu characterized totally geodesic complex projective spaces among Kähler submanifolds in a complex projective space by an extrinsic property of geodesics in his paper [N]. He gives the following: A Kähler submanifold $M_n$ in a complex projective space $\mathbb{C}P^{n+p}$ is locally congruent to $\mathbb{C}P^n$, which is a totally geodesic Kähler submanifold of $\mathbb{C}P^{n+p}$, if every geodesic on $M_n$ is locally contained in a complex projective line $\mathbb{C}P^1$ in $\mathbb{C}P^{n+p}$.

Motivated by this characterization, we shall characterize all totally geodesic Kähler immersions into an arbitrary Kähler manifold by the extrinsic shape of Kähler circles on the submanifold.

2. Results.

For the characterization of totally geodesic Kähler immersions, we review the definition of a Frenet curve of order 2 on a Riemannian manifold.

A smooth curve $\gamma = \gamma(s)$ parametrized by its arclength $s$ on a Riemannian manifold $M$ is called a Frenet curve of order 2 if there exist a field $V = V(s)$ of unit vectors along $\gamma$ and a positive function $\kappa = \kappa(s)$ satisfying

$$\begin{cases} \nabla \dot{\gamma} = \kappa V, \\ \nabla \gamma V = -\kappa \dot{\gamma}. \end{cases}$$

The function $\kappa$ is called the curvature of $\gamma$. Of course a circle of positive curvature is a Frenet curve of order 2 in a trivial sense and a Frenet curve of order 1 is nothing
We obtain from (4), (6) and (7) the equations

\[ \nabla_\gamma \hat{\gamma} = \kappa V, \quad \nabla_\gamma V = -\kappa \hat{\gamma}, \]

where \( \nabla \) denotes the covariant differentiation on \( \widetilde{M} \) and \( \kappa = ||\nabla_\gamma \hat{\gamma}|| \). Here for simplicity we usually denote the curve \( f \circ \gamma \) by \( \gamma \). We get from (3)

\[ \kappa \nabla_\gamma \nabla_\gamma \hat{\gamma} = \kappa \nabla_\gamma (\kappa V) = \kappa' \nabla_\gamma \hat{\gamma} - \kappa^2 \hat{\gamma}, \]

where \( \kappa' = \frac{d}{ds} \kappa \) and \( s \) is the arclength parameter of \( \gamma \). So we obtain equation

\[ (\kappa(\nabla_\gamma \nabla_\gamma \hat{\gamma} + \kappa^2 \hat{\gamma}) = \kappa' \nabla_\gamma \hat{\gamma}. \]

We calculate the covariant differentiation \( \nabla \) by use of the formulae of Gauss and Weingarten :

\[ \nabla_X Z = \nabla_X Z + \sigma(X, Z), \quad \nabla_X \xi = D_X \xi - A_\xi X \]

where \( \sigma \) denotes the second fundamental form of \( f \). Here, we define the covariant differentiation \( \nabla \) of the second fundamental form \( \sigma \) with respect to the connection in (tangent bundle) + (normal bundle) as follows:

\[ (\nabla_X \sigma)(Y, Z) = D_X (\sigma(Y, Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z). \]

It follows from (5) that

\[ \nabla_\gamma \nabla_\gamma \hat{\gamma} = \nabla_\gamma \nabla_\gamma \hat{\gamma} + \sigma(\nabla_\gamma \hat{\gamma}, \hat{\gamma}) - A_{\sigma(\hat{\gamma}, \hat{\gamma})} \hat{\gamma} + D_\gamma (\sigma(\hat{\gamma}, \hat{\gamma})) \]

\[ = -k^2 \hat{\gamma} + 3k J(\sigma(\hat{\gamma}, \hat{\gamma})) - A_{\sigma(\hat{\gamma}, \hat{\gamma})} \hat{\gamma} + (\nabla_\gamma \sigma)(\hat{\gamma}, \hat{\gamma}) \]

and

\[ \kappa^2 = \langle \nabla_\gamma \hat{\gamma}, \nabla_\gamma \hat{\gamma} \rangle = k^2 + ||\sigma(\hat{\gamma}, \hat{\gamma})||^2. \]

We obtain from (4), (6) and (7)

\[ \kappa \{ ||\sigma(\hat{\gamma}, \hat{\gamma})||^2 \hat{\gamma} + 3k J(\sigma(\hat{\gamma}, \hat{\gamma})) - A_{\sigma(\hat{\gamma}, \hat{\gamma})} \hat{\gamma} + (\nabla_\gamma \sigma)(\hat{\gamma}, \hat{\gamma}) \} \]

\[ = \kappa' \{ \pm k J \hat{\gamma} + \sigma(\hat{\gamma}, \hat{\gamma}) \}. \]
Taking the tangential and the normal components for the submanifold $M$ in equation (8), we obtain the following:

\begin{align}
\tilde{\kappa}(-A_{\sigma(\dot{\gamma},\dot{\gamma})}\dot{\gamma} + \|\sigma(\dot{\gamma},\dot{\gamma})\|^2\dot{\gamma}) &= \pm \tilde{\kappa}'k\dot{\gamma}, \\
\tilde{\kappa}\{\pm 3k.J(\sigma(\dot{\gamma},\dot{\gamma})) + (\nabla_{\dot{\gamma}}\sigma)(\dot{\gamma},\dot{\gamma})\} &= \tilde{\kappa}'\sigma(\dot{\gamma},\dot{\gamma}).
\end{align}

(9) \hspace{3cm} (10)

Take the inner product of both hand-sides of (9) with $J\dot{\gamma}$. We then find

\[\pm \tilde{\kappa}'k = -\tilde{\kappa}\langle A_{\sigma(\dot{\gamma},\dot{\gamma})}\dot{\gamma}, J\dot{\gamma} \rangle = -\tilde{\kappa}\langle \sigma(\dot{\gamma},\dot{\gamma}), \sigma(\dot{\gamma}, J\dot{\gamma}) \rangle = -\tilde{\kappa}\langle \sigma(\dot{\gamma},\dot{\gamma}), J(\sigma(\dot{\gamma},\dot{\gamma})) \rangle = 0.\]

This implies that the only curvature function $\tilde{\kappa}$ of the curve $f \circ \gamma$ is constant, so that this curve is a circle of positive curvature in the ambient manifold $\tilde{M}$. This shows (c) implies (b).

For an arbitrary unit tangent vector $v \in TM$ we denote by $\gamma_i (i = 1, 2)$ Kähler circles of curvature $k$ with $\dot{\gamma}_i(0) = v \in TM$:

\[\nabla_{\dot{\gamma}_1}\dot{\gamma}_1 = kJ\dot{\gamma}_1, \quad \nabla_{\dot{\gamma}_2}\dot{\gamma}_2 = -kJ\dot{\gamma}_2.\]

Following (10) we have

\[3k.J(\sigma(\dot{\gamma}_1,\dot{\gamma}_1)) + (\nabla_{\dot{\gamma}_1}\sigma)(\dot{\gamma}_1,\dot{\gamma}_1) = 0,\]
\[-3k.J(\sigma(\dot{\gamma}_2,\dot{\gamma}_2)) + (\nabla_{\dot{\gamma}_2}\sigma)(\dot{\gamma}_2,\dot{\gamma}_2) = 0.\]

Evaluating these at $s = 0$, we obtain

\[3k.J(\sigma(v,v)) + (\nabla_v\sigma)(v,v) = 0 = -3k.J(\sigma(v,v)) + (\nabla_v\sigma)(v,v),\]

which lead us to $\sigma(v,v) = 0$. As $v$ is an arbitrary unit vector we find that $M$ is totally geodesic in $\tilde{M}$. \hfill \square

A curve is said to be a plane curve in a Riemannian manifold $M$ if it is locally contained on some real 2-dimensional totally geodesic submanifold of $M$.

As an immediate consequence of this Theorem, we obtain the following characterization of totally geodesic Kähler submanifolds in a complete simply connected complex space form $\tilde{M}_m(c)$.

**Corollary.** Let $f$ be a Kähler isometric immersion of a Kähler manifold $M_n$ into a complete simply connected complex space form $\tilde{M}_m(c)$. Then the following conditions are equivalent:

(a) $f$ is a totally geodesic immersion.

(b) There exists $k > 0$ satisfying that $f$ maps every Kähler circle of curvature $k$ on $M_n$ to a circle in $\tilde{M}_m(c)$.

(c) There exists $k > 0$ satisfying that $f$ maps every Kähler circle of curvature $k$ on $M_n$ to a Frenet curve of order 2 in $\tilde{M}_m(c)$.

(d) There exists $k > 0$ satisfying that $f$ maps every Kähler circle of curvature $k$ on $M_n$ to a plane curve in $\tilde{M}_m(c)$.
3. Remarks.

We first claim following.

**Proposition.** In a nonflat complex space form \( \tilde{M}_m(c) (= \mathbb{C}P^m(c) \) or \( \mathbb{C}H^m(c) \) \)

generally, a Frenet curve \( \gamma \) of order 2 is not a plane curve.

**Proof.** We take a Frenet curve \( \gamma \) of order 2 in a Kähler manifold \( M \) with complex torsion \( \tau \) which is defined by

\[
\tau = \langle \dot{\gamma}, J\dot{V} \rangle.
\]

This complex torsion \( \tau \) \(( -1 \leq \tau \leq 1) \) is constant from (2). In fact by direct computation, we find that

\[
\nabla \dot{\gamma} \langle \dot{\gamma}, J\dot{V} \rangle = \langle \nabla \dot{\gamma} \dot{\gamma}, J\dot{V} \rangle + \langle \dot{\gamma}, J\nabla \dot{\gamma} V \rangle = \kappa \langle V, J\dot{V} \rangle - \kappa \langle \dot{\gamma}, J\dot{\gamma} \rangle = 0.
\]

Needless to say we can take a Frenet curve of order 2 with each complex torsion \( \tau \) \(( -1 \leq \tau \leq 1) \) in an arbitrary Kähler manifold. Indeed, for any unit vector \( X \in T\tilde{M}_m(c) \) and any constant \( \tau \) \(( |\tau| < 1, \tau \neq 0) \) we can take a unit vector \( Y \in T\tilde{M}_m(c) \) satisfying \( \langle X, Y \rangle = 0 \) and \( \langle X, JY \rangle = \tau \) as follows: For a unit vector \( Y_1 \in T\tilde{M}_m(c) \) which satisfies \( \langle X, Y_1 \rangle = \langle JX, Y_1 \rangle = 0 \) we put

\[
Y := -\tau JX + \sqrt{1 - \tau^2} Y_1.
\]

Then the vector \( Y \) satisfies \( \|Y\| = 1 \) and

\[
\langle X, Y \rangle = \langle X, -\tau JX + \sqrt{1 - \tau^2} Y_1 \rangle = 0,
\]

\[
\langle X, JY \rangle = \langle X, \tau X + \sqrt{1 - \tau^2} JY_1 \rangle = \tau.
\]

Note that for a Frenet curve \( \gamma \) of order 2 in \( \tilde{M}_m(c) \), \( \gamma \) is a plane curve if and only if \( \tau = 0, \pm 1 \).

When \( \tau = 0 \), this plane curve \( \gamma \) is as follows:

\[
\gamma \subset \mathbb{R}P^2(\xi) \xrightarrow{\text{totally real}} \text{totally geodesic} \mathbb{C}P^n(c),
\]

\[
\gamma \subset \mathbb{R}H^2(\xi) \xrightarrow{\text{totally real}} \text{totally geodesic} \mathbb{C}H^n(c).
\]

When \( \tau = \pm 1 \), this plane curve \( \gamma \) is as follows:

\[
\gamma \subset \mathbb{C}P^1(c) \xrightarrow{\text{Kähler}} \text{totally geodesic} \mathbb{C}P^n(c),
\]

\[
\gamma \subset \mathbb{C}H^1(c) \xrightarrow{\text{Kähler}} \text{totally geodesic} \mathbb{C}H^n(c).
\]

In the condition (d) of Corollary, the plane curve \( f \circ \gamma \) is nothing but a Frenet curve of order 2 with complex torsion \( \tau = 1 \) or \( \tau = -1 \) in \( \tilde{M}_m(c) \).

**Remark.** If we put \( k = 0 \) in the statements of Theorem and Corollary, these results are no longer true. For example, we consider the second Veronese imbedding \( f : \mathbb{C}P^n(c/2) \longrightarrow \mathbb{C}P(n^2+3n)/2(c) \) which is defined by

\[
(z_0, \ldots, z_n) \longmapsto (z_0^2, \sqrt{2} z_0 z_1, \ldots, z_n^2),
\]
where $z_0, \ldots, z_n$ is the homogeneous coordinates of $\mathbb{C}P^n$. This non-totally geodesic Kähler isometric imbedding $f$ maps every geodesic on $\mathbb{C}P^n(c/2)$ to a circle of curvature $\sqrt{c}/2$ in a real projective plane $\mathbb{R}P^2(c/4)$.

REFERENCES


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