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INTEGRAL CURVES OF KILLING VECTOR FIELDS IN A COMPLEX PROJECTIVE SPACE

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Abstract. In this article we treat a complex projective space $\mathbb{C}P^n$ of constant holomorphic sectional curvature 4 as a model space. By using submanifold theory of $\mathbb{C}P^n$ we shall investigate geometric properties about curves generated by some Killing vector fields on this space.

1. Introduction.

As a model space $\mathbb{C}P^n$ is a nice Riemannian manifold. It admits many homogeneous submanifolds, that is, submanifolds which are given as orbits under subgroups of the projective unitary group $PU(n+1)$ through equivariant isometric immersions. In this article we particularly consider two homogeneous Riemannian submanifolds of $\mathbb{C}P^n$.

In section 3, we consider a Riemannian symmetric space $M = S^1 \times S^{n-1}/\sim$ of rank 2 imbedded in $\mathbb{C}P^n$ (through the isometric imbedding, say f) as an isotropic submanifold with parallel second fundamental form (for details, see (3.1), (3.2) and [N]). This submanifold M has various geometric properties. For example, for each geodesic γ of M, the curve $f \circ \gamma$ is a circle of the same curvature $1/\sqrt{2}$ in $\mathbb{C}P^n$. We here remark that there exist many geodesics γ_1 and γ_2 on M such that the curves $f \circ \gamma_1$ and $f \circ \gamma_2$ are not congruent with respect to isometries of $\mathbb{C}P^n$. By virtue of the isometric imbedding $f : M = S^1 \times S^{n-1}/\sim \rightarrow \mathbb{C}P^n$ we obtain an interesting family of open circles and closed circles of the same curvature $1/\sqrt{2}$ in $\mathbb{C}P^n$. This interesting fact leads us to the study on circles in $\mathbb{C}P^n$. Note that every circle in a Riemannian symmetric space M of rank one is an integral curve of a Killing vector field on M (see [MT]). The purpose of this section is to give an answer to the problem "When is a circle closed in $\mathbb{C}P^n$?".

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In section 4, motivated by the study in section 3, we are interested in the problem "In a complex projective space $\mathbb{C}P^n$, for each positive ℓ does there exist a unique closed circle γ whose length is ℓ up to isometries of $\mathbb{C}P^n$?". In order to give an answer to this problem we study length spectrum of *circles* of $\mathbb{C}P^n$, that is, we investigate how lengths of closed *circles* are distributed on the real line. In this section we use a notation which is similar to that in geometry of length spectrum (of closed geodesics).

In section 5, we study a geodesic sphere $M = G_m(r)$ (through the isometric inclusion mapping, say g), that is, a distance sphere with center $m \in \mathbb{C}P^n$ and radius r $(0 < r < \pi/2)$ imbedded as a real hypersurface in $\mathbb{C}P^n$. These spheres are diffeomorphic (but not isometric) to standard spheres. Geodesic spheres in $\mathbb{C}P^n$ are nice objects in intrinsic geometry as well as extrinsic geometry, that is, submanifold theory (cf. [W]). Our study about geodesics on M tells us the fact that for each geodesic γ on M, the curve $g \circ \gamma$ is an integral curve of a Killing vector field on $\mathbb{C}P^n$, and moreover gives us many important information on length spectrum of $G_m(r)$. For example, on a geodesic sphere $G_m(r)$ $(0 < r < \pi/2)$ there exist infinitely many congruency classes of closed geodesics with respect to the isometry group of $G_m(r)$. In sections 4 and 5, some results on length spectrum come from classical number theory (see Theorems 4.5 and 5.12).

In section 6, we determine all integral curves of Killing vector fields on a 2-dimensional holomorphic totally geodesic submanifold $\mathbb{C}P^2$ of $\mathbb{C}P^n$. Our study here is motivated by the fact that for each geodesic γ on $S^1 \times S^{n-1}/\sim$ (resp. $G_m(r)$), the curve $f \circ \gamma$ (resp. $g \circ \gamma$) lies on $\mathbb{C}P^2$, and moreover that all of the curves $f \circ \gamma$ and $g \circ \gamma$ are generated by some Killing vector fields on $\mathbb{C}P^2$.

In the last section we shall construct a certain class of *closed helices with self-intersections* in $\mathbb{C}P^n$. Needless to say, these curves are *not* integral curves of Killing vector fields on $\mathbb{C}P^n$. We note that in any Riemannian manifold M, every integral curve γ of a Killing vector field is a helix, that is, all Frenet curvatures of γ are constant along the curve γ. Moreover, this curve γ is a simple curve, namely it does not have any self-intersection points. To obtain closed helices with self-intersections, we adopt the same isometric imbedding $f: M = S^1 \times S^{n-1}/\sim \longrightarrow \mathbb{C}P^n$ as in section 3. Let γ be a circle of curvature $\kappa(> 0)$ on M. Then for each positive κ, the curve $f \circ \gamma$ is a closed helix with length $2\pi/\kappa$ in $\mathbb{C}P^n$. By virtue of results in this section we know that the curve $f \circ \gamma$ has self-intersections if and only if $\kappa \leqq 3/(\sqrt{2}\pi)$.

Through out of this paper we suppose that a complex projective space $\mathbb{C}P^n$ is furnished with the standard metric of constant holomorphic sectional curvature 4.

2. Preliminaries.

In the first place we recall the Frenet formula for a smooth curve in a Riemannian manifold M with Riemannian metric $\langle\ ,\ \rangle$. A smooth curve $\gamma = \gamma(s)$ parametrized by its arclength s is called a *Frenet curve of proper order d* if there exist orthonormal frame fields $\{V_1 = \dot\gamma, \dots, V_d\}$ along γ and positive functions $\kappa_1(s), \dots, \kappa_{d-1}(s)$ which satisfy the following system of ordinary equations

$$\nabla_{\dot\gamma} V_j(s) = -\kappa_{j-1}(s)V_{j-1}(s) + \kappa_j(s)V_{j+1}(s), \quad j = 1, \dots, d, \tag{2.1}$$

where $V_0 \equiv V_{d+1} \equiv 0$ and $\nabla_{\dot\gamma}$ denotes the covariant differentiation along γ with respect to the Riemannian connection ∇ of M. Equation (2.1) is called the Frenet formula for the Frenet curve γ. The functions $\kappa_j(s)$ $(j = 1, \dots, d-1)$ and the orthonormal frame $\{V_1, \dots, V_d\}$ are called the *curvatures* and the *Frenet frame* of γ, respectively.

A Frenet curve is called a *Frenet curve of order d* if it is a Frenet curve of proper order $r(\leqq d)$. For a Frenet curve of order d which is of proper order $r(\leqq d)$, we use the convention in (2.1) that $\kappa_j \equiv 0$ $(r \leqq j \leqq d-1)$ and $V_j \equiv 0$ $(r+1 \leqq j \leqq d)$. In this paper a curve means a smooth Frenet curve. We call a curve a *helix* when all its curvatures are constant. A helix of order 1 is nothing but a geodesic. A helix of order 2, namely a curve which satisfies $\nabla_{\dot\gamma}V_1(s) = \kappa V_2(s), \nabla_{\dot\gamma}V_2(s) = -\kappa V_1(s)$ and $V_1(s) = \dot\gamma(s)$, is called a *circle of curvature* κ.

We now restrict ourselves to Frenet curves on Kähler manifolds. Let M be an n-dimensional Kähler manifold with complex structure J and Riemannian metric $\langle\ ,\ \rangle$. For a Frenet curve $\gamma = \gamma(s)$ in M of order $d(\leqq 2n)$ with associated Frenet frame $\{V_1, \dots, V_d\}$, we set $\tau_{ij}(s) = \langle V_i(s), JV_j(s)\rangle$ for $1 \leqq i < j \leqq d$ and call them its *complex torsions.* In the study of Frenet curves on a Kähler manifold their complex torsions play an important role. We call γ a *holomorphic helix* if all the curvatures and all the complex torsions of γ are constant functions along γ.

We here pay particular attention to Frenet curves in an n-dimensional complete simply connected non-flat complex space form $M_n(c)(= \mathbb{C}P^n(c)$ or $\mathbb{C}H^n(c))$ of constant holomorphic sectional curvature $c(\neq 0)$. The congruence theorem for Frenet curves in a non-flat complex space form is stated as follows (cf. Theorem 5.1 in [MOh]):

Theorem A. *Let $\gamma = \gamma(s)$ and $\delta = \delta(s)$ be two Frenet curves of orders p and q in a non-flat complex space form $M_n(c)$, respectively. Let $\{V_1, \dots, V_p\}$ (resp. $\{W_1, \dots, W_q\}$) denote the Frenet frame of γ (resp. δ) and $\{\lambda_1(s), \dots, \lambda_{p-1}(s)\}$ (resp. $\{\mu_1(s), \dots, \mu_{q-1}(s)\}$) be the curvature functions of γ (resp. δ). Then the curves γ and δ are congruent, that is, there exist an isometry φ of $M_n(c)$ and constant s_0 such that $\gamma(s) = (\varphi \circ \delta)(s + s_0)$ for every s if and only if they have the following conditions.*

(1) $p = q$.
(2) *There exists a constant s_0 with the following properties*:
  i) $\lambda_i(s) = \mu_i(s + s_0)$ $(i = 1, \dots, p-1)$ for every s,
  ii) the complex torsions of γ and δ satisfy either
  $\tau_\gamma^{ij}(0) = \tau_\delta^{ij}(s_0)$ $(1 \leqq i < j \leqq p)$ or $\tau_\gamma^{ij}(0) = -\tau_\delta^{ij}(s_0)$ $(1 \leqq i < j \leqq p)$.

Here, in the condition (2)ii), the former holds if γ, δ are congruent with respect to some holomorphic isometry, and the latter holds if they are congruent with respect to some anti-holomorphic isometry.

It is well-known that in a complete simply connected n-dimensional real space form $M^n(c)(= S^n(c),\ \mathbb{R}^n$ or $H^n(c))$ of constant sectional curvature c, a curve γ is a helix if and only if γ is an integral curve of a Killing vector field on $M^n(c)$. The following is a complex version of this fact (see [MOh]):

Theorem B. *In a complex space form $M_n(c)(= \mathbb{C}P^n(c),\ \mathbb{C}^n$ or $\mathbb{C}H^n(c))$, a curve γ is a holomorphic helix if and only if γ is an integral curve of a holomorphic Killing vector field on $M_n(c)$.*

Remark. It is known that if M is a complex space form of nonzero constant holomorphic sectional curvature, then any Killing vector field on M is a holomorphic vector field.

In general, in a Kähler manifold M (with complex structure J) a circle $\gamma = \gamma(s)$ (with $\nabla_{\dot\gamma}V_1(s) = \kappa V_2(s), \nabla_{\dot\gamma}V_2(s) = -\kappa V_1(s)$ and $V_1(s) = \dot\gamma$) is a holomorphic helix. Indeed,

$$\begin{aligned}\nabla_{\dot\gamma}\langle V_1(s), JV_2(s)\rangle &= \langle \nabla_{\dot\gamma}V_1(s), JV_2(s)\rangle + \langle V_1(s), J\nabla_{\dot\gamma}V_2(s)\rangle \\ &= \kappa \cdot \langle V_2(s), JV_2(s)\rangle - \kappa \cdot \langle V_1(s), JV_1(s)\rangle = 0.\end{aligned}$$

In the following, for a circle $\gamma = \gamma(s)$ in M we denote its complex torsion by τ for simplicity.

We finally note that there are many helices but not holomorphic helices of proper order $d(\geqq 3)$ in a Kähler manifold M. For example, let $\gamma = \gamma(s)$ be a helix of proper order 3 on M. Then the complex torsions of γ satisfy the following equations (see [AM2, MA]):

$$\begin{cases} \tau_{12}' = \kappa_2\tau_{13}, \\ \tau_{13}' = -\kappa_2\tau_{12} + \kappa_1\tau_{23}, \\ \tau_{23}' = -\kappa_1\tau_{13}, \end{cases}$$

where κ_1, κ_2 denote the curvatures of the helix γ. By solving them, we have

$$\begin{cases} \tau_{12}(s) = \alpha_1 \sin\sqrt{\kappa_1^2+\kappa_2^2}s + \alpha_2\cos\sqrt{\kappa_1^2+\kappa_2^2}s + \alpha_3, \\ \tau_{13}(s) = \dfrac{\sqrt{\kappa_1^2+\kappa_2^2}}{\kappa_2}\left(\alpha_1\cos\sqrt{\kappa_1^2+\kappa_2^2}s - \alpha_2\sin\sqrt{\kappa_1^2+\kappa_2^2}s\right), \\ \tau_{23}(s) = -\dfrac{\kappa_1}{\kappa_2}\left(\alpha_1\sin\sqrt{\kappa_1^2+\kappa_2^2}s + \alpha_2\cos\sqrt{\kappa_1^2+\kappa_2^2}s\right) + \dfrac{\kappa_2}{\kappa_1}\alpha_3 \end{cases}$$

for some constants α_1, α_2 and α_3. This implies that the curve γ is a holomorphic helix if and only if $\alpha_1 = \alpha_2 = 0$.

3. When is a circle closed in $\mathbb{C}P^n$?

In [AMU, AM1] we concentrated on the study about circles in $\mathbb{C}P^n$. We first consider a Riemannian symmetric space $S^1 \times S^{n-1}/\sim$ of rank 2. Here two points $(e^{i\theta}, (a_1, \dots, a_n))$ and $(e^{i\psi}, (b_1, \dots, b_n))$ on $S^1 \times S^{n-1}$ are identified if $(e^{i\theta}, a_1, \dots, a_n) = (-e^{i\psi}, -b_1, \dots . -b_n)$. The Riemannian metric on $S^1 \times S^{n-1}/\sim$ is given by

$$\langle (v, \xi), (w, \eta) \rangle = \frac{2}{9} \langle v, w \rangle_{S^1} + \frac{2}{3} \langle \xi, \eta \rangle_{S^{n-1}}$$

for tangent vectors $v, w \in TS^1$ and $\xi, \eta \in TS^{n-1}$, where $\langle\ ,\ \rangle_{S^1}$ and $\langle\ ,\ \rangle_{S^{n-1}}$ denote the canonical metrics on standard spheres S^1 and S^{n-1}, respectively. We define a parallel isometric imbedding $f : S^1 \times S^{n-1}/\sim \longrightarrow \mathbb{C}P^n(4)$ by

$$f([(e^{i\theta}, (a_1, \dots, a_n))]) = \pi\left(\begin{pmatrix} \frac{1}{3}(e^{-2i\theta/3} + 2a_1 e^{i\theta/3}) \\ \frac{\sqrt{2}}{3}(e^{-2i\theta/3} - a_1 e^{i\theta/3}) \\ \frac{2}{\sqrt{6}} i a_2 e^{i\theta/3} \\ \vdots \\ \frac{2}{\sqrt{6}} i a_n e^{i\theta/3} \end{pmatrix} \right) \tag{3.1}$$

with the Hopf fibration $\pi : S^{2n+1}(1) \to \mathbb{C}P^n(4)$. The second fundamental form σ_f of f is expressed as

$$\begin{gathered} \sigma_f((u,0),(u,0)) = -\frac{1}{\sqrt{2}} J(u,0), \quad \sigma_f((0,\xi),(0,\xi)) = \frac{1}{\sqrt{2}} J(u,0), \\ \sigma_f((u,0),(0,\xi)) = \frac{1}{\sqrt{2}} J(0,\xi) \end{gathered} \tag{3.2}$$

for each unit tangent vector $\xi \in TS^{n-1}$ and the normalized vector u of $\partial/\partial\theta$, where J denotes the complex structure on $\mathbb{C}P^n(4)$. Since f is parallel, we find by (3.2) that it maps every geodesic γ on $S^1 \times S^{n-1}/\sim$ to a circle of curvature $1/\sqrt{2}$ in $\mathbb{C}P^n(4)$:

$$\nabla_{\dot{\tilde{\gamma}}} \nabla_{\dot{\tilde{\gamma}}} \dot{\tilde{\gamma}} = \nabla_{\dot{\tilde{\gamma}}} \sigma_f(\dot{\gamma}, \dot{\gamma}) = -\|\sigma_f(\dot{\gamma}, \dot{\gamma})\| \dot{\tilde{\gamma}},$$

where $\tilde{\gamma} = f \circ \gamma$ and ∇ is the Riemannian connection on $\mathbb{C}P^n(4)$. This suggests us a kind of importance in study of circles.

We call a smooth curve $\gamma = \gamma(s)$ closed if there exists a positive s_1 with $\gamma(s + s_1) = \gamma(s)$ for every s. For a circle γ, the definition of closedness of γ can be rewritten as follows: A circle γ is said to be closed if there exists a positive s_1 with

$$\gamma(s_1) = \gamma(0),\ \dot{\gamma}(s_1) = \dot{\gamma}(0) \quad \text{and} \quad (\nabla_{\dot{\gamma}} \dot{\gamma})(s_1) = (\nabla_{\dot{\gamma}} \dot{\gamma})(0). \tag{3.3}$$

The minimum positive number s_1 satisfying (3.3) is called the length of a closed circle γ and is denoted by length(γ).

For each geodesic γ on $S^1 \times S^{n-1}/\sim$ we can compute the complex torsion of $f \circ \gamma$ by (3.2). Noticing the metric on $S^1 \times S^{n-1}/\sim$, we find the length of $f \circ \gamma$ and obtain the following theorem which gives us information on all circles of curvature $1/\sqrt{2}$ in $\mathbb{C}P^n$.

Theorem 3.1. *For each unit vector $X = (\alpha u, v) \in T_x(S^1 \times S^{n-1}/\sim) \backsimeq T_{x_1}S^1 \oplus T_{x_2}S^{n-1}$ at a point x, we denote by γ_X the geodesic along X on $S^1 \times S^{n-1}/\sim$. Then the circle $f \circ \gamma_X$ on $\mathbb{C}P^n(4)$ satisfies the following properties*:

(1) *The curvature of $f \circ \gamma_X$ is $1/\sqrt{2}$.*

(2) *The complex torsion of $f \circ \gamma_X$ is $4\alpha^3 - 3\alpha$ for $-1 \leqq \alpha \leqq 1$.*

(3) *The circle $f \circ \gamma_X$ is closed if and only if either $\alpha = 0$ or $\sqrt{(1-\alpha^2)/(3\alpha^2)}$ is rational.*

(4) *When $\alpha = 0$, the length of the closed circle $f \circ \gamma_X$ is $2\sqrt{6}\pi/3$.*

(5) *When $\alpha \neq 0$ and $\sqrt{(1-\alpha^2)/(3\alpha^2)}$ is rational, we denote by p/q the irreducible fraction defined by $\sqrt{(1-\alpha^2)/(3\alpha^2)}$. Then the length ℓ of the closed circle $f \circ \gamma_X$ is as follows*;

($5_{\rm i}$) *When pq is even, ℓ is the least common multiple of $2\sqrt{2}\pi/(3|\alpha|)$ and $2\sqrt{2}\pi/\sqrt{3(1-\alpha^2)}$. In particular, when $\alpha = \pm 1$, $\ell = 2\sqrt{2}\pi/3$.*

($5_{\rm ii}$) *When pq is odd, ℓ is the least common multiple of $\sqrt{2}\pi/(3|\alpha|)$ and $\sqrt{2}\pi/\sqrt{3(1-\alpha^2)}$.*

Next, we prepare the following in order to consider circles of arbitrary positive curvature. Let N be the outward unit normal on $S^{2n+1}(1)$ in $\mathbb{R}^{2n+2}(=\mathbb{C}^{n+1})$. We here mix the complex structures of $\mathbb{C}^{n+1}$ and $\mathbb{C}P^n(4)$. We shall study circles in $\mathbb{C}P^n(4)$ by use of the Hopf fibration $\pi : S^{2n+1}(1) \to \mathbb{C}P^n(4)$. For the sake of simplicity we identify a vector field X on $\mathbb{C}P^n(4)$ with its horizontal lift X^* on $S^{2n+1}(1)$. Then the relation between the Riemannian connection ∇ of $\mathbb{C}P^n(4)$ and the Riemannian connection $\widetilde{\nabla}$ of $S^{2n+1}(1)$ is as follows:

$$\widetilde{\nabla}_X Y = \nabla_X Y + \langle X, JY \rangle JN$$

for any vector fields X and Y on $\mathbb{C}P^n(4)$, where $\langle\ ,\ \rangle$ is the canonical metric on $\mathbb{C}^{n+1}$. By direct calculation with making use of this relation, we can see that for each circle γ of positive curvature every horizontal lift $\tilde{\gamma}$ of γ in $S^{2n+1}(1)$ is a helix in $S^{2n+1}(1)$.

Proposition 3.2. *Let γ denote a circle with curvature $\kappa(>0)$ and complex torsion τ in $\mathbb{C}P^n(4)$ satisfying that $\nabla_{\dot\gamma}\dot\gamma = \kappa Y_s$ and $\nabla_{\dot\gamma} Y_s = -\kappa\dot\gamma$. Then every horizontal lift $\tilde{\gamma}$ of γ in $S^{2n+1}(1)$ is a helix of order $2, 3$ or 5 corresponding to $\tau = 0, \tau = \pm 1$ or $\tau \neq 0, \pm 1$, respectively. Moreover, it satisfies the following* *differential equations*:

$$
(3.4)\qquad
\begin{cases}
\widetilde{\nabla}_{\dot\gamma}\dot\gamma = \kappa Y_s, \\
\widetilde{\nabla}_{\dot\gamma}Y_s = -\kappa\dot\gamma + \tau JN, \\
\widetilde{\nabla}_{\dot\gamma}(JN) = -\tau Y_s + \sqrt{1-\tau^2}Z_s, \\
\widetilde{\nabla}_{\dot\gamma}Z_s = -\sqrt{1-\tau^2}JN + \kappa W_s, \\
\widetilde{\nabla}_{\dot\gamma}W_s = -\kappa Z_s,
\end{cases}
$$

where $Z_s = \frac{1}{\sqrt{1-\tau^2}}(J\dot\gamma + \tau Y_s), W_s = \frac{1}{\sqrt{1-\tau^2}}(JY_s - \tau\dot\gamma)$.

Note that a curve $\gamma = \gamma(s)$ in $\mathbb{C}P^n(4)$ is closed if and only if there exists a positive constant s_* such that a horizontal lift $\tilde\gamma = \tilde\gamma(s)$ of γ in $S^{2n+1}(1)$ satisfies $\tilde\gamma(s+s_*) = e^{i\theta_s}\tilde\gamma(s)$ with some $\theta_s \in [0, 2\pi)$ for every s. Then by solving ordinary differential equation (3.4) for a horizontal lift $\tilde\gamma$ of each circle γ in $\mathbb{C}P^n(4)$ we establish the following.

Theorem 3.3. *Let γ be a circle of curvature $\kappa(>0)$ and of complex torsion τ in a complex projective space $\mathbb{C}P^n(4)$. Then the following hold*:

(1) *When $\tau = 0$, a circle γ is a simple closed curve with length $2\pi/\sqrt{\kappa^2+1}$.*

(2) *When $\tau = \pm 1$, a circle γ is a simple closed curve with length $2\pi/\sqrt{\kappa^2+4}$.*

(3) *When $\tau \neq 0, \pm 1$, we denote by a, b and d $(a < b < d)$ the nonzero solutions for*

$$\lambda^3 - (\kappa^2+1)\lambda + \kappa\tau = 0.$$

Then we find the following:

(i) *If one of* (*hence each of*) *the three ratios a/b, b/d and d/a is rational, then γ is a simple closed curve. Its length is the least common multiple of $2\pi/(b-a)$ and $2\pi/(d-a)$.*

(ii) *If each of the three ratios a/b, b/d and d/a is irrational, then γ is a simple open curve.*

Let γ be a circle of curvature κ in a Riemannian manifold $(M, \langle\ ,\ \rangle)$. When we change the metric $\langle\ ,\ \rangle$ homothetically to $m^2 \cdot \langle\ ,\ \rangle$ for some positive constant m, the curve $\sigma(s) = \gamma(s/m)$ is a circle of curvature κ/m in $(M, m^2 \cdot \langle\ ,\ \rangle)$. Under the operation $\langle\ ,\ \rangle \to m^2 \cdot \langle\ ,\ \rangle$, the length of a closed curve changes to m-times of the original length. Needless to say, the sectional curvature of M changes to $1/m^2$-times of the original sectional curvature under this operation. Hence, by virtue of Theorem 3.3 we can conclude the following which is the main result in this section.

Theorem 3.4. *Let γ be a circle with curvature $\kappa(>0)$ and with complex torsion τ in a complex projective space $\mathbb{C}P^n(c)$ of constant holomorphic sectional curvature c. Then the following hold*:

(1) *When $\tau = 0$, a circle γ is a simple closed curve with length $4\pi/\sqrt{4\kappa^2+c}$.*

(2) *When* $\tau = \pm 1$, *a circle* γ *is a simple closed curve with length* $2\pi/\sqrt{\kappa^2 + c}$.

(3) *When* $\tau \neq 0, \pm 1$, *we denote by* a, b *and* d $(a < b < d)$ *the nonzero solutions for*

$$c\lambda^3 - (4\kappa^2 + c)\lambda + 2\sqrt{c}\kappa\tau = 0.$$

Then we find the following:

(i) *If one of* (*hence each of*) *the three ratios* a/b, b/d *and* d/a *is rational,* γ *is a simple closed curve. Its length is the least common multiple of* $4\pi/\{\sqrt{c}(b-a)\}$ *and* $4\pi/\{\sqrt{c}(d-a)\}$.

(ii) *If each of the three ratios* a/b, b/d *and* d/a *is irrational,* γ *is a simple open curve.*

Remarks. A circle $\gamma = \gamma(s)$ with complex torsion τ is a plane curve in $\mathbb{C}P^n(c)$ (that is, γ is locally contained on some real 2-dimensional totally geodesic submanifold of $\mathbb{C}P^n(c)$) if and only if $\tau = 0$ or $\tau = \pm 1$.

(1) When $\tau = 0$, the circle γ lies on $\mathbb{R}P^2(c/4)$ which is a totally real totally geodesic submanifold of $\mathbb{C}P^n(c)$.

(2) When $\tau = 1$ or -1, the circle γ lies on $\mathbb{C}P^1(c)$ which is a holomorphic totally geodesic submanifold of $\mathbb{C}P^n(c)$.

Circles of complex torsion ±1 are called *holomorphic circles*, and circles of null complex torsion are called *totally real circles.*

4. Length spectrum of circles in $\mathbb{C}P^n(c)$.

In this section, we study the length spectrum of circles in $\mathbb{C}P^n(c)$. Rewriting Theorem 3.1, we find the following which is our main tool in this section.

Proposition 4.1. *In* $\mathbb{C}P^n(c)$ *a circle* γ *of curvature* $\sqrt{2c}/4$ *is closed if and only if its complex torsion is of the form*

$$\tau(p,q) = \frac{q(9p^2 - q^2)}{(3p^2 + q^2)^{3/2}}$$

for some relatively prime positive integers p *and* q *with* $p > q$. *In this case its length is*

$$\text{length}(\gamma) = \begin{cases} \frac{4}{3\sqrt{c}}\pi\sqrt{2(3p^2+q^2)}, & \text{if } pq \text{ is even,} \\ \frac{2}{3\sqrt{c}}\pi\sqrt{2(3p^2+q^2)}, & \text{if } pq \text{ is odd.} \end{cases}$$

In order to get rid of the influence of the action of the full isometry group, we shall consider the moduli space of circles under the action of isometries. The moduli space $\mathrm{Cir}(M)$ of circles is the quotient space of the set of all circles in a Riemannian manifold M under this congruence relation. The *length spectrum* of circles in M is the map $\mathcal{CL} : \mathrm{Cir}(M) \to \mathbb{R} \cup \{\infty\}$ defined by $\mathcal{CL}([\gamma]) = \text{length}(\gamma)$. Here, for an open circle γ, a circle which is not closed, we put $\text{length}(\gamma) = \infty$.

Sometimes we also call the image $\mathrm{CLSpec}(M) = \mathcal{CL}(\mathrm{Cir}(M)) \cap \mathbb{R}$ in the real line the length spectrum of circles on M.

For circles on a complex projective space $\mathbb{C}P^n(c)$ $(n \geqq 2)$ we have the following congruence theorem, which is a direct consequence of Theorem A.

Proposition 4.2. *Two circles in $\mathbb{C}P^n(c)$ are congruent if and only if they have the same curvatures and the same absolute values of complex torsions.*

We denote by $[\gamma_{\kappa,\tau}]$ the congruency class of circles of curvature κ (> 0) and complex torsion $\tau (\geqq 0)$ in $\mathbb{C}P^n(c)$ and by $[\gamma_0]$ the congruency class of geodesics in $\mathbb{C}P^n(c)$. The moduli space of circles in $\mathbb{C}P^n(c)$ is hence

$$\mathrm{Cir}(\mathbb{C}P^n(c)) = \{[\gamma_{\kappa,\tau}] \mid \kappa > 0, 0 \leqq \tau \leqq 1\} \cup \{[\gamma_0]\}.$$

The moduli space of circles has a natural stratification by their curvatures. We denote by $\mathrm{Cir}_\kappa(M)$ the moduli space of circles of curvature κ in M and by $\mathcal{CL}_\kappa$ the restriction of $\mathcal{CL}$ on this space. For a complex projective space we can define for each positive κ a canonical transformation

$$\Phi_\kappa : \mathrm{Cir}_\kappa(\mathbb{C}P^n(c)) \backslash \{[\gamma_{\kappa,1}]\} \to \mathrm{Cir}_{\sqrt{2c}/4}(\mathbb{C}P^n(c)) \backslash \{[\gamma_{\sqrt{2c}/4,1}]\}$$

by

$$\Phi_\kappa([\gamma_{\kappa,\tau}]) = [\gamma_{\sqrt{2c}/4, 3\sqrt{3}c\kappa\tau(4\kappa^2+c)^{-3/2}}].$$

The following lemma guarantees that the structure of the length spectrum $\mathcal{CL}_\kappa$ of circles of curvature κ essentially does not depend on κ.

Lemma 4.3. *The canonical transformation Φ_κ satisfies*

$$\mathcal{CL}([\gamma_{\kappa,\tau}]) = \sqrt{\frac{3c}{2(4\kappa^2+c)}} \cdot \mathcal{CL}(\Phi_\kappa([\gamma_{\kappa,\tau}]))$$

for every τ $(0 \leqq \tau < 1)$.

We denote by $\mathrm{CLSpec}_\kappa(M) = \mathcal{CL}\left(\mathrm{Cir}_\kappa(M)\right) \cap \mathbb{R}$ the length spectrum of circles of curvature κ in M. This lemma yields that

$$\begin{aligned} \mathrm{CLSpec}_\kappa(\mathbb{C}P^n(c)) = & \left\{ \frac{2\pi}{\sqrt{\kappa^2+c}}, \frac{4\pi}{\sqrt{4\kappa^2+c}} \right\} \\ & \bigcup \left\{ 4\pi\sqrt{\frac{3p^2+q^2}{3(4\kappa^2+c)}} \;\middle|\; \begin{array}{c} p \text{ and } q \text{ are relatively prime} \\ \text{integers which satisfy} \\ pq \text{ is even and } p > \alpha_\kappa q > 0 \end{array} \right\} \\ & \bigcup \left\{ 2\pi\sqrt{\frac{3p^2+q^2}{3(4\kappa^2+c)}} \;\middle|\; \begin{array}{c} p \text{ and } q \text{ are relatively prime} \\ \text{integers which satisfy} \\ pq \text{ is odd and } p > \alpha_\kappa q > 0 \end{array} \right\}, \end{aligned}$$

where α_κ ($\geqq 1$) denotes the number with

$$\frac{3\sqrt{3}c\kappa}{(4\kappa^2+c)^{3/2}} = \frac{9\alpha_\kappa^2-1}{(3\alpha_\kappa^2+1)^{3/2}}.$$

Note that the constant α_κ satisfies

i) $\alpha_{\sqrt{2c}/4} = 1$,
ii) monotone decreasing when $0 < \kappa \leqq \sqrt{2c}/4$, and monotone increasing when $\kappa \geqq \sqrt{2c}/4$,
iii) $\lim_{\kappa\to 0}\alpha_\kappa = \lim_{\kappa\to\infty}\alpha_\kappa = \infty$.

Lemma 4.3 also guarantees that

$$\mathrm{CLSpec}(\mathbb{C}P^n(c)) = \left(0, \frac{4\pi}{\sqrt{c}}\right) \cup \bigcup \left\{ I_{p,q} \,\middle|\, \begin{array}{c} p > q, p \text{ and } q \text{ are relatively} \\ \text{prime positive integers} \end{array} \right\},$$

where

$$I_{p,q} = \begin{cases} \left(\frac{4\pi}{3\sqrt{c}}\sqrt{2q(3p+q)}, \frac{4\pi}{3\sqrt{c}}\sqrt{9p^2-q^2}\right), & \text{if } pq \text{ is even,} \\ \left(\frac{2\pi}{3\sqrt{c}}\sqrt{2q(3p+q)}, \frac{2\pi}{3\sqrt{c}}\sqrt{9p^2-q^2}\right), & \text{if } pq \text{ is odd.} \end{cases}$$

For a spectrum $\lambda \in \mathrm{CLSpec}(M)$ the cardinality $m_c(\lambda)$ of the set $\mathcal{CL}^{-1}(\lambda)$ is called the *multiplicity* of the length spectrum $\mathcal{CL}$ at λ. When $m_c(\lambda) = 1$, we say that λ is *simple*. For example, every length spectrum of circles in a real space form is simple. When the multiplicity of $\mathcal{CL}$ is greater than one at some point λ, this means that we can find circles which are not congruent each other but have the same length λ. We denote by $\mathrm{Cir}^\tau(M)$ the moduli space of circles with complex torsion τ in a Kähler manifold M and by $\mathcal{CL}^\tau$ the restriction of $\mathcal{CL}$ onto this space. From those expressions on length spectrum of circles we establish the following main result.

Theorem 4.4. *For a complex projective space* $\mathbb{C}P^n(c)$ ($n \geqq 2$) *of constant holomorphic sectional curvature c, the length spectrum of circles has the following properties.*

(1) *Both the sets*

$$\mathrm{CLSpec}_\kappa(\mathbb{C}P^n(c)) = \mathcal{CL}(\mathrm{Cir}_\kappa(\mathbb{C}P^n(c))) \cap \mathbb{R}$$

and

$$\mathrm{CLSpec}^\tau(\mathbb{C}P^n(c)) = \mathcal{CL}(\mathrm{Cir}^\tau(\mathbb{C}P^n(c))) \cap \mathbb{R}$$

are unbounded discrete subsets of $\mathbb{R}$ *for each* $\kappa(>0)$ *and* $0 < \tau < 1$.

(2) *The length spectrum* $\mathrm{CLSpec}(\mathbb{C}P^n(c))$ *of circles coincides with the real positive line* $(0, \infty)$.

(3) *For $\kappa > 0$ the bottom of* $\mathrm{CLSpec}_\kappa(\mathbb{C}P^n(c))$ *is* $2\pi/\sqrt{\kappa^2+c}$, *which is the length of the holomorphic circle of curvature κ. The second lowest spectrum of* $\mathrm{CLSpec}_\kappa(\mathbb{C}P^n(c))$ *is* $4\pi/\sqrt{4\kappa^2+c}$, *which is the length of the totally real circle of curvature κ. They are simple for $\mathcal{CL}_\kappa$.*

(4) *The multiplicity m_c of $\mathcal{CL}$ is finite at each point $\lambda \in \mathbb{R}$. It satisfies*

$$\lim_{\lambda\to\infty} \frac{m_c(\lambda)}{\lambda^2 \log\lambda} = \frac{9c}{8\pi^4}.$$

(5) $\lambda (\in \mathbb{R})$ *is simple for $\mathcal{CL}$ if and only if λ is contained in the interval* $\left(\frac{2\pi}{\sqrt{c}}, \frac{4}{3}\sqrt{\frac{5}{c}}\pi\right]$.

(6) *The multiplicity of $\mathcal{CL}_\kappa$ $(\kappa > 0)$ is not uniformly bounded*;

$$\limsup_{\lambda\to\infty} \sharp(\mathcal{CL}_\kappa^{-1}(\lambda)) = \infty.$$

The growth order of the multiplicity with respect to λ is not so rapid. It satisfies $\lim_{\lambda\to\infty} \lambda^{-\delta}\sharp(\mathcal{CL}_\kappa^{-1}(\lambda)) = 0$ *for an arbitrary positive δ.*

The statements (2) and (5) in our theorem give the complete answer to the problem in the introduction.

Remark. We find that the length spectrum $\mathcal{CL}_{\sqrt{2c}/4}$ is not simple at the following points for example.

(i) Let γ_1 be a circle of curvature $\sqrt{2c}/4$ and complex torsion $\tau = \tau(27,7) = \frac{5698}{559\sqrt{559}}$ and γ_2 be a circle of curvature $\sqrt{2c}/4$ and complex torsion $\tau = \tau(25,19) = \frac{12502}{559\sqrt{559}}$. Then these two closed circles have the same curvature and the same length $\frac{4\sqrt{1118}}{3\sqrt{c}}\pi$. But they are not congruent.

(ii) Let γ_i be a circle of the same curvature $\sqrt{2c}/4$ and complex torsion $\tau_i = \tau(p_i,q_i)$, $i = 1,2,3$. Here we set $(p_1,q_1) = (129,71)$, $(p_2,q_2) = (131,59)$ and $(p_3,q_3) = (135,17)$. Note that $3p_i^2 + q_i^2 = 54964$ for $i = 1,2,3$. Then these three circles have the same curvature and the same length. But these three circles are not congruent each other.

Finally we investigate the asymptotic behavior of the number of congruency classes of closed circles of curvature κ. Let $n_c(\lambda;\kappa)$ denote the number of congruency classes of closed circles of curvature κ in M with length not greater than λ.

Theorem 4.5. *For a complex projective space $\mathbb{C}P^n(c)$ $(n \geqq 2)$ of constant holomorphic sectional curvature c, we have for $\kappa > 0$*

$$\lim_{\lambda\to\infty} \frac{n_c(\lambda;\kappa)}{\lambda^2} = \frac{3\sqrt{3}(4\kappa^2+c)}{8\pi^4}\tan^{-1}\left(\frac{1}{\sqrt{3}\alpha_\kappa}\right),$$

where α_κ $(\geqq 1)$ *denotes the number with*

$$\frac{3\sqrt{3}c\kappa}{(4\kappa^2+c)^{3/2}} = \frac{9\alpha_\kappa^2-1}{(3\alpha_\kappa^2+1)^{3/2}}.$$

In particular,

$$\lim_{\lambda\to\infty}\frac{n_c(\lambda;\sqrt{2c}/4)}{\lambda^2} = \frac{3\sqrt{3}c}{32\pi^3}.$$

The constant $c(\kappa)=\lim_{\lambda\to\infty}\lambda^{-2}n_c(\lambda;\kappa)$ *satisfies*

$$\lim_{\kappa\to 0}c(\kappa)=0 \quad \textit{and} \quad \lim_{\kappa\to\infty}c(\kappa)=\frac{9c}{16\pi^4}.$$

We finally pose some problems on length spectrum $\mathcal{CL}^\tau$ $(0<\tau<1)$ of circles.

Problems.

1. Are there non-simple spectrum for $\mathcal{CL}^\tau$ $(0<\tau<1)$?
2. Whether is the multiplicity of $\mathcal{CL}^\tau$ $(0<\tau<1)$ uniformly bounded or not?
3. Give an explicit formula of the first spectrum for $\mathcal{CL}^\tau$ $(0<\tau<1)$.
4. Study the asymptotic behavior of the number of congruency classes of closed circles of complex torsion $\tau(\neq 0,1)$ with respect to length.
5. Study the behavior of $c(\kappa)$. What is the maximum value of this function $c(\kappa)$?
6. Study the geometric meaning of the constants $\lim_{\lambda\to\infty}m_c(\lambda)/(\lambda^2\log\lambda)$ and $\lim_{\kappa\to\infty}c(\kappa)$.

5. Length spectrum of geodesics spheres in $\mathbb{C}P^n$.

In this section we study lengths of closed geodesics on geodesic spheres in a complex projective space. We first note that each geodesic sphere $G_m(2r/\sqrt{c})$ of radius $2r/\sqrt{c}$ $(0<r<\pi/2)$ with center $m\in\mathbb{C}P^n(c)$ which is imbedded as a real hypersurface in $\mathbb{C}P^n(c)$ is congruent to a tube of radius $(\pi-2r)/\sqrt{c}$ around totally geodesic complex hyperplane $\mathbb{C}P^{n-1}(c)$ in $\mathbb{C}P^n(c)$.

In general, each real hypersurface M admits an almost contact metric structure $(\phi,\xi,\eta,\langle\ ,\ \rangle)$ from the Kähler structure J of $\mathbb{C}P^n(c)$, which satisfies

$$\phi^2=-I+\eta\otimes\xi,\quad \eta(\xi)=1 \quad\text{and}\quad \langle\phi X,\phi Y\rangle=\langle X,Y\rangle-\eta(X)\eta(Y),$$

where I denotes the identity map of the tangent bundle TM of M. It is known that

$$(\nabla_X\phi)Y=\eta(Y)AX-\langle AX,Y\rangle\xi \quad\text{and}\quad \nabla_X\xi=\phi AX,$$

where ∇ is the Riemannian connection of M induced from the Fubini-Study metric of $\mathbb{C}P^n$.

We recall the following characterization of geodesic spheres in $\mathbb{C}P^n(c)$ (see [MOg]).

Proposition 5.1. *Let M be a real hypersurface of $\mathbb{C}P^n$. Then M is locally congruent to a geodesic sphere $G_m(r)$ if and only if there exist orthonormal vectors $v_1, \cdots, v_{2n-2}$ orthogonal to ξ at each point p of M such that all geodesics of M through p in the direction $v_i + v_j$ $(1 \leqq i \leqq j \leqq 2n-2)$ are circles in $\mathbb{C}P^n$ with positive curvature.*

Motivated by this proposition, we shall investigate the extrinsic shape of *every geodesic* of a geodesic sphere in $\mathbb{C}P^n$. It is enough to study the case of $c = 4$. The shape operator A of $G_m(r)$ in $\mathbb{C}P^n(4)$ is expressed as:

$$A\xi = (2\cot 2r)\xi \quad \text{and} \quad Au = (\cot r)u$$

for every tangent vector $u \in TG_m(r)$ orthogonal to ξ. Moreover, this real hypersurface $G_m(r)$ satisfies the following (cf. [NR]):

1) The structure tensor ϕ and the shape operator A of $G_m(r)$ in $\mathbb{C}P^n(4)$ are commutative: $\phi A = A\phi$.
2) The covariant derivative of the shape operator A satisfies

$$(\nabla_X A)Y = -\{\langle \phi X, Y\rangle \xi + \eta(Y)\phi X\}.$$

We remark that $\langle \dot\gamma(s), \xi\rangle$ is constant along γ. Indeed,

$$\nabla_{\dot\gamma}\langle \dot\gamma(s), \xi\rangle = \langle \dot\gamma(s), \phi A\dot\gamma\rangle = \langle \dot\gamma, A\phi\dot\gamma\rangle = -\langle \phi A\dot\gamma, \dot\gamma\rangle = 0.$$

We shall call this constant the *structure torsion* of γ and denote by $\sin\theta$ with $0 \leqq |\theta| \leqq \pi/2$.

By direct computation we obtain the following:

Proposition 5.2. *Let g be an isometric imbedding of a geodesic sphere $G_m(r)$ $(0 < r < \pi/2)$ into $\mathbb{C}P^n(4)$. Then the extrinsic shape $g \circ \gamma$ of a geodesic γ on $G_m(r)$ is as follows*:

(1) *Suppose the radius r satisfies $\pi/4 \leqq r < \pi/2$. If the structure torsion of γ is $\pm\cot r$, then the curve $g\circ\gamma$ is a geodesic.*
(2) *When $r \neq \pi/4$, if the structure torsion of γ is ± 1 (i.e. $\dot\gamma = \pm\xi$), then the curve $g\circ\gamma$ is a circle of curvature $2|\cot 2r|$ and of complex torsion ∓ 1 in $\mathbb{C}P^n(4)$. This circle lies on a totally geodesic $\mathbb{C}P^1(4)$.*
(3) *If γ has null structure torsion (i.e. $\dot\gamma$ is orthogonal to ξ), then the curve $g\circ\gamma$ is a circle of curvature $\cot r$ and null complex torsion in $\mathbb{C}P^n(4)$. This circle lies on a totally geodesic $\mathbb{R}P^2(1)$.*
(4) *Generally, if the structure torsion of γ is of the form $\sin\theta$ $(0 < |\theta| < \pi/2, \sin\theta \neq \pm\cot r)$, then the curve $g\circ\gamma$ is a holomorphic helix of proper order 4 whose curvatures are described as*

$$\kappa_1 = |\cot r - \tan r \cdot \sin^2\theta|,\ \kappa_2 = \tan r \cdot |\sin\theta|\cos\theta,\ \kappa_3 = \cot r.$$

Its complex torsions are described as

$$\tau_{12} = \begin{cases} -\ \sin\theta, & \text{if} \quad \cot r - \tan r \cdot \sin^2\theta > 0 \\ \sin\theta, & \text{if} \quad \cot r - \tan r \cdot \sin^2\theta < 0, \end{cases}$$
$$\tau_{14} = \begin{cases} -\ \mathrm{sgn}(\sin\theta)\cos\theta, & \text{if} \quad \cot r - \tan r \cdot \sin^2\theta > 0 \\ \mathrm{sgn}(\sin\theta)\cos\theta, & \text{if} \quad \cot r - \tan r \cdot \sin^2\theta < 0, \end{cases}$$
$$\tau_{23} = \mathrm{sgn}(\sin\theta)\cos\theta, \quad \tau_{34} = \sin\theta, \quad \tau_{13} = \tau_{24} = 0,$$

where $\mathrm{sgn}(a)$ *denotes the signature of a real number* a*. This helix* $g \circ \gamma$ *lies on a totally geodesic* $\mathbb{C}P^2(4)$.

It follows from Theorem B and Proposition 5.2 that

Proposition 5.3. *Every geodesic* γ *on a geodesic sphere* $G_m(r)$ *in* $\mathbb{C}P^n(c)$ *lies on* $\mathbb{C}P^2(c)$ (*which is a complex* 2*-dimensional complex linear subspace of* $\mathbb{C}P^n(c)$) *as a curve generated by a holomorphic Killing vector field on* $\mathbb{C}P^2(c)$*, so that* γ *is a simple curve lying on* $\mathbb{C}P^2(c)$*.*

In order to study lengths of closed geodesics on $G_m(r)$ in a complex projective space $\mathbb{C}P^n(4)$, we use the same idea as in section 3, which lies on considering a horizontal lift of a holomorphic helix $g \circ \gamma$ for every geodesic γ on $G_m(r)$. Regarding the curve $g \circ \gamma$ as a curve in a Euclidean space $\mathbb{C}^{n+1}$, we obtain an ordinary differential equation:

$$\begin{aligned}(g\circ\gamma)^{(4)} + (\cot^2 r + \cos^2\theta + \tan^2 r \sin^2\theta)(g\circ\gamma)'' \\ + \sin^2\theta(\tan^2 r\cos^2\theta + 1)g\circ\gamma = 0.\end{aligned}$$

Thus we can see that $g \circ \gamma$ is of the form

$$\begin{aligned}g\circ\gamma(s) =& A\exp(\sqrt{-1}s\tan r\sin\theta) + B\exp(-\sqrt{-1}s\tan r\sin\theta) \\ &+ C\exp(\sqrt{-1}s\sqrt{\cot^2 r + \cos^2\theta}) + D\exp(-\sqrt{-1}s\sqrt{\cot^2 r + \cos^2\theta})\end{aligned}$$

with some non-zero vectors $A, B, C, D \in \mathbb{C}^{n+1}$.

Lemma 5.4. *Let* σ *be a smooth simple curve on* $\mathbb{C}P^n(4)$*. Suppose a horizontal lift* $\tilde\sigma$ *of* σ *on* $S^{2n+1}(1)$ *is represented as*

$$\tilde\sigma(s) = Ae^{\sqrt{-1}as} + Be^{\sqrt{-1}bs} + Ce^{\sqrt{-1}cs} + De^{\sqrt{-1}ds},$$

which is a curve on $\mathbb{C}^{n+1}$ *with non-zero vectors* $A, B, C, D \in \mathbb{C}^{n+1}$ *and mutually distinct real numbers* a, b, c, d *which satisfy* $a + b + c + d = 0$ *and* $a \neq 0$*. Then* σ *is closed if and only if all the ratios* $b/a, c/a, d/a$ *are rational. In this case, its length is*

$$\mathrm{length}(\sigma) = 2\pi \times \mathrm{L.C.M.}\left(\frac{1}{|b-a|}, \frac{1}{|c-a|}, \frac{1}{|d-a|}\right).$$

Here for positive numbers $\alpha_1, \alpha_2, \alpha_3$, we denote by L.C.M.$\{\alpha_1, \alpha_2, \alpha_3\}$ *the minimum value of the set* $\{j\alpha_1 \mid j = 1, 2, \dots\} \cap \{j\alpha_2 \mid j = 1, 2, \dots\} \cap \{j\alpha_3 \mid j = 1, 2, \dots\}$.

Applying this lemma to our case, we obtain the following:

Proposition 5.5. *For a geodesic γ on a geodesic sphere $G_m(r)$ of radius r ($0 < r < \pi/2$) in $\mathbb{C}P^n$ of holomorphic sectional curvature 4 we have the following*:

1. *If the structure torsion of γ is ± 1, then γ is closed and its length is $\pi \sin 2r$.*
2. *If γ has null structure torsion, then γ is also closed and its length is $2\pi \sin r$.*
3. *When the structure torsion of γ is of the form* $\sin\theta$ $(0 < |\theta| < \pi/2)$, *it is closed if and only if*

$$\sin\theta = \frac{\pm q}{\sin r\sqrt{p^2 \tan^2 r + q^2}}$$

with some relatively prime positive integers p and q with $q < p\tan^2 r$. In this case, its length is

$$\text{length}(\gamma) = \begin{cases} 2\pi\sqrt{p^2 \sin^2 r + q^2 \cos^2 r}, & \textit{if } pq \textit{ is even} \\ \pi\sqrt{p^2 \sin^2 r + q^2 \cos^2 r}, & \textit{if } pq \textit{ is odd.} \end{cases}$$

Changing the metric homothetically, we obtain the following (Recall the lines after Theorem 3.3).

Theorem 5.6. *For a geodesic γ on a geodesic sphere $G_m(2r/\sqrt{c})$ of radius $2r/\sqrt{c}$ $(0 < r < \pi/2)$ in $\mathbb{C}P^n(c)$ of holomorphic sectional curvature c, we have the following*:

1. *If the structure torsion of γ is ± 1, then γ is closed and its length is* $(2\pi/\sqrt{c})\sin 2r$.
2. *If γ has null structure torsion, then γ is also closed and its length is* $(4\pi/\sqrt{c})\sin r$.
3. *When the structure torsion of γ is of the form* $\sin\theta$ $(0 < |\theta| < \pi/2)$, *it is closed if and only if*

$$\sin\theta = \frac{\pm q}{\sin r\sqrt{p^2 \tan^2 r + q^2}}$$

with some relatively prime positive integers p and q with $q < p\tan^2 r$. In this case, its length is

$$\text{length}(\gamma) = \begin{cases} 4\pi\sqrt{\frac{1}{c}\left(p^2 \sin^2 r + q^2 \cos^2 r\right)}, & \textit{if } pq \textit{ is even} \\ 2\pi\sqrt{\frac{1}{c}\left(p^2 \sin^2 r + q^2 \cos^2 r\right)}, & \textit{if } pq \textit{ is odd.} \end{cases}$$

We are now in a position to study the length spectrum of geodesic spheres in a complex projective space. We denote by $\mathrm{Geod}(N)$ the quotient space of the set of all geodesics on a Riemannian manifold N under the congruent relation with respect to the isometry group $\mathrm{Iso}(N)$ of N. We define the *length spectrum* $\mathcal{L} : \mathrm{Geod}(N) \to \mathbb{R} \cup \{\infty\}$ of N by $\mathcal{L}([\gamma]) = \mathrm{length}(\gamma)$, where $[\gamma]$ denotes the congruency class containing a geodesic γ. We also call the image $\mathrm{LSpec}(N) = \mathcal{L}(\mathrm{Geod}(N)) \cap \mathbb{R}$ the length spectrum of N. For example, the length spectrum of a standard unit sphere is $\mathrm{LSpec}(S^m(1)) = \{2\pi\}$. In order to study the length spectrum of a geodesic sphere $G_m(r)$ in a complex projective space, we need to study its isometry group. For a non-zero tangent vector $v \in T_xG_m(r)$ we denote by $\langle v\rangle$ the 1-dimensional linear subspace of $T_xG_m(r)$ spanned by v, and by $\langle v\rangle^{\perp}$ the orthogonal complement of $\langle v\rangle$ in $T_xG_m(r)$.

Lemma 5.7. *For any unit tangent vectors $u \in \langle \xi_x\rangle^{\perp}, v \in \langle \xi_y\rangle^{\perp}$ of $G_m(r)$ orthogonal to ξ at arbitrary points x, y, there exist isometries $\tilde{\varphi}^{+}, \tilde{\varphi}^{-}$ of $\mathbb{C}P^n$ with*

i) $\tilde{\varphi}^{+}(G_m(r)) = \tilde{\varphi}^{-}(G_m(r)) = G_m(r)$ *and* $\tilde{\varphi}^{+}(x) = \tilde{\varphi}^{-}(x) = y$,
ii) $d\tilde{\varphi}^{+}_x(u) = d\tilde{\varphi}^{-}_x(u) = v$ *and* $d\tilde{\varphi}^{+}_x(\xi_x) = \xi_y$, $d\tilde{\varphi}^{-}_x(\xi_x) = -\xi_y$.

This lemma guarantees that two geodesics on a geodesic sphere in $\mathbb{C}P^n$ are congruent if they have the same absolute values of the structure torsion. On the other hand, Theorem A and Proposition 5.2 show that two geodesics on a geodesic sphere in $\mathbb{C}P^n$ are not congruent if they do not have the same absolute values of the structure torsions. Hence we have

Proposition 5.8. *On a geodesic sphere $G_m(r)$ in a complex projective space, two geodesics are congruent with respect to the isometry group of $G_m(r)$ if and only if the absolute values of their structure torsions coincide.*

As a direct consequence of Theorem 5.6 we find that the length spectrum of a geodesic sphere $G_m(2r/\sqrt{c})$ in a complex projective space $\mathbb{C}P^n(c)$ is of the following form.

$$
\begin{aligned}
\mathrm{LSpec}\Big(G_m\Big(\frac{2r}{\sqrt{c}}\Big)\Big) &= \Big\{\frac{2\pi}{\sqrt{c}}\sin 2r\Big\} \bigcup \Big\{\frac{4\pi}{\sqrt{c}}\sin r\Big\} \\
&\bigcup \left\{ 4\pi\sqrt{\frac{1}{c}\left(p^2\sin^2 r + q^2\cos^2 r\right)} \;\middle|\; \begin{array}{c} p \text{ and } q \text{ are relatively} \\ \text{prime positive integers} \\ \text{which satisfy} \\ pq \text{ is even and } q < p\tan^2 r \end{array} \right\} \\
&\bigcup \left\{ 2\pi\sqrt{\frac{1}{c}\left\{p^2\sin^2 r + q^2\cos^2 r\right\}} \;\middle|\; \begin{array}{c} p \text{ and } q \text{ are relatively} \\ \text{prime positive integers} \\ \text{which satisfy} \\ pq \text{ is odd and } q < p\tan^2 r \end{array} \right\}.
\end{aligned}
$$

Therefore we obtain the following.

Theorem 5.9. *On a geodesic sphere $G_m(r)$ in $\mathbb{C}P^n$, there exist infinitely many congruency classes of closed geodesics. Moreover the length spectrum* $\mathrm{LSpec}(G_m(r))$ *of $G_m(r)$ is a discrete unbounded subset in the real line $\mathbb{R}$.*

By the expression of $\mathrm{LSpec}(G_m(r))$ we can also see that the multiplicity $m_{G_m(r)}(\lambda)$ of a spectrum λ, which is the cardinality of the set $\mathcal{L}^{-1}(\lambda)$, is finite at each λ. We here point out the first, the second and the third length spectrum, that is the minimum, the second minimum and the third minimum of the length spectrum.

Proposition 5.10. *For a geodesic sphere $G_m(2r/\sqrt{c})$ $(0 < r < \pi/2)$ in $\mathbb{C}P^n(c)$ of holomorphic sectional curvature c, we obtain the following*:

1. *The first length spectrum of $G_m(2r/\sqrt{c})$ is $(2\pi/\sqrt{c})\sin 2r$, which is the length of geodesics with structure torsion ± 1. It is simple.*
2. *The second length spectrum of $G_m(2r/\sqrt{c})$ is also simple.* *When $0 < r \leqq \pi/4$, it is $(4\pi/\sqrt{c})\sin r$, which is the length of geodesics with null structure torsion. When $\pi/4 < r < \pi/2$, it is $2\pi/\sqrt{c}$, which is the length of geodesics with structure torsion $\pm\cot r$.*
3. *The third length spectrum is also simple. When $\pi/4 < r < \pi/2$, it is $(4\pi/\sqrt{c})\sin r$, which is the length of geodesics with null structure torsion. When $\sqrt{2m-1} \leqq \cot r < \sqrt{2m+1}$ $(m = 1, 2, \dots)$, in particular, $0 < r \leqq \pi/4$, it is $2\pi\sqrt{\left\{4m(m+1)\sin^2 r + 1\right\}/c}$, which is the length of geodesics with structure torsion $\pm 1/(\sin r\sqrt{(2m+1)^2\tan^2 r + 1})$.*

We remark that the sectional curvature K of $G_m(r)$ in $\mathbb{C}P^n(4)$ lies in the interval $[\cot^2 r, 4 + \cot^2 r]$. Hence, when $\tan^2 r > 2$, we find that there exists some $\delta \in (0, 1/9)$ satisfying $\delta(4 + \cot^2 r) \leqq K \leqq (4 + \cot^2 r)$ and, moreover that the first length spectrum of $G_m(r)$ is smaller than $2\pi/\sqrt{4 + \cot^2 r}$. This implies that when $\tan^2 r > 2$, $G_m(r)$ is an example of so called a Bereger sphere, as was pointed out in [W]. But for other length spectrum, by virtue of the above argument we find that the following statement of Klingenberg's type holds:

Corollary. *Except geodesics with structure torsion ± 1, every geodesic γ on $G_m(r)$ $(0 < r < \pi/2)$ in $\mathbb{C}P^n(4)$ satisfies* $\mathrm{length}(\gamma) > 2\pi/\sqrt{4 + \cot^2 r}$.

Length spectrum is of course not necessarily simple. For example, in $\mathbb{C}P^n(4)$ we have

$$\mathrm{LSpec}\left(G_m\left(\frac{\pi}{4}\right)\right) = \Big\{\pi, \sqrt{2}\pi, \sqrt{5}\pi, \sqrt{10}\pi, \sqrt{13}\pi, \sqrt{17}\pi, 5\pi, \sqrt{26}\pi, \sqrt{29}\pi, \sqrt{34}\pi, \sqrt{37}\pi, \sqrt{41}\pi, \sqrt{50}\pi, \sqrt{53}\pi, \sqrt{58}\pi, \sqrt{61}\pi, \sqrt{65}\pi, \sqrt{73}\pi \dots\Big\}$$

and find that the multiplicity of $\sqrt{65}\pi$ is two; it is the common length of geodesics of structure torsions $3/\sqrt{65}$ and $7/\sqrt{65}$. Every spectrum which is smaller than $\sqrt{65}\pi$ is simple. Our aim here is to establish the following:

Theorem 5.11. *For a geodesic sphere* $G_m(2r/\sqrt{c})$ $(0 < r < \pi/2)$ *in* $\mathbb{C}P^n(c)$ *of holomorphic sectional curvature* c*, we obtain the following*:

(1) *If* $\tan^2 r$ *is irrational, every length spectrum of* $G_m(2r/\sqrt{c})$ *is simple.*
(2) *If* $\tan^2 r$ *is rational, the multiplicity of each length spectrum of* $G_m(2r/\sqrt{c})$ *is finite. But it is not uniformly bounded*;
$\limsup_{\lambda\to\infty} m_{G_m(2r/\sqrt{c})}(\lambda) = \infty$.
In this case, the growth order of $m_{G_m(2r/\sqrt{c})}$ *is not so rapid. It satisfies* $\lim_{\lambda\to\infty} \lambda^{-\delta} m_{G_m(2r/\sqrt{c})}(\lambda) = 0$ *for arbitrary positive* δ*.*

This theorem guarantees that on a geodesic sphere $G_m(r)$ with irrational $\tan^2 r$ in a complex projective space, two closed geodesics are congruent if and only if they have the same length. On the other hand, if $\tan^2 r$ is rational, this theorem shows that we can not classify congruency classes of geodesics only by their length.

Finally we make mention of the growth of the number of congruency classes of closed geodesics with respect to their length spectrum for a geodesic sphere in a complex projective space. For a Riemannian manifold N we denote by $n_N(\lambda)$ the cardinality of the set $\{[\gamma] \in \mathrm{Geod}(N) \mid \mathcal{L}_N([\gamma]) \leqq \lambda\}$.

Theorem 5.12. *For a geodesic sphere* $G_m(2r/\sqrt{c})$ *in* $\mathbb{C}P^n(c)$ *of holomorphic sectional curvature* c *we have*

$$\lim_{\lambda\to\infty} \frac{n_{G_m(2r/\sqrt{c})}(\lambda)}{\lambda^2} = \frac{3cr}{4\pi^4 \sin 2r}.$$

6. Holomorphic helices in a complex space form.

We shall show that the moduli of all holomorphic helices of order 3 in an n-dimensional complex space form is parametrized by three real numbers or two real numbers according as $n \geqq 3$ or $n = 2$. Moreover, we investigate the moduli of all holomorphic helices in a 2-dimensional complex space form.

Let γ be a helix in a Kähler manifold M (with complex structure J) of order $d(\leqq 2n)$ satisfying (2.1). Note that every helix is a real analytic curve in M. All the complex torsions $\tau_{ij}(s) = \langle V_i(s), JV_j(s)\rangle$ $(1 \leqq i < j \leqq d)$ satisfy the following differential equation

$$\frac{d}{ds}\tau_{ij}(s) = -\kappa_{i-1}\tau_{i-1,j}(s) + \kappa_i\tau_{i+1,j}(s) - \kappa_{j-1}\tau_{i,j-1}(s) + \kappa_j\tau_{i,j+1}(s), \tag{6.1}$$

where $\tau_{k\ell} = 0$ when $k = \ell$ or $k = 0$ or ℓ is greater than the proper order of γ. We hence from (6.1) get the following.

Proposition 6.1. *The complex torsions of a holomorphic helix of odd proper* *order d on a Kähler manifold satisfy the following relations*:

$$\begin{aligned}
\tau_{i,i+2k} &= 0 \quad \textit{for } i = 1, 2, \dots, d-2k, \\
&\qquad \textit{where } k = 1, 2, \dots, (d-1)/2, \\
\kappa_1\tau_{2d} &= \kappa_{d-1}\tau_{1,d-1}, \\
\kappa_1\tau_{2j} + \kappa_j\tau_{1,j+1} &= \kappa_{j-1}\tau_{1,j-1} \quad \textit{for } j = 3, 5, \dots, d-2, \\
\kappa_{i-1}\tau_{i-1,d} + \kappa_{d-1}\tau_{i,d-1} &= \kappa_i\tau_{i+1,d} \quad \textit{for } i = 3, 5, \dots, d-2, \\
\kappa_{i-1}\tau_{i-1,j} + \kappa_{j-1}\tau_{i,j-1} &= \kappa_i\tau_{i+1,j} + \kappa_j\tau_{i,j+1} \\
&\qquad \textit{for } i = 2, 3, \dots, d-3,\ j = i+2, i+4, \dots, d-1.
\end{aligned}$$

Proposition 6.2. *The complex torsions of a holomorphic helix of even proper order d on a Kähler manifold satisfy the following relations*:

$$\begin{aligned}
\tau_{i,i+2k} &= 0 \quad \textit{for } i = 1, 2, \dots, d-2k, \\
&\qquad \textit{where } k = 1, 2, \dots, (d-2)/2, \\
\kappa_1\tau_{2d} &= \kappa_{d-1}\tau_{1,d-1}, \\
\kappa_1\tau_{2j} + \kappa_j\tau_{1,j+1} &= \kappa_{j-1}\tau_{1,j-1} \quad \textit{for } j = 3, 5, \dots, d-1, \\
\kappa_{i-1}\tau_{i-1,d} + \kappa_{d-1}\tau_{i,d-1} &= \kappa_i\tau_{i+1,d} \quad \textit{for } i = 2, 4, \dots, d-2, \\
\kappa_{i-1}\tau_{i-1,j} + \kappa_{j-1}\tau_{i,j-1} &= \kappa_i\tau_{i+1,j} + \kappa_j\tau_{i,j+1} \\
&\qquad \textit{for } i = 2, 3, \dots, d-3,\ j = i+2, i+4, \dots, d-1.
\end{aligned}$$

Conversely, if the Frenet frame of a helix γ in a Kähler manifold satisfies the above relations at one point, then all derivatives of its complex torsions vanish at this point. Since γ is real analytic, we find that it is a holomorphic helix. We therefore have

Proposition 6.3. *For orthonormal vectors $v_1, \dots, v_d$ at a point p of a Kähler manifold M with complex structure J, we set $\tau_{ij} = \langle v_i, Jv_j\rangle$ $(1 \leqq i < j \leqq d)$. If positive constants $\kappa_1, \dots, \kappa_{d-1}$ and the vectors $v_1, \dots, v_d$ satisfy the relations in Proposition* 6.1 *or* 6.2, *then there exists a unique holomorphic helix with curvatures $\kappa_1, \dots, \kappa_{d-1}$ satisfying that the initial value of its Frenet frame is $(v_1, \dots, v_d)$.*

The following is easily verified.

Proposition 6.4. *The complex torsions τ_{ij} of a holomorphic helix of proper order d in a Kähler manifold M satisfy $\sum_{j=1}^{i-1}\tau_{ji}^2 + \sum_{j=i+1}^{d}\tau_{ij}^2 \leqq 1$ for each i.*

We here investigate holomorphic helices of order 3. We need to choose orthonormal vectors $v_1, v_2, v_3 \in T_pM$ which satisfy

$$\kappa_1\langle v_2, Jv_3\rangle = \kappa_2\langle v_1, Jv_2\rangle,\ \langle v_1, Jv_3\rangle = 0.$$

Identifying T_pM with $\mathbb{C}^n$, we set v_1, v_2 and v_3 as

$$\begin{cases} v_1 = (1, 0, \dots, 0), \\ v_2 = (-i\tau, \sqrt{1-\tau^2}, 0, \dots, 0), \\ v_3 = (0, -i\rho/\sqrt{1-\tau^2}, \sqrt{1-\tau^2-\rho^2}/\sqrt{1-\tau^2}, 0, \dots, 0) \end{cases}$$

for positive constants τ and ρ with $\tau^2 + \rho^2 \leqq 1$. Then they are orthonormal and satisfy $\langle v_1, Jv_2 \rangle = \tau, \langle v_2, Jv_3 \rangle = \rho, \langle v_1, Jv_3 \rangle = 0$. We therefore have

Theorem 6.5. *Let M be a Kähler manifold of dimension greater than 2. Then the following hold*:

(1) *Every holomorphic helix of order 3 satisfies*

$$\kappa_1 \tau_{23} = \kappa_2 \tau_{12},\ \tau_{13} = 0,\ |\tau_{12}| \leqq \frac{\kappa_1}{\sqrt{\kappa_1^2 + \kappa_2^2}}.$$

(2) *Conversely, if nonnegative constants κ_1, κ_2 and a constant τ satisfy $|\tau| \leqq \kappa_1/\sqrt{\kappa_1^2 + \kappa_2^2}$, then there exists a holomorphic helix of order 3 on M with the first curvature κ_1 and the second curvature κ_2, and with the first complex torsion $\tau_{12} = \tau$.*

(3) *If $|\tau| > \kappa_1/\sqrt{\kappa_1^2 + \kappa_2^2}$, then we have no such a holomorphic helix of order 3 on M.*

Theorem 6.6. *Let M be a 2-dimensional Kähler manifold. Then the following hold*:

(1) *The complex torsions of each holomorphic helix of proper order 3 in M are*

$$\tau_{12} = \frac{\kappa_1}{\sqrt{\kappa_1^2 + \kappa_2^2}},\ \tau_{13} = 0,\ \tau_{23} = \frac{\kappa_2}{\sqrt{\kappa_1^2 + \kappa_2^2}} \tag{6.2}$$

or

$$\tau_{12} = -\frac{\kappa_1}{\sqrt{\kappa_1^2 + \kappa_2^2}},\ \tau_{13} = 0,\ \tau_{23} = -\frac{\kappa_2}{\sqrt{\kappa_1^2 + \kappa_2^2}}, \tag{6.3}$$

where its curvatures are κ_1 and κ_2.

(2) *Conversely for given positive constants κ_1 and κ_2, there exists a holomorphic helix of proper order 3 with curvatures κ_1 and κ_2, and with complex torsions defined by* (6.2) *or* (6.3)*.*

Such a description as above for holomorphic helices of order 4 is much more complicated. We restrict ourselves here to holomorphic helices in a 2-dimensional

Kähler manifold M. For given constants τ and ρ with $\tau^2+\rho^2=1$, we choose vectors

$$v_1=(1,0),\ v_2=(-i\tau,\rho),\ v_3=(0,-i),\ v_4=\mp(i\rho,\tau)$$

in $T_pM\cong\mathbb{C}^2$. Then they are orthonormal and satisfy

$$\langle v_1,Jv_2\rangle=\tau,\ \langle v_2,Jv_3\rangle=\rho,\ \langle v_1,Jv_4\rangle=\pm\rho,$$
$$\langle v_1,Jv_3\rangle=\langle v_2,Jv_4\rangle=0,\ \langle v_3,Jv_4\rangle=\pm\tau.$$

On the other hand, Proposition 6.2 shows that a helix is a holomorphic helix if and only if

$$\tau_{13}(0)=\tau_{24}(0)=0,\ \kappa_1\tau_{23}(0)+\kappa_3\tau_{14}(0)=\kappa_2\tau_{12}(0),$$
$$\kappa_1\tau_{14}(0)+\kappa_3\tau_{23}(0)=\kappa_2\tau_{34}(0).$$

We therefore have

Theorem 6.7. *Let M be a 2-dimensional Kähler manifold. Then the following hold*:

(1) *The complex torsions of each holomorphic helix of proper order 4 with curvatures κ_1,κ_2 and κ_3 on M satisfy one of the following*:

$$\tau_{12}=\tau_{34}=\tau,\ \tau_{23}=\tau_{14}=\frac{\kappa_2\tau}{\kappa_1+\kappa_3},\ \tau_{13}=\tau_{24}=0, \tag{6.4}$$

where $\tau=\pm(\kappa_1+\kappa_3)/\sqrt{\kappa_2^2+(\kappa_1+\kappa_3)^2}$,

$$\tau_{12}=-\tau_{34}=\tau,\ \tau_{23}=-\tau_{14}=\frac{\kappa_2\tau}{\kappa_1-\kappa_3},\ \tau_{13}=\tau_{24}=0, \tag{6.5}$$

when $\kappa_1\neq\kappa_3$, where $\tau=\pm(\kappa_1-\kappa_3)/\sqrt{\kappa_2^2+(\kappa_1-\kappa_3)^2}$, or

$$\tau_{12}=\tau_{34}=\tau_{13}=\tau_{24}=0,\ \tau_{23}=-\tau_{14}=\pm1, \tag{6.5'}$$

when $\kappa_1=\kappa_3$.

(2) *Conversely, for given any positive constants κ_1,κ_2 and κ_3, there exist holomorphic helices of proper order 4 in M with curvatures κ_1,κ_2 and κ_3, and with complex torsions defined by* (6.4), (6.5) *or* (6.4), (6.5′).

Remark. The complex torsions of the holomorphic helices in (4) of Proposition 5.2 satisfy

i) (6.4) when $\cot r-\tan r\cdot\sin^2\theta<0$, and
ii) (6.5) when $\cot r-\tan r\cdot\sin^2\theta>0$.

We here rewrite Theorem 6.7 in the case where the ambient Kähler manifold M is a complex space form $M_n(c)(=\mathbb{C}^n,\ \mathbb{C}P^n(c)$ or $\mathbb{C}H^n(c))$ of constant holomorphic sectional curvature c. We denote by $Hh^d(M_n(c))$ the set of all equivalence classes of all holomorphic helices of order $d(\leqq 2n)$ in $M_n(c)$ with respect to holomorphic isometries of $M_n(c)$. By virtue of Theorem A the set $Hh^d(M_n(c))$ can be naturally regarded as a set of $[0,\infty)^{d-1}\times[-1,1]^{d(d-1)/2}\subset\mathbb{R}^{(d+2)(d-1)/2}$.

Theorem 6.8. *For given positive constants* κ_1, κ_2 *and* κ_3*, there exist four equivalence classes of holomorphic helices of proper order* 4 *with curvatures* κ_1, κ_2 *and* κ_3 *with respect to holomorphic isometries of* $M_2(c)$*. In addition, these four equivalence classes are given by* (6.4), (6.5) *or* (6.4), (6.5′).

Finally we shall investigate the moduli spaces $Hh^d(M_n(c))$ $(d = 1, 2, 3)$. The moduli space $Hh^1(M_n(c))$ clearly consists of one point. As an immediate consequence of the above discussion we can establish the following.

Theorem 6.9. (1) *The moduli space* $Hh^2(M_n(c))$ *is homeomorphic to a cone in* $\mathbb{R}^2$ *or a half line according as* $n \geqq 2$ *or* $n = 1$*. More precisely,* $Hh^2(M_n(c))$ *is* $[0,\infty) \times [-1,1]/\sim$ *or* $[0,\infty)$ *according as* $n \geqq 2$ *or* $n = 1$*, where the equivalence relation* $\sim$ *means that* $(0,\tau) \sim (0,\rho)$ *if* $\tau,\ \rho \in [-1,1]$.
(2) *The moduli space* $Hh^3(M_n(c))$ *is connected and*

$$Hh^3(M_n(c)) = \begin{cases} \Big\{(\kappa_1,\kappa_2,\tau) \in [0,\infty) \times [0,\infty) \times [-1,1] \Big| \\ \qquad\qquad\qquad\qquad \tau^2 \leqq \frac{\kappa_1^2}{(\kappa_1^2+\kappa_2^2)} \Big\}\Big/\sim, & n \geqq 3, \\ \Bigg([0,\infty) \times \{0\} \times [-1,1] \\ \qquad \cup \Big\{ \Big(\kappa_1,\kappa_2,\pm\frac{\kappa_1}{\sqrt{\kappa_1^2+\kappa_2^2}}\Big) \ \Big|\ \kappa_1 > 0, \kappa_2 > 0 \Big\} \Bigg)\Big/\sim, & n = 2, \end{cases}$$

where the equivalence relation $\sim$ *means that* $(0,\kappa,\tau) \sim (0,\ell,\rho)$ *if* $\kappa,\ell \in [0,\infty)$ *and* $\tau,\rho \in [-1,1]$.

7. Closed helices with self-intersections in $\mathbb{C}P^n$.

In this section we give a class of closed helices with self-intersections in a complex projective plane $\mathbb{C}P^2$ with the aid of the isometric imbedding f in the case of $n = 2$ in section 3. Namely we consider the isometric imbedding $f : M = (S^1 \times S^1)/\sim \longrightarrow \mathbb{C}P^2(4)$ defined by
(7.1)

$$f([e^{i\theta},(a_1,a_2)]) = \pi\left(\frac{1}{3}(e^{-\frac{2i\theta}{3}} + 2a_1e^{\frac{i\theta}{3}}), \frac{\sqrt{2}}{3}(e^{-\frac{2i\theta}{3}} - a_1e^{\frac{i\theta}{3}}), \frac{2}{\sqrt{6}}ia_2e^{\frac{i\theta}{3}}\right),$$

where $\pi : S^5(1) \to \mathbb{C}P^2(4)$ is the Hopf fibration and $(a_1)^2 + (a_2)^2 = 1$.

We here study images of circles in M under this isometric parallel imbedding. As we see in section 3, the imbedding f maps each geodesic of M to a circle of curvature $1/\sqrt{2}$ in $\mathbb{C}P^2(4)$. This circle does not have self-intersections, but it is not necessarily closed in $\mathbb{C}P^2(4)$. For images of circles on M through f we have the following.

Proposition 7.1. *For a circle γ of curvature $\kappa(>0)$ on M, the curve $f\circ\gamma$ is a helix of order* 4 *in $\mathbb{C}P^2(4)$. More precisely, we have the following.*

(1) *When $\kappa=1/2$, it is a helix of proper order* 3 *with curvatures*
$\kappa_1=\frac{\sqrt{3}}{2}$, $\kappa_2=\sqrt{\frac{3}{2}}$.

(2) *When $\kappa\neq 1/2$, it is a helix of proper order* 4 *with curvatures*
$\kappa_1=\sqrt{\kappa^2+\frac{1}{2}}$, $\kappa_2=\frac{3\kappa}{\sqrt{2\kappa^2+1}}$, $\kappa_3=\frac{|4\kappa^2-1|}{\sqrt{2(2\kappa^2+1)}}$.

We now compute the complex torsions of $f\circ\gamma$ for a circle γ which satisfies the following equations:

$$\nabla_X X=\kappa Y \text{ and } \nabla_X Y=-\kappa X, \text{ with } X=V_1=\dot{\gamma}.$$

We can represent the orthonormal pair $\{X,Y\}$ as

$$\begin{cases} X=\cos\psi\cdot(u,0)+\sin\psi\cdot(0,w), \\ Y=-\sin\psi\cdot(u,0)+\cos\psi\cdot(0,w) \quad (0\leqq\psi<2\pi) \end{cases} \tag{7.2}$$

at each point $\gamma(s)$, where $w\in TS^1(1)$ is a unit tangent vector of the second component, and u is the normalized vector of $\partial/\partial\theta$. We here make use of the representation (7.2). Straightforward computation yields the following.

Proposition 7.2. *Let γ be a circle of curvature κ (>0) in M. Then the complex torsions $\tau_{ij}(s)=\langle V_i(s),JV_j(s)\rangle$ $(1\leqq i<j\leqq 4)$ of $f\circ\gamma$ are described as follows*:

(1) *When $\kappa>1/2$, we have*

$$\tau_{12}=\tau_{34}=\frac{1}{\sqrt{2\kappa^2+1}}\cos 3(\kappa s+\psi_0),\quad \tau_{13}=-\tau_{24}=-\sin 3(\kappa s+\psi_0),$$
$$\tau_{14}=\tau_{23}=-\frac{\sqrt{2}\kappa}{\sqrt{2\kappa^2+1}}\cos 3(\kappa s+\psi_0).$$

(2) *When $\kappa=1/2$, we have*

$$\tau_{12}=\sqrt{\frac{2}{3}}\cos 3(\frac{1}{2}s+\psi_0),\ \tau_{13}=-\sin 3(\frac{1}{2}s+\psi_0),\ \tau_{23}=-\frac{1}{\sqrt{3}}\cos 3(\frac{1}{2}s+\psi_0).$$

(3) *When $\kappa<1/2$, we have*

$$\tau_{12}=-\tau_{34}=\frac{1}{\sqrt{2\kappa^2+1}}\cos 3(\kappa s+\psi_0),\quad \tau_{13}=\tau_{24}=-\sin 3(\kappa s+\psi_0),$$
$$\tau_{14}=-\tau_{23}=\frac{\sqrt{2}\kappa}{\sqrt{2\kappa^2+1}}\cos 3(\kappa s+\psi_0).$$

Here, ψ_0 is the angle between $\dot\gamma(0)$ and the unit vector u tangent to the first component of M.

This proposition shows that $f \circ \gamma$ is not generated by any Killing vector field on $\mathbb{C}P^2(4)$.

We now consider the universal Riemannian covering $p : \mathbb{R}^2 \to M$. Regarding the Riemannian metric of M, we can choose a fundamental region for N in $\mathbb{R}^2$ as $\mathfrak{F} = [0, 2\sqrt{2}\pi/3) \times [0, \sqrt{6}\pi/3)$. Two points (x_1, x_2) and (y_1, y_2) on $\mathbb{R}^2$ satisfy $p((x_1, x_2)) = p((y_1, y_2))$ if and only if either

i) $x_1 - y_1 = 2\sqrt{2}m_1\pi/3$, $x_2 - y_2 = 2\sqrt{6}m_2\pi/3$ for some $m_1, m_2 \in \mathbb{Z}$, or
ii) $x_1 - y_1 = \sqrt{2}(2m_1+1)\pi/3$, $x_2 - y_2 = \sqrt{6}(2m_2+1)\pi/3$ for some $m_1, m_2 \in \mathbb{Z}$.

Let $\tilde\gamma$ denote a covering circle of γ in $\mathbb{R}^2$. Then it is a circle of radius $1/\kappa$ in the sense of Euclidean Geometry. This guarantees that γ is a closed curve of length $2\pi/\kappa$ and moreover that γ has self-intersections in the case of $\kappa \leqq 3/(\sqrt{2}\pi)$. We remark that if $\gamma(s_0) = \gamma(0)$ for some $s_0 \neq 0$, then $\tilde\gamma(s_0)$ and $\tilde\gamma(0)$ satisfy either the condition i) or ii). Therefore we obtain the following.

Theorem 7.3. *Let $f : M \longrightarrow \mathbb{C}P^2(4)$ denote the imbedding defined by* (7.1) *and γ be any circle of curvature κ (> 0) in M. Then we have the following*:

(1) *The helix $f \circ \gamma$ is closed of length $2\pi/\kappa$, and is not generated by any Killing vector field on $\mathbb{C}P^2(4)$.*
(2) *The helix $f \circ \gamma$ has self-intersections if and only if $\kappa \leqq 3/(\sqrt{2}\pi)$. The number of intersection points is greater than* 2.

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