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CIRCLES AND TOTALLY GEODESIC KÄHLER SUBMANIFOLDS

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ABSTRACT. The purpose of this paper is to characterize all totally geodesic Kähler submanifolds by some circles.

1. INTRODUCTION.

This paper is a part of a joint work with Professors T. Adachi and S. Maeda [SMA].

We first review the definition of circles. Let $\gamma = \gamma(s)$ be a regular curve parametrized by its arclength s in a Riemannian manifold M. Then the curve γ is called a *circle* if there exist a field $V = V(s)$ of unit vectors along γ and a constant k (≥ 0) satisfying

$$\begin{cases} \nabla_{\dot\gamma}\dot\gamma = kV, \\ \nabla_{\dot\gamma}V = -k\dot\gamma, \end{cases} \tag{1}$$

where $\nabla_{\dot\gamma}$ denotes the covariant differentiation along γ with respect to the Riemannian connection ∇ of M. The constant k is called the *curvature* of the circle. A circle of curvature zero is nothing but a geodesic. For each point $x \in M$, each orthonormal pair (u, v) of vectors at x and each positive constant k, there exists locally a unique circle $\gamma = \gamma(s)$ on M with initial condition that $\gamma(0) = x$, $\dot\gamma(0) = u$ and $\nabla_{\dot\gamma}\dot\gamma(0) = kv$. For details, see [NY].

We here recall the following two parallel surfaces in Euclidean space. Let f_1 be a totally umbilic imbedding of a 2-dimensional standard sphere $S^2(c)$ of curvature c into Euclidean space $\mathbb{R}^5$, and let $f_2 = \iota \circ f$ be an isometric parallel immersion of $S^2(c)$ into $\mathbb{R}^5$. Here f is the second standard minimal immersion of $S^2(c)$ into $S^4(3c)$ and ι is a totally umbilic imbedding of $S^4(3c)$ into $\mathbb{R}^5$. We know that for each great circle γ on $S^2(c)$, both of the curves $f_1 \circ \gamma$ and $f_2 \circ \gamma$ are circles in the ambient space $\mathbb{R}^5$. This implies that we cannot distinguish f_1 from f_2 by the extrinsic shape of *geodesics* of $S^2(c)$ in $\mathbb{R}^5$. However we emphasize that we can distinguish these two isometric immersions f_1 and f_2 by the extrinsic shape of

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(small) *circles* of $S^2(c)$ in $\mathbb{R}^5$. In fact, for each small circle γ on $S^2(c)$, the curve $f_1 \circ \gamma$ is also a circle in $\mathbb{R}^5$ but the curve $f_2 \circ \gamma$ is a helix of proper order 4 in the ambient $\mathbb{R}^5$ (for details, see [S]).

In this context we are interested in the extrinsic shape of *circles* of the submanifold.

We recall a Kähler circle in a Kähler manifold (M, J) with complex structure J. Let γ be a circle in a Kähler manifold M. Then we see from (1) that $\langle \dot{\gamma}, JV \rangle$ is constant along γ. Therefore it makes sense to define a *Kähler circle* as a circle γ satisfying the condition that $\dot{\gamma}$ and Y span a holomorphic plane, that is, $Y = J\dot{\gamma}$ or $Y = -J\dot{\gamma}$. Note that if γ is a Kähler circle, then (1) reduces to $\nabla_{\dot{\gamma}}\dot{\gamma} = kJ\dot{\gamma}$ or $\nabla_{\dot{\gamma}}\dot{\gamma} = -kJ\dot{\gamma}$.

Let M be a complex n-dimensional complex space form $M_n(c)$, which is locally either a complex projective space $\mathbb{C}P^n(c)$ of holomorphic sectional curvature c (> 0), a complex Euclidean space $\mathbb{C}^n$ or a complex hyperbolic space $\mathbb{C}H^n(c)$ of holomorphic sectional curvature c (< 0). Kähler circles γ of curvature k on M are the following plane curves:

$$\begin{aligned}
&\gamma \overset{\text{circle}}{\subset} \mathbb{C}P^1(c) \xrightarrow[\text{totally geodesic}]{\text{Kähler}} \mathbb{C}P^n(c),\\
&\gamma \overset{\text{circle}}{\subset} \mathbb{C}^1 \xrightarrow[\text{totally geodesic}]{\text{Kähler}} \mathbb{C}^n,\\
&\gamma \overset{\text{circle}}{\subset} \mathbb{C}H^1(c) \xrightarrow[\text{totally geodesic}]{\text{Kähler}} \mathbb{C}H^n(c).
\end{aligned}$$

In this paper we pay particular attention to Kähler circles. Nomizu characterized totally geodesic complex projective spaces among Kähler submanifolds in a complex projective space by an extrinsic property of *geodesics* in his paper [N]. He gives the following : A Kähler submanifold M_n in a complex projective space $\mathbb{C}P^{n+p}$ is locally congruent to $\mathbb{C}P^n$, which is a totally geodesic Kähler submanifold of $\mathbb{C}P^{n+p}$, if every geodesic on M_n is locally contained in a complex projective line $\mathbb{C}P^1$ in $\mathbb{C}P^{n+p}$.

Motivated by this characterization, we shall characterize all totally geodesic Kähler immersions into an arbitrary Kähler manifold by the extrinsic shape of *Kähler circles* on the submanifold.

2. Results.

For the characterization of totally geodesic Kähler immersions, we review the definition of a Frenet curve of order 2 on a Riemannian manifold.

A smooth curve $\gamma = \gamma(s)$ parametrized by its arclength s on a Riemannian manifold M is called a *Frenet curve of order* 2 if there exist a field $V = V(s)$ of unit vectors along γ and a positive function $\kappa = \kappa(s)$ satisfying

$$\begin{cases} \nabla_{\dot{\gamma}}\dot{\gamma} = \kappa V, \\ \nabla_{\dot{\gamma}}V = -\kappa\dot{\gamma}. \end{cases} \tag{2}$$

The function κ is called the *curvature* of γ. Of course a circle of positive curvature is a Frenet curve of order 2 in a trivial sense and a Frenet curve of order 1 is nothing but a geodesic.

Theorem. *Let f be a Kähler isometric immersion of a Kähler manifold M into an arbitrary Kähler manifold $\widetilde{M}$. Then the following conditions are equivalent*:

(a) *f is a totally geodesic immersion.*
(b) *There exists $k > 0$ satisfying that f maps every Kähler circle of curvature k on M to a circle in $\widetilde{M}$.*
(c) *There exists $k > 0$ satisfying that f maps every Kähler circle of curvature k on M to a Frenet curve of order 2 in $\widetilde{M}$.*

Proof. It suffices to show that (c) implies (a). Let γ be a Kähler circle of curvature k on M which satisfies $\nabla_{\dot\gamma}\dot\gamma = \pm kJ\dot\gamma$. By the assumption it is mapped to a Frenet curve $f\circ\gamma$ of order 2 in the ambient manifold $\widetilde{M}$ satisfying the equations

$$\begin{cases} \widetilde{\nabla}_{\dot\gamma}\dot\gamma = \widetilde{\kappa}V, \\ \widetilde{\nabla}_{\dot\gamma}V = -\widetilde{\kappa}\dot\gamma, \end{cases} \tag{3}$$

where $\widetilde{\nabla}$ denotes the covariant differentiation on $\widetilde{M}$ and $\widetilde{\kappa} = \|\widetilde{\nabla}_{\dot\gamma}\dot\gamma\|$. Here for simplicity we usually denote the curve $f\circ\gamma$ by γ. We get from (3)

$$\widetilde{\kappa}\widetilde{\nabla}_{\dot\gamma}\widetilde{\nabla}_{\dot\gamma}\dot\gamma = \widetilde{\kappa}\widetilde{\nabla}_{\dot\gamma}(\tilde\kappa V) = \widetilde{\kappa}'\widetilde{\nabla}_{\dot\gamma}\dot\gamma - \widetilde{\kappa}^3\dot\gamma,$$

where $\widetilde{\kappa}' = \frac{d}{ds}\widetilde{\kappa}$ and s is the arclength parameter of γ. So we obtain equation

$$\widetilde{\kappa}(\widetilde{\nabla}_{\dot\gamma}\widetilde{\nabla}_{\dot\gamma}\dot\gamma + \widetilde{\kappa}^2\dot\gamma) = \widetilde{\kappa}'\widetilde{\nabla}_{\dot\gamma}\dot\gamma. \tag{4}$$

We calculate the covariant differentiation $\widetilde{\nabla}$ by use of the formulae of Gauss and Weingarten :

$$\widetilde{\nabla}_X Z = \nabla_X Z + \sigma(X,Z), \quad \widetilde{\nabla}_X\xi = D_X\xi - A_\xi X \tag{5}$$

where σ denotes the second fundamental form of f. Here, we define the covariant differentiation $\bar\nabla$ of the second fundamental form σ with respect to the connection in (tangent bundle) + (normal bundle) as follows:

$$(\bar\nabla_X\sigma)(Y,Z) = D_X(\sigma(Y,Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z).$$

It follows from (5) that

$$\begin{aligned} \widetilde{\nabla}_{\dot\gamma}\widetilde{\nabla}_{\dot\gamma}\dot\gamma &= \nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma + \sigma(\nabla_{\dot\gamma}\dot\gamma, \dot\gamma) - A_{\sigma(\dot\gamma,\dot\gamma)}\dot\gamma + D_{\dot\gamma}(\sigma(\dot\gamma,\dot\gamma)) \\ &= -k^2\dot\gamma \pm 3kJ(\sigma(\dot\gamma,\dot\gamma)) - A_{\sigma(\dot\gamma,\dot\gamma)}\dot\gamma + (\bar\nabla_{\dot\gamma}\sigma)(\dot\gamma,\dot\gamma) \end{aligned} \tag{6}$$

and

$$\widetilde{\kappa}^2 = \langle\widetilde{\nabla}_{\dot\gamma}\dot\gamma, \widetilde{\nabla}_{\dot\gamma}\dot\gamma\rangle = k^2 + \|\sigma(\dot\gamma,\dot\gamma)\|^2. \tag{7}$$

We obtain from (4),(6) and (7)

$$\begin{aligned} &\widetilde{\kappa}\{\|\sigma(\dot\gamma,\dot\gamma)\|^2\dot\gamma \pm 3kJ(\sigma(\dot\gamma,\dot\gamma)) - A_{\sigma(\dot\gamma,\dot\gamma)}\dot\gamma + (\bar\nabla_{\dot\gamma}\sigma)(\dot\gamma,\dot\gamma)\} \\ &\quad = \widetilde{\kappa}'\{\pm kJ\dot\gamma + \sigma(\dot\gamma,\dot\gamma)\}. \end{aligned} \tag{8}$$

Taking the tangential and the normal components for the submanifold M in equation (8), we obtain the following:

$$\widetilde{\kappa}(-A_{\sigma(\dot\gamma,\dot\gamma)}\dot\gamma + \|\sigma(\dot\gamma,\dot\gamma)\|^2\dot\gamma) = \pm\widetilde{\kappa}' kJ\dot\gamma, \tag{9}$$

$$\widetilde{\kappa}\{\pm 3kJ(\sigma(\dot\gamma,\dot\gamma)) + (\bar\nabla_{\dot\gamma}\sigma)(\dot\gamma,\dot\gamma)\} = \widetilde{\kappa}'\sigma(\dot\gamma,\dot\gamma). \tag{10}$$

Take the inner product of both hand-sides of (9) with $J\dot\gamma$. We then find

$$\pm\widetilde{\kappa}'k = -\widetilde{\kappa}\langle A_{\sigma(\dot\gamma,\dot\gamma)}\dot\gamma, J\dot\gamma\rangle = -\widetilde{\kappa}\langle\sigma(\dot\gamma,\dot\gamma),\sigma(\dot\gamma,J\dot\gamma)\rangle = -\widetilde{\kappa}\langle\sigma(\dot\gamma,\dot\gamma),J(\sigma(\dot\gamma,\dot\gamma))\rangle = 0.$$

This implies that the only curvature function $\widetilde{\kappa}$ of the curve $f\circ\gamma$ is constant, so that this curve is a circle of positive curvature in the ambient manifold $\widetilde{M}$. This shows (c) implies (b).

For an arbitrary unit tangent vector $v \in TM$ we denote by γ_i $(i = 1, 2)$ Kähler circles of curvature k with $\dot\gamma_i(0) = v \in TM$:

$$\nabla_{\dot\gamma_1}\dot\gamma_1 = kJ\dot\gamma_1,\ \nabla_{\dot\gamma_2}\dot\gamma_2 = -kJ\dot\gamma_2.$$

Following (10) we have

$$3kJ(\sigma(\dot\gamma_1,\dot\gamma_1)) + (\bar\nabla_{\dot\gamma_1}\sigma)(\dot\gamma_1,\dot\gamma_1) = 0,$$
$$-3kJ(\sigma(\dot\gamma_2,\dot\gamma_2)) + (\bar\nabla_{\dot\gamma_2}\sigma)(\dot\gamma_2,\dot\gamma_2) = 0.$$

Evaluating these at $s = 0$, we obtain

$$3kJ(\sigma(v,v)) + (\bar\nabla_v\sigma)(v,v) = 0 = -3kJ(\sigma(v,v)) + (\bar\nabla_v\sigma)(v,v),$$

which lead us to $\sigma(v,v) = 0$. As v is an arbitrary unit vector we find that M is totally geodesic in $\widetilde{M}$. □

A curve is said to be a *plane curve* in a Reimannian manifold M if it is locally contained on some real 2-dimensional totally geodesic submanifold of M.

As an immediate consequence of this Theorem, we obtain the following characterization of totally geodesic Kähler submanifolds in a complete simply connected complex space form $\widetilde{M}_m(c)$.

Corollary. *Let f be a Kähler isometric immersion of a Kähler manifold M_n into a complete simply connected complex space form $\widetilde{M}_m(c)$. Then the following conditions are equivalent*:

- (a) *f is a totally geodesic immersion.*
- (b) *There exists $k > 0$ satisfying that f maps every Kähler circle of curvature k on M_n to a circle in $\widetilde{M}_m(c)$.*
- (c) *There exists $k > 0$ satisfying that f maps every Kähler circle of curvature k on M_n to a Frenet curve of order 2 in $\widetilde{M}_m(c)$.*
- (d) *There exists $k > 0$ satisfying that f maps every Kähler circle of curvature k on M_n to a plane curve in $\widetilde{M}_m(c)$.*

3. Remarks.

We first claim following.

Proposition. *In a nonflat complex space form* $\widetilde{M}_m(c)(=\mathbb{C}P^m(c)$ *or* $\mathbb{C}H^m(c))$ *generally, a Frenet curve* γ *of order* 2 *is not a plane curve.*

Proof. We take a Frenet curve γ of order 2 in a Kähler manifold M with complex torsion τ which is defined by $\tau=\langle\dot\gamma, JV\rangle$. This complex torsin $\tau(-1\le\tau\le 1)$ is constant from (2). In fact by direct computation, we find that

$$\nabla_{\dot\gamma}\langle\dot\gamma, JV\rangle=\langle\nabla_{\dot\gamma}\dot\gamma, JV\rangle+\langle\dot\gamma, J\nabla_{\dot\gamma}V\rangle=\kappa\langle V, JV\rangle-\kappa\langle\dot\gamma, J\dot\gamma\rangle=0.$$

Needless to say we can take a Frenet curve of order 2 with each complex torsion $\tau(-1\le\tau\le 1)$ in an arbitrary Kähler manifold. Indeed, for any unit vector $X\in T\widetilde{M}_m(c)$ and any constant τ $(|\tau|<1, \tau\neq 0)$ we can take a unit vector $Y\in T\widetilde{M}_m(c)$ satisfying $\langle X, Y\rangle=0$ and $\langle X, JY\rangle=\tau$ as follows: For a unit vector $Y_1\in T\widetilde{M}_m(c)$ which satisfies $\langle X, Y_1\rangle=\langle JX, Y_1\rangle=0$ we put

$$Y:=-\tau JX+\sqrt{1-\tau^2}Y_1.$$

Then the vector Y satisfies $\|Y\|=1$ and

$$\langle X, Y\rangle=\langle X, -\tau JX+\sqrt{1-\tau^2}Y_1\rangle=0,$$
$$\langle X, JY\rangle=\langle X, \tau X+\sqrt{1-\tau^2}JY_1\rangle=\tau.$$

Note that for a Frenet curve γ of order 2 in $\widetilde{M}_m(c)$, γ is a plane curve if and only if $\tau=0,\pm1$. □

When $\tau=0$, this plane curve γ is as follows:

$$\gamma\ \subset\ \mathbb{R}P^2(\tfrac{c}{4})\ \xrightarrow[\text{totally geodesic}]{\text{totally real}}\ \mathbb{C}P^n(c),$$

$$\gamma\ \subset\ \mathbb{R}H^2(\tfrac{c}{4})\ \xrightarrow[\text{totally geodesic}]{\text{totally real}}\ \mathbb{C}H^n(c).$$

When $\tau=\pm1$, this plane curve γ is as follows:

$$\gamma\ \subset\ \mathbb{C}P^1(c)\ \xrightarrow[\text{totally geodesic}]{\text{Kähler}}\ \mathbb{C}P^n(c),$$

$$\gamma\ \subset\ \mathbb{C}H^1(c)\ \xrightarrow[\text{totally geodesic}]{\text{Kähler}}\ \mathbb{C}H^n(c).$$

In the condition (d) of Corollary, the plane curve $f\circ\gamma$ is nothing but a Frenet curve of order 2 with complex torsion $\tau=1$ or $\tau=-1$ in $\widetilde{M}_m(c)$.

Remark. If we put $k=0$ in the statements of Theorem and Corollary, these results are no longer true. For example, we consider the second Veronese imbedding $f:\mathbb{C}P^n(c/2)\longrightarrow\mathbb{C}P^{(n^2+3n)/2}(c)$ which is defined by

$$(z_0,\dots,z_n)\longmapsto(z_0^2,\sqrt{2}z_0z_1,\dots,z_n^2),$$

where $z_0, \dots, z_n$ is the homogeneous coordinates of $\mathbb{C}P^n$. This non-totally geodesic Kähler isometric imbedding f maps every geodesic on $\mathbb{C}P^n(c/2)$ to a circle of curvature $\sqrt{c}/2$ in a real projective plane $\mathbb{R}P^2(c/4)$.

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