Mem. Fac. Sci. Eng. Shimane Univ.
Series B: Mathematical Science
34 (2001), pp. 45–52

A WEIGHTED SOBOLEV-POINCARÉ'S INEQUALITY ON INFINITE NETWORKS

Atsushi Murakami and Maretsugu Yamasaki

(Received: January 15, 2001)

Abstract. Inequalities on networks have played important roles in the theory of networks. We study the famous Sobolev-Poincaré's inequality on infinite networks in the weighted form. This inequality is closely related to the smallest eigenvalue of a weighted discrete Laplacian. We give a dual characterization for the smallest eigenvalue.

1. PROBLEM SETTING

Let X be a countable set of nodes, Y be a countable set of arcs and K be the node-arc incidence matrix. Assume that the graph $G := \{X, Y, K\}$ is locally finite and connected and has no self-loop. For a strictly positive real valued function r on Y, $N := \{G, r\}$ is called a network.

Let $L(X)$ be the set of all real valued functions on X, $L^+(X)$ be the set of all non-negative $u \in L(X)$ and $L_0(X)$ be the set of all $u \in L(X)$ with finite support. We denote by ε_A the characteristic function of the subset A of X and put $\varepsilon_x := \varepsilon_A$ in case $A = \{x\}$.

The discrete derivative du and the discrete Laplacian $\Delta u(x)$ of $u \in L(X)$ are defined by

$$\begin{aligned} du(y) &:= -r(y)^{-1} \sum\nolimits_{x \in X} K(x,y)u(x), \\ \Delta u(x) &:= \sum\nolimits_{y \in Y} K(x,y)[du(y)]. \end{aligned}$$

The mutual Dirichlet sum $D(u, v)$ of $u, v \in L(X)$ is defined by

$$D(u,v) := \sum\nolimits_{y \in Y} r(y)[du(y)][dv(y)]$$

1991 *Mathematics Subject Classification.* 31C20, 39A10.

Key words and phrases. Infinite network, Sobolev-Poincaré's inequality, the smallest eigenvalue, the discrete Laplacian .

This work was supported in part by Grant-in-Aid for Scientific Research (C)(No. 11640202), Japanese Ministry of Education, Science and Culture.

if the sum on the right hand side converges. We call $D(u) := D(u,u)$ the Dirichlet sum of u and put

$$D(N) := \{u \in L(X); D(u) < \infty\}.$$

Notice that $D(N)$ is a Hilbert space with the inner product

$$((u,v))_D := D(u,v) + u(x_0)v(x_0),$$

where x_0 is a fixed node. We set $\|u\|_D = ((u,u))_D^{1/2}$. We shall use the set of Dirichlet potentials $D_0(N)$ which is defined as the closure of $L_0(X)$ in $D(N)$.

Let m be a strictly positive real valued function on X and put

$$((u,v))_m := \sum_{x \in X} m(x)u(x)v(x)$$

if the sum on the right hand side converges. We put $\|u\|_m := [((u,u))_m]^{1/2}$ and

$$L_2(X;m) := \{u \in L(X); \|u\|_m < \infty\}.$$

We shall be concerned with the following weighted Sobolev-Poincaré's inequality on N:

(C; m) There exists a constant $c > 0$ such that

$$\|u\|_m^2 \le cD(u) \quad \text{for all} \quad u \in L_0(X).$$

For simplicity, we use the function $\chi_m(u)$ on $D(N)$ defined by

$$\chi_m(u) := \frac{D(u)}{\|u\|_m^2} \quad \text{for} \quad u \neq 0$$

and $\chi_m(u) = \infty$ for $u = 0$.

We shall consider the following extremum problem:

$$\lambda_m(N) := \inf\{\chi_m(u); u \in L_0(X)\}.$$

Then it is easily seen that $\lambda_m(N)$ is the best possible value of $1/c$. Therefore the weighted Sobolev-Poincaré's inequality (C; m) is equivalent to the fact that $\lambda_m(N) > 0$.

Let $E_+(\Delta)$ be the set of all $\lambda > 0$ such that there exists $u \in L(X)$ satisfying the condition:

(E) $$\Delta u + \lambda m u = 0 \quad \text{on } X \quad \text{and } u > 0 \text{ on } X.$$

We shall give a characterization of $\lambda_m(N)$ with the aid of $E_+(\Delta)$. Namely it will be shown that $E_+(\Delta)$ is equal to the interval $(0, \lambda_m(N)]$ and $\lambda_m(N) = \max E_+(\Delta)$ if $\lambda_m(N) > 0$.

For notation and terminology, we mainly follow [7].

2. Preliminaries

Given a finite subnetwork $N' =< X', Y' >$ of N, we consider the following extremum problem:

$$\lambda_m(N') := \inf\{\chi_m(u); u \in S(N')\},$$

where we set

$$S(N') := \{u \in L(X); u = 0 \text{ on } X \setminus X'\}.$$

As in [8], we have

Lemma 2.1. *For every finite subnetwork $N' =< X', Y' >$ of N, there exists a unique $\tilde{u} \in S(N')$ which has the following properties:*
(1) $\lambda_m(N') = \chi_m(\tilde{u})$,
(2) $\Delta\tilde{u}(x) = -\lambda_m(N')m(x)\tilde{u}(x)$ *on* X'.
(3) $\tilde{u}(x) > 0$ *on* X' *and* $\|\tilde{u}\|_m = 1$.

Theorem 2.1. *Let $\{N_n\}(N_n =< X_n, Y_n >)$ be an exhaustion of N. Then the sequence $\{\lambda_m(N_n)\}$ converges to $\lambda_m(N)$.*

Proof. We have

$$\lambda_m(N) \le \lambda_m(N_{n+1}) \le \lambda_m(N_n).$$

For any $\varepsilon > 0$ we can find $u \in L_0(X)$ such that $\chi_m(u) < \lambda_m(N) + \varepsilon$. There exists n_0 such that $u = 0$ on $X \setminus X_n$ for all $n \ge n_0$. Thus $\lambda_m(N_n) \le \chi_m(u)$ for all $n \ge n_0$. Hence $\{\lambda_m(N_n)\}$ converges to $\lambda_m(N)$. □

3. A characterization of $\lambda_m(N)$

Let $E_+(\Delta)$ be the set of all $\lambda > 0$ such that there exists $u \in L(X)$ satisfying the condition:

(E) $$\Delta u + \lambda m u = 0 \quad \text{on } X \quad \text{and } u > 0 \text{ on } X.$$

We shall prove

Theorem 3.1. *Assume that $E_+(\Delta) \ne \emptyset$. Then $\sup E_+(\Delta) \le \lambda_m(N)$.*

Proof. Let $\lambda \in E_+(\Delta)$. There exists $u \in L(X)$ which satisfies Condition (E). Consider an exhaustion $\{N_n\}(N_n =< X_n, Y_n >)$ of N. By Lemma 2.1, there exists $v_n \in L(X)$ such that $v_n = 0$ on $X \setminus X_n, v_n > 0$ on X_n and $\Delta v_n + \lambda_m(N_n) m v_n = 0$ on X_n. Put

$$P := (\lambda - \lambda_m(N_n)) \sum_{x \in X_n} m(x)u(x)v_n(x).$$

Since $\Delta u + \lambda m u = 0$ on X_n, we have

$$\begin{aligned} P &= -\sum_{x \in X_n} v_n(x)[\Delta u(x)] + \sum_{x \in X_n} u(x)[\Delta v_n(x)] \\ &= -\sum_{x \in X} v_n(x)[\Delta u(x)] + \sum_{x \in X} u(x)[\Delta v_n(x)] - \sum_{x \in X \setminus X_n} u(x)[\Delta v_n(x)] \\ &= D(v_n, u) - D(u, v_n) - \sum_{x \in X \setminus X_n} u(x)[\Delta v_n(x)] \\ &= -\sum_{x \in X \setminus X_n} u(x)[\Delta v_n(x)]. \end{aligned}$$

For each boundary node x of X_n(i.e., $x \notin X_n$ and x is a neighboring node of X_n), we have

$$\Delta v_n(x) = \sum_{z \in X_n} t(x, z)v_n(z) \ge 0,$$

where

$$t(x, z) := \sum_{y \in Y} |K(x, y)K(z, y)|r(y)^{-1}.$$

Therefore $P \leq 0$. Since $u(x)v_n(x) > 0$ on X_n, we obtain $\lambda \leq \lambda_m(N_n)$. Our assertion follows from Theorem 2.1. $\square$

This result was proved in [3] in case $r = 1$ and $m = 1$. To prove the converse of the above result, we prepare

Lemma 3.1. *Let $0 < \lambda < \lambda_m(N)$. For each $a \in X$, there exists a unique $\pi_a \in L(X)$ which satisfies the following conditions:*
(1) $\pi_a(x) > 0$ on X.
(2) $\Delta\pi_a(x) + \lambda m(x)\pi_a(x) = -\varepsilon_a(x)$ on X.

Proof. Notice that N is of hyperbolic type by Theorem 3.3 in [7]. Since $0 < \lambda < \lambda_m(N)$, we see that $D(u) > \lambda\|u\|_m^2$ for every $u \in D_0(N)$ with $u \neq 0$. Let $a \in X$ and consider the following minimizing problem:

$$\text{(P)} \qquad \rho(a) := \inf\{D(u) - \lambda\|u\|_m^2; u \in D_0(N),\ u(a) = 1\}.$$

Let $\{u_n\}$ be a minimizing sequence, i.e., $u_n \in D_0(N)$, $u_n(a) = 1$ and $D(u_n) - \lambda\|u_n\|_m^2 \to \rho(a)$ as $n \to \infty$. Since $\lambda_m(N)\|u_n\|_m^2 \leq D(u_n)$, we have

$$D(u_n) - \lambda\|u_n\|_m^2 \geq (1 - \frac{\lambda}{\lambda_m(N)})D(u_n),$$

so that $\{D(u_n)\}$ is bounded. For every $x \in X$, there exists a constant $M(x) > 0$ such that $|u_n(x)| \leq M(x)[D(u_n)]^{1/2}$ for all n (cf. [10]). Therefore $\{u_n(x)\}$ is bounded. By choosing a subsequence if necessary, we may assume that $\{u_n\}$ converges pointwise to $\tilde{u} \in L(X)$. It follows that $\tilde{u} \in D_0(N)$, $\tilde{u}(a) = 1$ and $\rho(a) = D(\tilde{u}) - \lambda\|\tilde{u}\|_m^2$. Notice that $\rho(a) > 0$. In fact, if $\rho(a) = 0$, then

$$\lambda = \frac{D(\tilde{u})}{\|\tilde{u}\|_m^2} \geq \lambda_m(N),$$

which is a contradiction.

Next we show that

$$\text{(Q)} \qquad \Delta\tilde{u}(x) + \lambda m(x)\tilde{u}(x) = -\rho(a)\varepsilon_a(x) \text{ on } X.$$

For any real number t and any $f \in D_0(N)$ with $f(a) = 0$, we have

$$\Phi(t) := D(\tilde{u} + tf) - \lambda\|\tilde{u} + tf\|_m^2 \geq \rho(a) = \Phi(0),$$

so that the derivative of $\Phi(t)$ at $t = 0$ vanishes, i.e., $\Phi'(0) = 0$. It follows that

$$-\sum\nolimits_{z \in X}[\Delta\tilde{u}(z)]f(z) - \lambda((\tilde{u}, f))_m = 0.$$

Taking $f = \varepsilon_x$ ($x \in X, x \neq a$), we obtain $\Delta\tilde{u}(x) + \lambda m(x)\tilde{u}(x) = 0$. For $f = \tilde{u} - \varepsilon_a$, we have

$$-\sum\nolimits_{z \in X}[\Delta\tilde{u}(z)](\tilde{u}(z) - \varepsilon_a(z)) - \lambda((\tilde{u}, \tilde{u} - \varepsilon_a))_m = 0,$$

so that

$$\Delta\tilde{u}(a) + \lambda m(a)\tilde{u}(a) = -D(\tilde{u}) + \lambda\|\tilde{u}\|_m^2 = -\rho(a).$$

Namely every optimal solution $\tilde{u}$ of the problem (P) satisfies the above equation (Q). We show the uniqueness of the solution of the equation (Q). Let $\tilde{u}_1, \tilde{u}_2$ be solutions of the equation (Q). Then $v := \tilde{u}_1 - \tilde{u}_2 \in D_0(N), v(a) = 0$ and $\Delta v(x) + \lambda m(x)v(x) = 0$ on X. Thus $D(v) = \lambda \|v\|_m^2$, and hence $v = 0$. Therefore $\tilde{u}_1 = \tilde{u}_2$.

We show that $\tilde{u} \geq 0$. Let $v := |\tilde{u}|$. Then v is a feasible solution of the problem (P). We have $D(v) \leq D(\tilde{u})$ and $\|v\|_m^2 = \|\tilde{u}\|_m^2$, so that

$$\rho(a) \leq D(v) - \lambda \|v\|_m^2 \leq D(\tilde{u}) - \lambda \|\tilde{u}\|_m^2.$$

Therefore $|\tilde{u}|$ is also an optimal solution of the problem (P). By the above observation, we conclude that $\tilde{u} = |\tilde{u}| \geq 0$.

It follows that $\tilde{u}$ is a nonnegative superharmonic function on X. By the minimum principle, we see that $\tilde{u}(x) > 0$ on X. Now we may conclude that $\pi_a(x) := \tilde{u}(x)/\rho(a)$ satisfies our requirement. □

Theorem 3.2. *Let* $0 < \lambda < \lambda_m(N)$. *Then there exists* $u^* \in L(X)$ *such that* $u^*(x) > 0$ *on* X *and*

$$\Delta u^*(x) + \lambda m(x) u^*(x) = 0 \quad \text{on } X.$$

Proof. Let $\{N_n\} (N_n =< X_n, Y_n >)$ be an exhaustion of N and define $u_n \in L(X)$ by

$$u_n(x) := \lambda \sum\nolimits_{a \in X_n} \pi_a(x) m(a).$$

Then $u_n > 0$ on X and

$$\Delta u_n(x) + \lambda m(x) u_n(x) = \lambda \left[\sum\nolimits_{a \in X_n} (\Delta \pi_a(x) + \lambda m(x) \pi_a(x)) m(a) \right] = -\lambda m(x) \varepsilon_{X_n}(x)$$

for every $x \in X$. Notice that u_n is superharmonic on X. First we consider the case where there exists $b \in X$ such that $\{u_n(b)\}$ is bounded. Notice that $\{u_n(x)\}$ is bounded for every $x \in X$ by Harnack's inequality (cf. Theorem 2.3 in [11]). By choosing subsequences if necessary, we may assume that $\{u_n(x)\}$ converges pointwise to $\tilde{u}(x)$. Then we have

$$\Delta \tilde{u}(x) + \lambda m(x) \tilde{u}(x) = -\lambda m(x),$$

so that $u^* := \tilde{u} + 1$ satisfies our requirement.

Next we consider the case where there exists $b \in X$ such that $u_n(b) \to \infty$ as $n \to \infty$. We put $v_n(x) := u_n(x)/u_n(b)$. Then v_n is positive and superharmonic and $v_n(b) = 1$. By Harnack's inequality, we see that $\{v_n(x)\}$ is bounded for each $x \in X$. Therefore we may assume that $\{v_n\}$ converges pointwise to $\tilde{v}$. We see easily that $\tilde{v}(b) = 1, \tilde{v} > 0$ on X and $\Delta \tilde{v}(x) + \lambda m(x) \tilde{v}(x) = 0$ on X. This completes the proof. □

J. Dodziuk and L. Karp [3] gave a proof for this theorem by using the reasoning for manifolds in [4]. Their reasoning seems to contain a gap which occurs in the discrete case.

By Theorems 3.1 and 3.2, we obtain

Theorem 3.3. *If* $\lambda_m(N) > 0$, *then* $E_+(\Delta)$ *is equal to the interval* $(0, \lambda_m(N)]$ *and* $\lambda_m(N) = \max E_+(\Delta)$.

Proof. By Theorem 3.2, we see that $(0, \lambda_m(N)) \subset E_+(\Delta)$. The fact that $\lambda_m(N) \in E_+(\Delta)$ follows from Theorem 6.3 in [7]. $\square$

Theorem 3.4. *Assume that $\lambda_m(N) > 0$ and that $u^* \in D_0(N)$ satisfies the difference equation:*

$$\Delta u^*(x) + \lambda_m(N)m(x)u^*(x) = 0 \quad on \quad X.$$

Then $\chi_m(u^*) = \lambda_m(N)$.

Proof. There exists a sequence $\{f_n\}$ in $L_0(X)$ such that $\|u^* - f_n\|_D \to 0$ as $n \to \infty$.

$$\lambda_m(N)\|u^* - f_n\|_m^2 \le D(u^* - f_n) \to 0$$

as $n \to \infty$, so that $\{f_n\}$ converges weakly to u^* both in $L_2(X; m)$ and $D_0(N)$ By our assumption, we have $D(u^*, f_n) = \lambda_m(N)((u^*, f_n))_m$, and hence $D(u^*) = \lambda_m(N)\|u^*\|_m^2$. $\square$

4. An Example

Let $G = \{X, Y, K\}$ be the binary tree rooted at x_0 and $r = 1$ on Y(cf. [9]). Denote by $d(x_0, x)$ the geodesic distance between x_0 and x (i.e., the number of arcs in the path connecting two nodes x_0 and x) and let $Z_k := \{x \in X; d(x_0, x) = k\}$ and

$$Q_k := Z_k \ominus Z_{k-1} = \{y \in Y; K(x, y)K(x', y) = -1 \text{ for } x \in Z_k \text{ and } x' \in Z_{k-1}\}.$$

Then we have $\mathrm{Card}(Z_k) = 2^k$ for $k \ge 0$ and $\mathrm{Card}(Q_k) = 2^k$ for $k \ge 1$, where $\mathrm{Card}(A)$ stands for the cardinality of a set A.

We shall determine $\lambda_m(N)$ in case $m = 1$. For simplicity, we put $\|u\| = \|u\|_m$. Let us define a subset $L(X; d)$ of $L(X)$ by $u \in L(X; d)$ if and only if $u(x) = u_k$ for all $x \in Z_k$. For $u \in L(X; d)$, we have

$$\begin{aligned} \Delta u(x_0) &= -2u_0 + 2u_1 \quad (x \in Z_0) \\ \Delta u(x) &= -3u_k + 2u_{k+1} + u_{k-1} \quad (x \in Z_k; k = 1, 2, \cdots). \end{aligned}$$

Let us find a contant $\lambda > 0$ and a function $u > 0$ on X which satisfy $\Delta u(x) + \lambda u(x) = 0$ on X. First consider the following difference equation:

$$\text{(DE)} \qquad 2u_{k+1} - (3 - \lambda)u_k + u_{k-1} = 0 \quad (k = 1, 2, \cdots).$$

Let us put $\lambda^* := 3 - 2\sqrt{2}$. This value gives a double solution for the equation:

$$2t^2 - (3 - \lambda^*)t + 1 = 0.$$

We see easily that

$$u_k^* = (\alpha + \beta k)(\frac{1}{\sqrt{2}})^k \quad (k = 0, 1, 2, \cdots)$$

is a general solution of the difference equation (DE) with $\lambda = \lambda^*$. Determine α and β so that $u_0 = 1$ and $-2u_0 + 2u_1 = -\lambda^* u_0$. Then we obtain

$$u_k^* := \left[1 + (1 - \frac{1}{\sqrt{2}})k\right](\frac{1}{\sqrt{2}})^k$$

for $k = 0, 1, 2, \cdots$. Define $u^* \in L(X; d)$ by $\{u_k^*\}$. Then λ^* and u^* satisfies our requirement. Therefore we have $\lambda_1(N) \geq 3 - 2\sqrt{2}$ by Theorem 3.1.

To prove the converse inequality, we consider a sequence $\{u^{(n)}\}$ in $L(X; d)$ defined by

$$u_k^{(n)} := \begin{cases} (1/\sqrt{2})^k & \text{for} \quad 0 \leq k \leq n \\ 0 & \text{for} \quad k \geq n+1 \end{cases}$$

Then $u^{(n)} \in L_0(X)$ and

$$\begin{aligned} \|u^{(n)}\|^2 &= \sum_{k=0}^{n} 2^k [u_k^{(n)}]^2 = n + 1 \\ D(u^{(n)}) &= \sum_{k=0}^{n} 2^{k+1} [u_k^{(n)} - u_{k+1}^{(n)}]^2 \\ &= 2(1 - \frac{1}{\sqrt{2}})^2 n + 2. \end{aligned}$$

Thus we have

$$\begin{aligned} \lambda_1(N) &\leq \frac{D(u^{(n)})}{\|u^{(n)}\|^2} \\ &= 2(1 - \frac{1}{\sqrt{2}})^2 \frac{n}{n+1} + \frac{2}{n+1} \\ &\to 2(1 - \frac{1}{\sqrt{2}})^2 = 3 - 2\sqrt{2} \quad (n \to \infty). \end{aligned}$$

Therefore we have

Theorem 4.1. *Let G be the binary tree rooted at x_0 and let $r = 1$ on Y and $m = 1$ on X. Then $\lambda_m(N) = 3 - 2\sqrt{2}$.*

Notice that we have $\|u^*\|^2 = \infty$.

REFERENCES

[1] N.L. Biggs, B. Mohar and J. Shawe-Taylor, The spectral radius of infinite graphs, Bull. London Math. Soc. **20**(1988), 116-120.

[2] J. Dodziuk, Difference equations, isoperimetric inequality and transience of certain random walks, Trans. Amer. Math. Soc. **284**(1984), 787-794.

[3] J. Dodziuk and L. Karp, Spectral and function theory for combinatorial Laplacians, Contemporary Math. **73**(1988), 25-40.

[4] D. Fisher-Colbrie and R. Schoen, The structure of complete stable minimal surfaces in 3-manifolds of non-negative scalar curvature, Comm. Pure Appl. Math. **33**(1980), 199-211.

[5] J. Friedman, Some geometric aspects of graphs and their eigenfunctions, Duke Math. J. **69**(1993), 487-525.

[6] P. Gerl, Random walks on graphs, Lecture Notes in Math. **1210**(l986), 285-303, Springer, Berlin.

[7] A. Murakami and M. Yamasaki, Inequalities on infinite networks, to appear in Mem. Fac. Sci. Eng. Shimane Univ. Ser B: Mathematical Science **33**(2000), 47-62.

[8] Y. Shogenji and M. Yamasaki, Hardy's inequality on finite netwoeks, Mem. Fac. Sci. Eng. Shimane Univ. Ser B: Mathematical Science **32**(1999), 75–84.

[9] M. Yamasaki, Parabolic and hyperbolic infinite networks, Hiroshima Math. J.**7**(1977), 135–146.
[10] M. Yamasaki, Discrete potentials on an infinite network, Mem. Fac. Shimane Univ. **13**(1979), 31–44.
[11] M. Yamasaki, The equation $\Delta u = qu$ on an infinite network, Mem. Fac. Shimane Univ. **21**(1987), 31–46.

Department of Mathematics, Hiroshima Institute of Technology, Hiroshima, 731-5193 JAPAN
E-mail address: `atsumura@cc.it-hiroshima.ac.jp`

Department of Mathematics and Computer Science, Shimane University, Matsue, 690-8504 JAPAN
E-mail address: `yamasaki@math.shimane-u.ac.jp`