Mem. Fac. Sci. Eng. Shimane Univ.
Series B: Mathematical Science
34 (2001), pp. 1–19

THE CATEGORY $\mathcal{MAP}$

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Communicated by T. Miwa

(Received: December 24, 2000)

Abstract. In this paper we introduce a new category $\mathcal{MAP}$ of continuous maps and their morphisms as an extension of both the category $\mathcal{TOP}_Y$ (of continuous maps into a fixed space Y and their morphisms) and $\mathcal{TOP}$ (of topological spaces and continuous maps as morphisms). Several operations in $\mathcal{MAP}$ are introduced such as products, fibrewise products, inverse limits, sums, fibrewise sums and direct limits. Finally we take a look at compact (perfect) maps as an object in the category $\mathcal{MAP}$.

1. Introduction

The study of General Topology is usually concerned with the category $\mathcal{TOP}$ of topological spaces as objects, and continuous maps as morphisms. It goes without saying that both of these concepts are equally important. Moreover, one can look at a space as a map from this space onto a singleton space and in this manner identify these two concepts.

Bearing this in mind, a branch of General Topology which has become known as General Topology of Continuous Maps, or Fibrewise General Topology, was initiated. This field of research is concerned most of all in extending the main notions and results concerning topological spaces to that of continuous maps. This is usually done in the following way. For an arbitrary topological space Y one considers the category $\mathcal{TOP}_Y$, the objects of which are continuous maps into the space Y, and for the objects $f : X \to Y$ and $g : Z \to Y$, a morphism from f into g is a continuous map $\lambda : X \to Z$ with the property $f = g \circ \lambda$. In defining properties of a continuous map $f : X \to Y$ one does not directly involve any properties on the spaces X and Y (except the existence of a topology). Such were the definitions given in [2, 3, 4, 12, 13] for the separation axioms, compactness, paracompactness,

1991 *Mathematics Subject Classification.* Primary 54C05, 54C99; Secondary 54C10, 54B30, 54B35.

Key words and phrases. Fibrewise Topology, Categorical Topology, Continuous Map.

The author wishes to thank Professor D. Shakhmatov of Ehime University for suggesting the idea of extending the category $\mathcal{TOP}_Y$ during a private conversation at the General Topology Symposium 1998, Matsuyama, Ehime, Japan.

metrization, weight and others. As noted above this situation is a generalization of the category $\mathcal{TOP}$, since the category $\mathcal{TOP}$ is isomorphic to the particular case of $\mathcal{TOP}_Y$ in which the space Y is a singleton space.

The carried out research showed a strong analogy in the behaviour of spaces and maps and although, in general, a map is a more complicated concept than a space (in the sense described in the first paragraph above), it was possible to extend many important notions and results of spaces to that of maps. Since the considered case is of a wider generality (compared to that of spaces), the results obtained for maps are technically more complicated. Moreover there are moments which are specific to maps. For example, there is no analogue to Urysohn's Lemma for maps and so normality and functional normality do not coincide.

Some results in the General Topology of Continuous Maps were obtained quite some time ago. For example, in 1947, I.A.Vainstein [16] proposed the name of compact maps to perfect maps. Here it is important to realize that the notion of perfect map is the analogue in $\mathcal{TOP}_Y$ of the notion compact space in $\mathcal{TOP}$. G.T.Whyburn in 1953 [17, 18], as did G.L.Cain, N.Krolevets, V.M.Ulyanov [15] and others, considered compactifications of maps. Completely regular and Tychonoff maps, as well as (functionally) normal maps, were defined by B.A.Pasynkov in 1984. These definitions made it possible to generalize and obtain an analogue to the theorem on the embedding of Tychonoff spaces of weight τ into I^τ and to the existence of a compactification for a Tychonoff space having the same weight. It was also possible to construct a maximal Tychonoff compactification for a Tychonoff map (i.e. construct an analogue to the Stone-Čech compactification). For more details and other results on the General Topology of Continuous Maps one can consult [2, 3, 4, 7, 8, 9, 12, 13].

The aim of this paper is to study a category of maps in which we do not restrain ourselves with a fixed base space Y. This gives rise to a new category which we denote as $\mathcal{MAP}$. In a following paper [1], the products in $\mathcal{MAP}$ defined in sections 4 and 5 are used to obtain universal type theorems for T_0, Tychonoff and zero-dimensional maps.

2. Preliminary notions

We begin this section by introducing the category $\mathcal{MAP}$.

The objects of $\mathcal{MAP}$ are continuous maps from any topological space into any topological space. For two objects $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$, a morphism from f_1 into f_2 is a pair of continuous maps $\{\lambda_T, \lambda_B\}$, where $\lambda_T : X_1 \to X_2$ and $\lambda_B : Y_1 \to Y_2$, such that the diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{\lambda_T} & X_2 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ Y_1 & \xrightarrow[\lambda_B]{} & Y_2 \end{array}$$

is commutative. It is not difficult to see that this definition of a morphism in $\mathcal{MAP}$ satisfies the necessary axioms that morphisms should satisfy in any category (see, for example, [14]).

Let $\mathcal{P}_T$ and $\mathcal{P}_B$ be two topological/set-theoretic properties of maps (for example: closed, open, 1-1, onto, etc.). If λ_T has property $\mathcal{P}_T$ and λ_B has property $\mathcal{P}_B$ then we say that $\{\lambda_T, \lambda_B\}$ is a *$\{\mathcal{P}_T, \mathcal{P}_B\}$-morphism*. If $\mathcal{P}_T$ is the *continuous* property, then we say that $\{\lambda_T, \lambda_B\}$ is a *$\{*, \mathcal{P}_B\}$-morphism*, similarly for $\mathcal{P}_B$. Therefore, a $\{*, *\}$-morphism is just a morphism. Also, if $\mathcal{P}_T = \mathcal{P}_B = \mathcal{P}$ then a $\{\mathcal{P}_T, \mathcal{P}_B\}$-morphism is called a *$\mathcal{P}$-morphism*.

As noted in the introduction, separation axioms for maps have already been defined in the category $\mathcal{TOP}_Y$ and since these axioms involve only one map, they have also been defined for the category $\mathcal{MAP}$. For completeness we now give these definitions.

Definition 2.1. A continuous map $f : X \to Y$ is said to be a *T_i-map*, $i = 0, 1, 2$, if for every two distinct points $x, x' \in X$ lying in the same fibre, the following condition is respectively satisfied:

$i = 0$: at least one of the points x, x' has a neighborhood in X which does not contain the other point;

$i = 1$: each of the points x, x' has a neighborhood in X which does not contain the other point;

$i = 2$: the points x and x' have disjoint neighborhoods in X.

A T_2-map is also called *Hausdorff*. Such maps were considered quite some time ago. It appears to be that T_0- and T_1-maps were defined for the first time in [12]. We note that for $i = 0, 1$ the property for a map $f : X \to Y$ to be a T_i-map, is equivalent to the property that all the fibres $f^{-1}y$, $y \in Y$, are T_i-spaces.

Definition 2.2. The subsets A and B of the space X are said to be respectively:

(a) *neighborhood separated in $U \subset X$*,
(b) *functionally separated in $U \subset X$*,

if the sets $A \cap U$ and $B \cap U$

(a) have disjoint neighborhoods in U,
(b) are functionally separated in U (that is, there exists a continuous function $\phi : U \to [0, 1]$ such that $A \cap U \subset \phi^{-1}(0)$ and $B \cap U \subset \phi^{-1}(1)$).

Definition 2.3. A continuous map $f : X \to Y$ is said to be *functionally Hausdorff or $T_{2\frac{1}{2}}$*, if for every two distinct points x and x' in X lying in the same fibre, there exists a neighborhood O of the point $f(x)$, such that the sets $\{x\}$ and $\{x'\}$ are functionally separated in $f^{-1}O$.

It is evident that every functionally Hausdorff map is Hausdorff.

Definition 2.4. A continuous map $f : X \to Y$ is said to be *completely regular* (resp. *regular*), if for every point $x \in X$ and every closed set F in X, not containing the point x, there exists a neighborhood O of the point $f(x)$, such that the sets $\{x\}$ and F are functionally separated (resp. neighborhood separated) in $f^{-1}O$.

Thus every completely regular map is also regular.

Definition 2.5. (a) A completely regular T_0-map is called a *Tychonoff* (or $T_{3\frac{1}{2}}$-) map.

(b) A regular T_0-map is called a T_3-map.

It can be easily verified that every T_j-map is a T_i-map for $j, i = 0, 1, 2, 3, 3\frac{1}{2}$ and $i \leqslant j$. We also have that every Tychonoff map is functionally Hausdorff.

Definition 2.6. For a continuous map $f : X \to Y$, the subsets A and B of the space X are said to be *f-functionally separated* (resp. *f-neighborhood separated*) *over the set* $K \subset Y$, if for every $y \in K$ there exists a neighborhood O of y in K, such that the sets A and B are functionally separated (resp. neighborhood separated) in $f^{-1}O$. The sets A and B are said to be *f-functionally separated* (resp. *f-neighborhood separated*), if they are f-functionally separated (resp. f-neighborhood separated) over all the space Y.

Definition 2.7. A continuous map $f : X \to Y$ is said to be *functionally prenormal* (resp. *prenormal*) if every two disjoint closed sets in X are f-functionally separated (resp. f-neighborhood separated).

Therefore, a functionally prenormal map is prenormal.

Definition 2.8. A continuous map $f : X \to Y$ is said to be *functionally normal* (resp. *normal*) if for every open set O in Y the map $f : f^{-1}O \to O$ is functionally prenormal (resp. prenormal).

It is evident that every normal map is prenormal. Also, a functionally normal map is normal and functionally prenormal. A normal T_3-map is called a T_4-map, and a functionally normal $T_{3\frac{1}{2}}$-map is called a $T_{4\frac{1}{2}}$-map. As noted in the introduction, completely regular and Tychonoff maps, as well as (functionally) normal maps, were defined by B.A.Pasynkov [12].

If the map f is closed we have the following two results.

Proposition 2.1. *For a closed map $f : X \to Y$ we have:*

1. *If every two disjoint closed sets in $f^{-1}y$ are neighborhood separated in X, for every $y \in Y$, then f is normal;*
2. *If every two disjoint closed sets in the $f^{-1}y$ are functionally separated in some neighborhood of $f^{-1}y$ in X, for every $y \in Y$, then f is functionally normal.*

Proposition 2.2. *For a closed map $f : X \to Y$ we have:*

1. *If for every $y \in Y$, every $x \in f^{-1}y$ and every closed set A in $f^{-1}y$ such that $x \notin A$, the sets $\{x\}$ and A are neighborhood separated in X, then f is regular;*
2. *If for every $y \in Y$, every $x \in f^{-1}y$ and every closed set A in $f^{-1}y$ such that $x \notin A$, the sets $\{x\}$ and A are functionally separated in some neighborhood of $f^{-1}y$, then f is completely regular.*

The above proposition shows that a closed normal T_1-map is T_3 and so is a T_4-map. For more results concerning the above separation axioms for maps one can consult [12, 13].

We now give the definition of collectionwise normal maps due to D.Buhagiar [4].

Definition 2.9. A T_1-map f is said to be *collectionwise prenormal* if for every discrete collection $\{F_s : s \in \mathcal{S}\}$ of closed subsets of X and for every $y \in Y$, there exist a neighborhood O_y of y in Y and a collection of open subsets $\{U_s : s \in \mathcal{S}\}$, such that $F_s \cap f^{-1}O_y \subset U_s$ and U_s are discrete in $f^{-1}O_y$. The map f is said to be *collectionwise normal* if for every open set O in Y, the map $f|_{f^{-1}O} : f^{-1}O \to O$ is collectionwise prenormal.

We note the following result [4].

Proposition 2.3. *A T_1-map f is collectionwise normal if and only if for every open set O in Y, every closed discrete (in $f^{-1}O$) collection $\{F_s : s \in \mathcal{S}\}$ and every $y \in O$, there exists a neighborhood $O_y \subset O$ of y such that $\{F_s \cap f^{-1}O_y : s \in \mathcal{S}\}$ are neighborhood separated.*

Further results concerning collectionwise normal maps can be found in [4, 3].

Finally, we give the definitions of base and weight for a continuous map, both given by B.A.Pasynkov [10, 12].

Definition 2.10. Let $f : X \to Y$ be a map of topological spaces. A collection $\mathfrak{B}_f$ of open subsets of X is called a *base for the map f*, if for every point $x \in X$ and every neighborhood U_x of x in X there exists a neighborhood O_y of the point $y = f(x)$ in Y and an element $V \in \mathfrak{B}_f$ such that $x \in f^{-1}O_y \cap V \subset U_x$.

Definition 2.11. A collection $\mathfrak{S}_f$ of open subsets of X is called a *subbase for the map f* if the intersection of finite subcollections of the collection $\mathfrak{S}_f$ constitute a base for the map f.

Definition 2.12. The minimal cardinal number of the form $|\mathfrak{B}_f|$, where $\mathfrak{B}_f$ is a base for the map f, is called the *weight of the continuous map f* and is denoted by $\mathfrak{w}(f)$.

3. Submaps

In this section we now give the definition of a submap as an analogue of subspace. Since we do not restrict ourselves to a fixed base space Y our definition slightly differs from that given in the category $\mathcal{TOP}_Y$ [12].

Definition 3.1. The map $g : A \to B$ is said to be a (closed, open, everywhere dense, etc.) submap of the map $f : X \to Y$, if g is the restriction of the map f on the (closed, open, everywhere dense, etc.) subset A of the space X and $g(A) = f(A) \subset B \subset Y$.

The following proposition is not difficult to prove.

Proposition 3.1. *Any submap of a T_i-map is a T_i-map for $i \leqslant 3\frac{1}{2}$. Prenormality, functional prenormality, normality, functional normality, collectionwise prenormality and collectionwise normality are hereditary with respect to closed submaps.*

4. Tychonoff products

In this section we introduce the notion of product in the category $\mathcal{MAP}$. This is the usual product of maps and we recall the definition.

Definition 4.1. Let $\{f_\alpha : \alpha \in \mathcal{A}\}$ be a collection of continuous maps, where $f_\alpha : X_\alpha \to Y_\alpha$. The Tychonoff product of the maps $\{f_\alpha : \alpha \in \mathcal{A}\}$, which is denoted by $\prod\{f_\alpha : \alpha \in \mathcal{A}\}$, is the continuous map which assigns to the point $x = \{x_\alpha\} \in \prod\{X_\alpha : \alpha \in \mathcal{A}\}$ the point $\{f_\alpha(x_\alpha)\} \in \prod\{Y_\alpha : \alpha \in \mathcal{A}\}$.

Let $pr_T^\alpha : \prod\{X_\alpha : \alpha \in \mathcal{A}\} \to X_\alpha$ and $pr_B^\alpha : \prod\{Y_\alpha : \alpha \in \mathcal{A}\} \to Y_\alpha$ be the projections. Then we have that the diagram

$$\begin{array}{ccc} \prod\{X_\alpha : \alpha \in \mathcal{A}\} & \xrightarrow{pr_T^\alpha} & X_\alpha \\ {\scriptstyle \prod\{f_\alpha:\alpha\in\mathcal{A}\}}\big\downarrow & & \big\downarrow{\scriptstyle f_\alpha} \\ \prod\{Y_\alpha : \alpha \in \mathcal{A}\} & \xrightarrow[pr_B^\alpha]{} & Y_\alpha \end{array}$$

is commutative. Therefore, the pair $\{pr_T^\alpha, pr_B^\alpha\}$ serve as a projective morphism from $\prod\{f_\alpha : \alpha \in \mathcal{A}\}$ onto f_α. It can be seen that this definition of product confers well with the notion of categorical product (see, for example [14]).

Proposition 4.1. *Let $f = \prod\{f_\alpha : \alpha \in \mathcal{A}\} : X = \prod\{X_\alpha : \alpha \in \mathcal{A}\} \to Y = \prod\{Y_\alpha : \alpha \in \mathcal{A}\}$ be the Tychonoff product of the maps $f_\alpha : X_\alpha \to Y_\alpha$. If $\mathfrak{B}_{f_\alpha}$ is a base for the map f_α for every $\alpha \in \mathcal{A}$, then the collection $\mathfrak{S}_f = \bigcup\{(pr_T^\alpha)^{-1}\mathfrak{B}_{f_\alpha} : \alpha \in \mathcal{A}\}$ is a subbase for the map f and the weight $\mathfrak{w}(f) \leqslant \max\{|\mathcal{A}|, \sup\{\mathfrak{w}(f_\alpha) : \alpha \in \mathcal{A}\}, \aleph_0\}$.*

Proof. Let $x = \{x_\alpha\}$ be an arbitrary point in X and consider a neighbourhood U of x in X. By definition of the topology in X, there exist $\alpha_1, \dots, \alpha_k \in \mathcal{A}$ and open in X_{α_i} sets $U_i, i = 1, \dots, k$, such that $x \in \bigcap\{(pr_T^{\alpha_i})^{-1}U_i : i = 1, \dots, k\} \subset U$.

We now choose open sets $V_i \in \mathfrak{B}_{f_{\alpha_i}}$ and open in Y_{α_i} sets O_i such that $x_{\alpha_i} \in (V_i \cap f_{\alpha_i}^{-1}O_i) \subset U_i$ for $i = 1, \dots, k$. Then

$$\begin{aligned} x &\in \bigcap\{(pr_T^{\alpha_i})^{-1}(V_i \cap f_{\alpha_i}^{-1}O_i) : i = 1, \dots, k\} \\ &= \left(\cap\{(pr_T^{\alpha_i})^{-1}V_i : i = 1, \dots, k\}\right) \bigcap \left(\cap\{(pr_T^{\alpha_i})^{-1}f_{\alpha_i}^{-1}O_i : i = 1, \dots, k\}\right) \\ &= \left(\cap\{(pr_T^{\alpha_i})^{-1}V_i : i = 1, \dots, k\}\right) \bigcap f^{-1}\left(\cap\{(pr_B^{\alpha_i})^{-1}O_i : i = 1, \dots, k\}\right). \end{aligned}$$

Since the set $\cap\{(pr_B^{\alpha_i})^{-1}O_i : i = 1, \dots, k\}$ is an open neighborhood of $y = f(x)$ in Y, the first part of the proposition is proved.

By choosing a base $\mathfrak{B}_{f_\alpha}$ for every $\alpha \in \mathcal{A}$ with $|\mathfrak{B}_{f_\alpha}| = \mathfrak{w}(f_\alpha)$, the required inequality follows. □

Proposition 4.2. *The Tychonoff product $f = \prod\{f_\alpha : \alpha \in \mathcal{A}\}$ of T_i-maps f_α is a T_i-map for $i \leqslant 3\frac{1}{2}$.*

Proof. We shall give a prove for the case $i = 3\frac{1}{2}$ since the other cases are simpler.

Now let the maps f_α be $T_{3\frac{1}{2}}$-maps for every $\alpha \in \mathcal{A}$. That the map f is a T_0-map is not difficult to prove. Consider an arbitrary point $x = \{x_\alpha\} \in X$ and a closed set

$F \subset X$ such that $x \notin F$. There exist $\alpha_i \in \mathcal{A}$ and open sets U_i in $X_{\alpha_i}, i = 1, \dots, k$, such that

$$x \in U = \bigcap\{(pr_T^{\alpha_i})^{-1}U_i : i = 1, \dots, k\} \subset X \setminus F.$$

By the hypothesis, there exists neighborhoods O_i of $f_{\alpha_i}x_{\alpha_i} = y_{\alpha_i}$ in Y_{α_i} for $i = 1, \dots, k$, and continuous functions $\phi_i : f_{\alpha_i}^{-1}O_i \to [0,1]$ such that $x_{\alpha_i} \in \phi_i^{-1}(0)$ and $\left(f_{\alpha_i}^{-1}O_i \setminus U_i\right) \subset \phi_i^{-1}(1)$.

Consider the neighborhood $O = \bigcap\{(pr_B^{\alpha_i})^{-1}O_i : i = 1, \dots, k\}$ of the point $y = f(x)$ in Y. Then the function $\phi = \phi_1 \circ pr_T^{\alpha_1}|_{f^{-1}O} + \dots + \phi_k \circ pr_T^{\alpha_k}|_{f^{-1}O}$ is equal 0 on x and is $\geqslant 1$ on $(f^{-1}O \setminus U) \supset F \cap f^{-1}O$. Therefore, $\psi = \min\{\phi, 1\}$ is the required function. □

We end this section with the following known results with respect to open and closed maps (see for example [6]).

Proposition 4.3. *Let $f = \prod\{f_\alpha : \alpha \in \mathcal{A}\} : X = \prod\{X_\alpha : \alpha \in \mathcal{A}\} \to Y = \prod\{Y_\alpha : \alpha \in \mathcal{A}\}$ be the Tychonoff product of the maps $f_\alpha : X_\alpha \to Y_\alpha$, where $X_\alpha \neq \emptyset$ for every $\alpha \in \mathcal{A}$. If the map f is closed then so are all the maps f_α.*

It is well known that the converse of the above proposition is not true.

Proposition 4.4. *Under the above conditions, the map f is open if and only if all the maps f_α are open and there exists a finite subset $\mathcal{A}_0 \subset \mathcal{A}$ such that $f_\alpha(X_\alpha) = Y_\alpha$ for every $\alpha \in \mathcal{A} \setminus \mathcal{A}_0$.*

5. Fan products

In this section we define the notion of fan product with respect to a collection of maps and an inverse system. For this purpose we need the notion of Inverse Systems.

For completeness we remind the reader of this notion. Suppose we are given a directed set $(\Sigma, \leqslant)$ and for every $\sigma \in \Sigma$ there corresponds a topological space Y_σ. Suppose also, that for every $\sigma, \rho \in \Sigma$ satisfying the relation $\rho \leqslant \sigma$ we are given a continuous map $\lambda_\rho^\sigma : Y_\sigma \to Y_\rho$ with the following properties:

$$\lambda_\tau^\rho \circ \lambda_\rho^\sigma = \lambda_\tau^\sigma \text{ for every } \sigma, \rho, \tau \in \Sigma \text{ satisfying } \tau \leqslant \rho \leqslant \sigma, \tag{5.1}$$

$$\lambda_\sigma^\sigma = \mathrm{id}_{Y_\sigma} \text{ for every } \sigma \in \Sigma. \tag{5.2}$$

In this case we say that the collection $\boldsymbol{S} = \{Y_\sigma, \lambda_\rho^\sigma, \Sigma\}$ is an *inverse system of the spaces* Y_σ, the maps λ_ρ^σ are called the *binding maps* of the inverse system $\boldsymbol{S}$. An element $\{y_\sigma\}$ of the Tychonoff product $\prod\{Y_\sigma : \sigma \in \Sigma\}$ is called a *thread* of $\boldsymbol{S}$ if $\lambda_\rho^\sigma(y_\sigma) = y_\rho$ for every $\sigma, \rho \in \Sigma$ satisfying $\rho \leqslant \sigma$, and subspace of $\prod\{Y_\sigma : \sigma \in \Sigma\}$ consisting of all threads of $\boldsymbol{S}$ is called the *limit of the inverse system* $\boldsymbol{S}$ and is denoted by $\varprojlim \boldsymbol{S}$ or by $\varprojlim\{Y_\sigma, \lambda_\rho^\sigma, \Sigma\}$. The restriction of the projection $pr_{Y_\sigma} : \prod\{Y_\sigma : \sigma \in \Sigma\} \to Y_\sigma$ on the subspace $\varprojlim \boldsymbol{S}$ is called the *projection of the limit of* $\boldsymbol{S}$ *to* Y_σ and is denoted by λ_σ

Now suppose we are given a collection of maps $f_\sigma : X_\sigma \to Y_\sigma$ for every $\sigma \in \Sigma$, where the indexing set Σ is directed by the relation $\leqslant$. We further suppose that we are given an inverse system $\{Y_\sigma, \lambda^\sigma_\rho, \Sigma\}$. We denote by P, the subspace of the Tychonoff product $\prod\{X_\sigma : \sigma \in \Sigma\}$ given by

$$\big\{\{x_\sigma\} : \lambda^\sigma_\rho(f_\sigma x_\sigma) = f_\rho x_\rho \text{ for every } \sigma, \rho \in \Sigma \text{ satisfying } \rho \leqslant \sigma\big\}\,. \tag{5.3}$$

We call this space, the *fan product of the spaces X_σ with respect to the maps f_σ and the inverse system* $\{Y_\sigma, \lambda^\sigma_\rho, \Sigma\}$. The space P is denoted by $\prod\big\{X_\sigma, f_\sigma, \{Y_\sigma, \lambda^\sigma_\rho, \Sigma\}\big\}$.

For every $\sigma \in \Sigma$, the restriction of the projection $pr_\sigma : \prod\{X_\sigma : \sigma \in \Sigma\} \to X_\sigma$ on the subspace P will be denoted by π_σ and is called the *projection of the fan product P to X_σ*. From the definition of fan product we have $\lambda^\sigma_\rho \circ f_\sigma \circ \pi_\sigma = f_\rho \circ \pi_\rho$ for every $\sigma, \rho \in \Sigma$ satisfying $\rho \leqslant \sigma$. In this way one can define a map $p : P \to \varprojlim\{Y_\sigma, \lambda^\sigma_\rho, \Sigma\}$, called the *projection of the fan product P to the limit of the inverse system* $\{Y_\sigma, \lambda^\sigma_\rho, \Sigma\}$, by

$$p = \prod\{f_\sigma \circ \pi_\sigma : \sigma \in \Sigma\}. \tag{5.4}$$

It is evident that the projections p and $\pi_\sigma, \sigma \in \Sigma$ are continuous maps. The projection p will be called the *fibrewise product of the maps f_σ with respect to the inverse system* $\{Y_\sigma, \lambda^\sigma_\rho, \Sigma\}$ and will be denoted by $\prod\big\{f_\sigma, \{Y_\sigma, \lambda^\sigma_\rho, \Sigma\}\big\}$. It is not difficult to see that for every point $y = \{y_\sigma\} \in \varprojlim\{Y_\sigma, \lambda^\sigma_\rho, \Sigma\}$, the preimage $p^{-1}y$ is homeomorphic to the Tychonoff product of the fibres $f_\sigma^{-1}y_\sigma$, that is $\prod\{f_\sigma^{-1}y_\sigma : \sigma \in \Sigma\}$.

Similar to limits of inverse systems we have the following two results. The proofs are analogous to their couterparts in the theory of inverse systems and so are omitted (see [6]).

Proposition 5.1. *If the space Y_σ is a Hausdorff space for every $\sigma \in \Sigma$, then the fan product $P = \prod\big\{X_\sigma, f_\sigma, \{Y_\sigma, \lambda^\sigma_\rho, \Sigma\}\big\}$ is a closed subset of the Tychonoff product $\prod\{X_\sigma : \sigma \in \Sigma\}$.*

Proposition 5.2. *If the space X_σ is a T_i-space for every $\sigma \in \Sigma$, then the fan product $P = \prod\big\{X_\sigma, f_\sigma, \{Y_\sigma, \lambda^\sigma_\rho, \Sigma\}\big\}$ is also a T_i-space for $i \leqslant 3\frac{1}{2}$.*

Example 5.1. In the above context let Y be a topological space and let $Y_\sigma = Y$ for every $\sigma \in \Sigma$, where Σ is any non-empty directed set. We further take the binding maps to be $\lambda^\sigma_\rho = \mathrm{id}_Y$ for every $\sigma, \rho \in \Sigma$ satisfying $\rho \leqslant \sigma$. In this case the inverse system $\boldsymbol{S}(Y, \Sigma) = \{Y_\sigma, \lambda^\sigma_\rho, \Sigma\}$ is called the *constant inverse system* of the space Y on the set Σ and we have that the limit $\varprojlim\{Y_\sigma, \lambda^\sigma_\rho, \Sigma\}$ is homeomorphic to Y. Therefore, the fan product $P = \prod\big\{X_\sigma, f_\sigma, \{Y_\sigma, \lambda^\sigma_\rho, \Sigma\}\big\}$ is the fan product $\prod\{X_\sigma \text{ rel } f_\sigma : \sigma \in \Sigma\}$ defined by B.A.Pasynkov, for a collection of maps $f_\sigma : X_\sigma \to Y$, and the fibrewise product $\prod\big\{f_\sigma, \{Y_\sigma, \lambda^\sigma_\rho, \Sigma\}\big\}$ is the fibrewise product $\prod\{f_\sigma : \sigma \in \Sigma\}$, also defined by B.A.Pasynkov (see [10, 11, 12]).

The folowing two results have their counterparts in the category $\mathcal{TOP}_Y$.

Proposition 5.3. *Let $p : P \to \varprojlim\{Y_\sigma, \lambda^\sigma_\rho, \Sigma\}$ be the fibrewise product of the maps f_σ with respect to the inverse system $\{Y_\sigma, \lambda^\sigma_\rho, \Sigma\}$. If $\mathfrak{B}_{f_\sigma}$ is a base for the map f_σ*

for every $\sigma \in \Sigma$, *then the collection* $\mathfrak{S}_p = \bigcup\{\pi_\sigma^{-1}\mathfrak{B}_{f_\sigma} : \sigma \in \Sigma\}$ *is a subbase for the map* p *and the weight* $\mathfrak{w}(p) \leqslant \max\{|\Sigma|, \sup\{\mathfrak{w}(f_\sigma) : \sigma \in \Sigma\}, \aleph_0\}$.

Proof. Let $x = \{x_\sigma\}$ be an arbitrary point in P and consider a neighbourhood U of x in P. By definition of the topology in P, there exist $\sigma_1, \dots, \sigma_k \in \Sigma$ and open in X_{σ_i} sets $U_i, i = 1, \dots, k$, such that $x \in \bigcap\{\pi_{\sigma_i}^{-1}U_i : i = 1, \dots, k\} \subset U$.

We now choose open sets $V_i \in \mathfrak{B}_{f_{\sigma_i}}$ and open in Y_{σ_i} sets O_i such that $x_{\sigma_i} \in (V_i \cap f_{\sigma_i}^{-1}O_i) \subset U_i$ for $i = 1, \dots, k$. Then

$$\begin{aligned} x &\in \bigcap\{\pi_{\sigma_i}^{-1}(V_i \cap f_{\sigma_i}^{-1}O_i) : i = 1, \dots, k\} \\ &= \left(\cap\{\pi_{\sigma_i}^{-1}V_i : i = 1, \dots, k\}\right) \bigcap \left(\cap\{\pi_{\sigma_i}^{-1}f_{\sigma_i}^{-1}O_i : i = 1, \dots, k\}\right). \end{aligned}$$

There exists a $\sigma_0 \geqslant \sigma_i, i = 1, \dots, k$, and an open set $O \in Y_{\sigma_0}$ such that $y = p(x) \in \lambda_{\sigma_0}^{-1}O \subset \prod\{O_\sigma : \sigma \in \Sigma\}$, where $O_\sigma = O_{\sigma_i}$ for $\sigma = \sigma_i, i = 1, \dots, k$ and $O_\sigma = Y_\sigma$ otherwise. It is not difficult to see that $x \in \left(\bigcap\{\pi_{\sigma_i}^{-1}V_i : i = 1, \dots, k\}\right) \cap p^{-1}O$, which proves the first part of the proposition.

By choosing a base $\mathfrak{B}_{f_\sigma}$ for every $\sigma \in \Sigma$ with $|\mathfrak{B}_{f_\sigma}| = \mathfrak{w}(f_\sigma)$, the required inequality follows. □

Proposition 5.4. *The fibrewise product* $p = \prod\left\{f_\sigma, \{Y_\sigma, \lambda_\rho^\sigma, \Sigma\}\right\}$ *of* T_i*-maps* f_σ *with respect to the inverse system* $\{Y_\sigma, \lambda_\rho^\sigma, \Sigma\}$ *is a* T_i*-map for* $i \leqslant 3\frac{1}{2}$.

Proof. We shall give proves for the cases $i = 2$ and $i = 3\frac{1}{2}$ since the other cases are similar.

Let the maps f_σ be T_2-maps for every $\sigma \in \Sigma$, and let $x = \{x_\sigma\}$ and $x' = \{x'_\sigma\}$ be two distinct points in some fibre $p^{-1}(y), y = \{y_\sigma\}$. There exists $\rho \in \Sigma$ such that $x_\rho \neq x'_\rho$. We have $\prod f_\sigma x_\sigma = \prod(f_\sigma \circ \pi_\sigma)x = p(x) = p(x') = \prod(f_\sigma \circ \pi_\sigma)x' = \prod f_\sigma x'_\sigma$ and so, in particular, $f_\rho x_\rho = f_\rho x'_\rho$. Thus, x_ρ and x'_ρ are points of some fibre under the map f_ρ. Since f_ρ is a T_2-map, there exist disjoint neighborhoods U and U' of the points x_ρ and x'_ρ in X_ρ and therefore the open sets $\pi_\rho^{-1}U$ and $\pi_\rho^{-1}U'$ are the needed disjoint neighborhoods in P of x and x'.

Now let the maps f_σ be $T_{3\frac{1}{2}}$-maps for every $\sigma \in \Sigma$. That the map p is a T_0-map is proved as above. Consider an arbitrary point $x = \{x_\sigma\} \in P$ and a closed set $F \subset P$ such that $x \notin F$. There exist $\sigma_i \in \Sigma$ and open sets U_i in $X_{\sigma_i}, i = 1, \dots, k$, such that

$$x \in U = \bigcap\{\pi_{\sigma_i}^{-1}U_i : i = 1, \dots, k\} \subset P \setminus F.$$

By the hypothesis, there exists neighborhoods O_i of $f_{\sigma_i}x_{\sigma_i} = y_{\sigma_i}$ in Y_{σ_i} for $i = 1, \dots, k$, and continuous functions $\phi_i : f_{\sigma_i}^{-1}O_i \to [0, 1]$ such that $x_{\sigma_i} \in \phi_i^{-1}(0)$ and $\left(f_{\sigma_i}^{-1}O_i \setminus U_i\right) \subset \phi_i^{-1}(1)$.

Consider the neighborhood $O = \bigcap\{\lambda_{\sigma_i}^{-1}O_i : i = 1, \dots, k\}$ of the point $y = p(x)$ in $\varprojlim\{Y_\sigma, \lambda_\rho^\sigma, \Sigma\}$. Then the function $\phi = \phi_1 \circ \pi_{\sigma_1}|_{p^{-1}O} + \dots + \phi_k \circ \pi_{\sigma_k}|_{p^{-1}O}$ is equal 0 on x and is $\geqslant 1$ on $(p^{-1}O \setminus U) \supset F \cap p^{-1}O$. Therefore, $\psi = \min\{\phi, 1\}$ is the required function. □

6. Inverse systems

In this section we define the notion of inverse systems in the category $\mathcal{MAP}$ which coincides with the standard notion of inverse systems of maps whose limit is called the limit map.

Suppose we are given a collection of maps $f_\sigma : X_\sigma \to Y_\sigma$ for every $\sigma \in \Sigma$, where the indexing set Σ is directed by the relation $\leqslant$. We further suppose that we are given two inverse systems $\boldsymbol{S}_T = \{X_\sigma, \pi^\sigma_\rho, \Sigma\}$ and $\boldsymbol{S}_B = \{Y_\sigma, \lambda^\sigma_\rho, \Sigma\}$ with the property that the diagram

$$\begin{array}{ccc} X_\sigma & \xrightarrow{\pi^\sigma_\rho} & X_\rho \\ {\scriptstyle f_\sigma}\big\downarrow & & \big\downarrow{\scriptstyle f_\rho} \\ Y_\sigma & \xrightarrow[\lambda^\sigma_\rho]{} & Y_\rho \end{array}$$

is commutative for every $\sigma, \rho \in \Sigma$ satisfying $\rho \leqslant \sigma$. In other words, $\{\pi^\sigma_\rho, \lambda^\sigma_\rho\}$ is a morphism from f_σ into f_ρ for every $\sigma, \rho \in \Sigma$ satisfying $\rho \leqslant \sigma$. This induces a continuous map of $\varprojlim \boldsymbol{S}_T$ into $\varprojlim \boldsymbol{S}_B$ which is called the *limit map of the inverse system* $\boldsymbol{S}_f = \big\{f_\sigma, \{\pi^\sigma_\rho, \lambda^\sigma_\rho\}, \Sigma\big\}$ and is denoted by $\varprojlim \boldsymbol{S}_f$ or by $\varprojlim \big\{f_\sigma, \{\pi^\sigma_\rho, \lambda^\sigma_\rho\}, \Sigma\big\}$. By definition, for every $x = \{x_\sigma\} \in \varprojlim \boldsymbol{S}_T$, the image under the map $\varprojlim \boldsymbol{S}_f$ is the point $\{f_\sigma(x_\sigma)\}$ which belongs to $\varprojlim \boldsymbol{S}_B$.

We note that the map $\varprojlim \boldsymbol{S}_f$ is usually denoted by $\varprojlim\{\mathrm{id}_\Sigma, f_\sigma\}$ and is called the *limit map induced by* $\{\mathrm{id}_\Sigma, f_\sigma\}$ (see for example [6]).

Let π_σ be the projection of the limit of $\boldsymbol{S}_T$ to X_σ and let λ_σ be the projection of the limit of $\boldsymbol{S}_B$ to Y_σ. Then the pair $\{\pi_\sigma, \lambda_\sigma\}$ is called the *projection of the limit of* $\boldsymbol{S}_f$ *to* f_σ. By definition, the following diagram is commutative:

$$\begin{array}{ccccccc} \varprojlim \boldsymbol{S}_T & \xrightarrow{\pi_\sigma} & X_\sigma & \xrightarrow{\pi^\sigma_\rho} & X_\rho & \xleftarrow{\pi_\rho} & \varprojlim \boldsymbol{S}_T \\ {\scriptstyle \varprojlim \boldsymbol{S}_f}\big\downarrow & & {\scriptstyle f_\sigma}\big\downarrow & & \big\downarrow{\scriptstyle f_\rho} & & \big\downarrow{\scriptstyle \varprojlim \boldsymbol{S}_f} \\ \varprojlim \boldsymbol{S}_B & \xrightarrow[\lambda_\sigma]{} & Y_\sigma & \xrightarrow[\lambda^\sigma_\rho]{} & Y_\rho & \xleftarrow[\lambda_\rho]{} & \varprojlim \boldsymbol{S}_B \end{array}$$

Example 6.1. In the above context let Y be a topological space and let $Y_\sigma = Y$ for every $\sigma \in \Sigma$, where Σ is any non-empty directed set. We further take $\{Y_\sigma, \lambda^\sigma_\rho, \Sigma\}$ to be the constant inverse system $\boldsymbol{S}(Y, \Sigma)$ of the space Y on the set Σ. Thus we can identify the limit $\varprojlim\{Y_\sigma, \pi^\sigma_\rho, \Sigma\}$ with the space Y. In this case, the limit map $\varprojlim \boldsymbol{S}_f : \{X_\sigma, \pi^\sigma_\rho, \Sigma\} \to \boldsymbol{S}(Y, \Sigma) \cong Y$ is the *limit map induced by* $\{f_\sigma : \sigma \in \Sigma\}$ and is denoted by $\varprojlim f_\sigma$. This can be looked upon as the limit of an inverse system in the category $\mathcal{TOP}_Y$.

As for fan products we have the following two results.

Proposition 6.1. *Let* $\varprojlim \boldsymbol{S}_f : \varprojlim \boldsymbol{S}_T \to \varprojlim \boldsymbol{S}_B$ *be the limit map of the inverse system* $\boldsymbol{S}_f = \big\{f_\sigma, \{\pi^\sigma_\rho, \lambda^\sigma_\rho\}, \Sigma\big\}$. *If* $\mathfrak{B}_{f_\sigma}$ *is a base for the map* f_σ *for every*

$\sigma \in \Sigma$, then the collection $\mathfrak{B}_{\varprojlim \boldsymbol{S}_f} = \bigcup\{\pi_\sigma^{-1}\mathfrak{B}_{f_\sigma} : \sigma \in \Sigma'\}$, where Σ' is a cofinal subset of Σ, is a base for the limit map $\varprojlim \boldsymbol{S}_f$ and the weight $\mathfrak{w}(\varprojlim \boldsymbol{S}_f) \leqslant$ $\max\{cf(|\Sigma|), \sup\{\mathfrak{w}(f_\sigma) : \sigma \in \Sigma\}, \aleph_0\}$.

Proof. Let $x = \{x_\sigma\}$ be an arbitrary point in $\varprojlim \boldsymbol{S}_T$ and consider a neighbourhood U of x in $\varprojlim \boldsymbol{S}_T$. By definition of the topology in $\varprojlim \boldsymbol{S}_T$, there exists $\sigma_0 \in \Sigma'$ and open in X_{σ_0} set U_0, such that $x \in \pi_{\sigma_0}^{-1}U_0 \subset U$.

Choose open sets $V_0 \in \mathfrak{B}_{f_{\sigma_0}}$ and O_0 in Y_{σ_0} such that $x_{\sigma_0} \in (V_0 \cap f_{\sigma_0}^{-1}O_0) \subset U_0$. Then $x \in \pi_{\sigma_0}^{-1}(V_0 \cap f_{\sigma_0}^{-1}O_0) = \pi_{\sigma_0}^{-1}V_0 \cap \pi_{\sigma_0}^{-1}f_{\sigma_0}^{-1}O_0 = \pi_{\sigma_0}^{-1}V_0 \cap \left(\varprojlim \boldsymbol{S}_f\right)^{-1} \lambda_{\sigma_0}^{-1}O_0$. Since $\lambda_{\sigma_0}^{-1}O_0$ is a neighborhood of the point $\varprojlim \boldsymbol{S}_f(x)$, the first part of the proposition is proved.

By choosing a base $\mathfrak{B}_{f_\sigma}$ for every $\sigma \in \Sigma$ with $|\mathfrak{B}_{f_\sigma}| = \mathfrak{w}(f_\sigma)$, the required inequality follows. □

The proof of the following proposition is analogous to the proof of the corresponding result for fan products and so is omitted.

Proposition 6.2. *The limit map $\varprojlim \boldsymbol{S}_f$ of the inverse system $\boldsymbol{S}_f = \left\{f_\sigma, \{\pi_\rho^\sigma, \lambda_\rho^\sigma\}, \Sigma\right\}$, where the maps f_σ are T_i-maps, is a T_i-map for $i \leqslant 3\frac{1}{2}$.*

7. Sums

In this section we introduce the notion of sum in the category $\mathcal{MAP}$. This is the standard sum of maps and we recall the definition.

Definition 7.1. Let $\{f_\alpha : \alpha \in \mathcal{A}\}$ be a collection of continuous maps, where $f_\alpha : X_\alpha \to Y_\alpha$ and $\{X_\alpha : \alpha \in \mathcal{A}\}, \{Y_\alpha : \alpha \in \mathcal{A}\}$ are two collections of pairwise disjoint topological spaces (note that if any of the collections, say $\{X_\alpha : \alpha \in \mathcal{A}\}$ is not a pairwise disjoint collection, one can always take a pairwise disjoint collection $\{X'_\alpha : \alpha \in \mathcal{A}\}$, where X'_α is homeomorphic to X_α for every $\alpha \in \mathcal{A}$). By letting $f(x) = f_\alpha(x)$ for $x \in X_\alpha$ we define a continuous map f of the sum $\bigoplus\{X_\alpha : \alpha \in \mathcal{A}\}$ to the sum $\bigoplus\{Y_\alpha : \alpha \in \mathcal{A}\}$ which is called the *sum of the maps* $\{f_\alpha : \alpha \in \mathcal{A}\}$ and is denoted by $\bigoplus\{f_\alpha : \alpha \in \mathcal{A}\}$.

Let $i_T^\alpha : X_\alpha \to \bigoplus\{X_\alpha : \alpha \in \mathcal{A}\}$ and $i_B^\alpha : Y_\alpha \to \bigoplus\{Y_\alpha : \alpha \in \mathcal{A}\}$ be the subspace embeddings. Then we have that the diagram

$$\begin{array}{ccc} X_\alpha & \xrightarrow{i_T^\alpha} & \bigoplus\{X_\alpha : \alpha \in \mathcal{A}\} \\ {\scriptstyle f_\alpha}\downarrow & & \downarrow{\scriptstyle \bigoplus\{f_\alpha : \alpha \in \mathcal{A}\}} \\ Y_\alpha & \xrightarrow[i_B^\alpha]{} & \bigoplus\{Y_\alpha : \alpha \in \mathcal{A}\} \end{array}$$

is commutative. Therefore, the pair $\{i_T^\alpha, i_B^\alpha\}$ serve as a submap embedding morphism from f_α into $\bigoplus\{f_\alpha : \alpha \in \mathcal{A}\}$. It can be seen that this definition of sum confers well with the notion of categorical sum (or coproduct) (see, for example [14]).

The following result follows from the definitions.

Proposition 7.1. *The sum* $\bigoplus\{f_\alpha : \alpha \in \mathcal{A}\}$ *is a closed map (resp. open map, homeomorphic embedding, 1-1 map, onto map) if and only if all the maps* f_α *are closed (resp. open, homeomorphic embeddings, 1-1, onto).*

The proof of the following result follows directly from the definitions and so is omitted.

Proposition 7.2. *Let* $f = \bigoplus\{f_\alpha : \alpha \in \mathcal{A}\} : X = \bigoplus\{X_\alpha : \alpha \in \mathcal{A}\} \to Y = \bigoplus\{Y_\alpha : \alpha \in \mathcal{A}\}$ *be the sum of the maps* $f_\alpha : X_\alpha \to Y_\alpha$*. If* $\mathfrak{B}_{f_\alpha}$ *is a base for the map* f_α *for every* $\alpha \in \mathcal{A}$*, then the collection* $\mathfrak{B}_f = \bigcup\{i_T^\alpha \mathfrak{B}_{f_\alpha} : \alpha \in \mathcal{A}\}$ *is a base for the map* f *and the weight* $\mathfrak{w}(f) \leqslant \max\{|\mathcal{A}|, \sup\{\mathfrak{w}(f_\alpha) : \alpha \in \mathcal{A}\}, \aleph_0\}$.

Proposition 7.3. *The sum* $f = \bigoplus\{f_\alpha : \alpha \in \mathcal{A}\}$ *of* T_i*-maps* f_α *is a* T_i*-map for* $i \leqslant 4\frac{1}{2}$.

Proof. We shall show that if f_α is a functionally prenormal map for every $\alpha \in \mathcal{A}$, then so is f. For other separation axioms the proof is similar.

Let the maps f_α be functionally prenormal for every $\alpha \in \mathcal{A}$ and let F, H be two disjoint closed subsets in $\bigoplus\{X_\alpha : \alpha \in \mathcal{A}\}$. Consider an arbitrary point $y \in Y = \bigoplus\{Y_\alpha : \alpha \in \mathcal{A}\}$ and w.l.g. assume that $y \in Y_{\sigma_0}$ for some $\sigma_0 \in \Sigma$. Since the map f_{σ_0} is functionally prenormal, there exists a neighborhood $O_y \subset Y_{\sigma_0}$ such that $F \cap X_{\sigma_0}$ and $H \cap X_{\sigma_0}$ are functionally separated in $f_{\sigma_0}^{-1} O_y$. Therefore, the sets F and H are functionally separated in $f^{-1}(i_B^{\sigma_0} O_y)$, which shows that the map f is functionally prenormal. ☐

8. Fibrewise sums

In this section we define the notion of fibrewise sum of a collection of maps with respect to a direct system. For this purpose we need the notion of Direct Systems and their limit, a less used twin of the limit of an inverse system.

We remind the reader of this notion. Suppose we are given a directed set $(\Sigma, \leqslant)$ and for every $\sigma \in \Sigma$ there corresponds a topological space Y_σ. Suppose also, that for every $\sigma, \rho \in \Sigma$ satisfying the relation $\rho \leqslant \sigma$ we are given a continuous map $\psi_\sigma^\rho : Y_\rho \to Y_\sigma$ with the following properties:

$$\psi_\sigma^\rho \circ \psi_\rho^\tau = \psi_\sigma^\tau \text{ for every } \tau, \rho, \sigma \in \Sigma \text{ satisfying } \tau \leqslant \rho \leqslant \sigma, \tag{8.1}$$

$$\psi_\sigma^\sigma = \mathrm{id}_{Y_\sigma} \text{ for every } \sigma \in \Sigma. \tag{8.2}$$

In this case we say that the collection $\boldsymbol{D} = \{Y_\sigma, \psi_\sigma^\rho, \Sigma\}$ is a *directed system of the spaces* Y_σ, the maps ψ_σ^ρ are called the *binding maps* of the directed system $\boldsymbol{D}$. Two elements y_τ and y_ρ of the sum $\bigoplus\{Y_\sigma : \sigma \in \Sigma\}$ are called *equivalent* if $y_\tau \in Y_\tau, y_\rho \in Y_\rho$ and $\psi_\sigma^\tau(y_\tau) = \psi_\sigma^\rho(y_\rho)$ for some $\sigma \in \Sigma$. It is not difficult to prove that this equivalence is an *equivalence relation* and the quotient space of $\bigoplus\{Y_\sigma : \sigma \in \Sigma\}$ with respect to this equivalence is called the *limit of the directed system* $\boldsymbol{D}$ and is denoted by $\varinjlim \boldsymbol{D}$ or by $\varinjlim\{Y_\sigma, \psi_\sigma^\rho, \Sigma\}$. The restriction of the projection $pr : \bigoplus\{Y_\sigma : \sigma \in \Sigma\} \to \varinjlim \boldsymbol{D}$ on the subspace Y_σ is called the *embedding of* Y_σ *into the limit of* $\boldsymbol{D}$ and is denoted by ψ^σ [5].

The element $y_\sigma \in Y_\sigma$ is said to be a *representative* of $y \in \varinjlim \boldsymbol{D}$ if $\psi^\sigma(y_\sigma) = y$. A given element $y \in \varinjlim \boldsymbol{D}$ need not have a representative in every Y_σ. However, if finitely many elements $y_1, \dots, y_k \in \varinjlim \boldsymbol{D}$ are given, it is always possible to find at least one $\sigma \in \Sigma$ such that Y_σ contains representatives for all the given y_i.

We now prove the following proposition regarding the openess and closedness of the projection map $pr : \bigoplus\{Y_\sigma : \sigma \in \Sigma\} \to \varinjlim \boldsymbol{D}$.

Proposition 8.1. *Let $\boldsymbol{D} = \{Y_\sigma, \psi^\rho_\sigma, \Sigma\}$ be a directed system. If all the binding maps ψ^ρ_σ are open (closed, Σ has a maximal element σ_0 and the collection $\{\psi^\rho_{\sigma_0} Y_\rho\}$ is locally finite in Y_{σ_0}), then the projection $pr : \bigoplus\{Y_\sigma : \sigma \in \Sigma\} \to \varinjlim \boldsymbol{D}$ is open (closed).*

Proof. We begin by considering the case when all the binding maps are open. Let A be an open set in the sum $\bigoplus\{Y_\sigma : \sigma \in \Sigma\}$. Then $A_\sigma = A \cap Y_\sigma$ is an open subset of Y_σ for every $\sigma \in \Sigma$. We have

$$pr^{-1}pr(A) = \{y_\sigma : \text{there exists } \rho, \tau \in \Sigma \text{ such that } \psi^\rho_\tau(y_\rho) = \psi^\sigma_\tau(y_\sigma) \text{ for some } y_\rho \in A_\rho\}.$$

Now let $G_{\rho\sigma} = \psi^\rho_\sigma(A_\rho)$, which is open in Y_σ, and $H_{\rho\tau\sigma} = (\psi^\tau_\sigma)^{-1} G_{\rho\sigma}$, which is open in Y_τ. Consider the set $B = \bigcup\{A_\rho, G_{\rho\sigma}, H_{\rho\tau\sigma}\} = \bigcup\{G_{\rho\sigma}, H_{\rho\tau\sigma}\} = \bigcup\{H_{\rho\tau\sigma}\}$. Therefore, B is open in $\bigoplus\{Y_\sigma : \sigma \in \Sigma\}$ and we are left to show that $pr^{-1}pr(A) = B$

Take any element $y_\sigma \in pr^{-1}pr(A)$. Then there exists $\rho, \tau \in \Sigma$ such that $\psi^\rho_\tau(y_\rho) = \psi^\sigma_\tau(y_\sigma)$ for some $y_\rho \in A_\rho$. Thus $\psi^\sigma_\tau(y_\sigma) \in G_{\rho\tau}$, which implies that $y_\sigma \in H_{\rho\sigma\tau} \subset B$. Conversely, if $y_\sigma \in H_{\rho\sigma\tau} \subset B$, then by definition $\psi^\rho_\tau(y_\rho) = \psi^\sigma_\tau(y_\sigma)$ for some $y_\rho \in A_\rho$, which shows that $y_\sigma \in pr^{-1}pr(A)$.

Next we consider the case when the binding maps are closed, Σ has a maximal element σ_0 and the collection $\{\psi^\rho_{\sigma_0} Y_\rho\}$ is locally finite in Y_{σ_0}. By changing the word *open* to the word *closed* in the above and noticing that in this case we have $B = \bigcup\{H_{\rho\tau\sigma_0}\}$ we get the desired result. □

Corollary 8.2. *Let $\boldsymbol{D} = \{Y_\sigma, \psi^\rho_\sigma, \Sigma\}$ be a directed system. If all the binding maps ψ^ρ_σ are closed and Σ is finite, then the projection $pr : \bigoplus\{Y_\sigma : \sigma \in \Sigma\} \to \varinjlim \boldsymbol{D}$ is closed.*

Now suppose we are given a collection of maps $f_\sigma : X_\sigma \to Y_\sigma$ for every $\sigma \in \Sigma$, where the indexing set Σ is directed by the relation $\leqslant$. We further suppose that we are given a directed system $\{Y_\sigma, \psi^\rho_\sigma, \Sigma\}$. The continuous map $q = pr \circ \bigoplus\{f_\sigma : \sigma \in \Sigma\} : \bigoplus\{X_\sigma : \sigma \in \Sigma\} \to \varinjlim\{Y_\sigma, \psi^\rho_\sigma, \Sigma\}$ will be called the *fibrewise sum of the maps f_σ with respect to the directed system* $\{Y_\sigma, \psi^\rho_\sigma, \Sigma\}$ and will be denoted by $\bigoplus\{f_\sigma, \{Y_\sigma, \psi^\rho_\sigma, \Sigma\}\}$.

It is not difficult to see that for every element $y \in \varinjlim \boldsymbol{D}$, the preimage $q^{-1}y$ is homeomorphic to the sum of the fibres $f^{-1}_\sigma(y_\sigma)$, where $y_\sigma \in Y_\sigma$ is a representative of y, i.e.

$$q^{-1}y \cong \bigoplus\{f^{-1}_\sigma(y_\sigma) : y_\sigma \in Y_\sigma \text{ is a representative of } y\}.$$

¿From Propositions 7.1, 8.1 and Corollary 8.2 we have the following results.

Proposition 8.3. *Let $\boldsymbol{D} = \{Y_\sigma, \psi_\sigma^\rho, \Sigma\}$ be a directed system. If all the maps $f_\sigma : X_\sigma \to Y_\sigma$ are open (closed) and all the binding maps ψ_σ^ρ are open (closed, Σ has a maximal element σ_0 and the collection $\{\psi_{\sigma_0}^\rho Y_\rho\}$ is locally finite in Y_{σ_0}), then the fibrewise sum $\bigoplus\{f_\sigma, \{Y_\sigma, \psi_\sigma^\rho, \Sigma\}\}$ is also open (closed).*

Corollary 8.4. *Let $\boldsymbol{D} = \{Y_\sigma, \psi_\sigma^\rho, \Sigma\}$ be a directed system. If all the maps $f_\sigma : X_\sigma \to Y_\sigma$ and all the binding maps $\psi_\sigma^\rho : Y_\rho \to Y_\sigma$ are closed and Σ is a finite set, then the fibrewise sum $\bigoplus\{f_\sigma, \{Y_\sigma, \psi_\sigma^\rho, \Sigma\}\}$ is also closed.*

Example 8.1. In the above context let Y be a topological space and let $Y_\sigma = Y$ for every $\sigma \in \Sigma$, where Σ is any non-empty directed set. We further take the binding maps to be $\psi_\sigma^\rho = \mathrm{id}_Y$ for every $\rho, \sigma \in \Sigma$ satisfying $\rho \leqslant \sigma$. In this case the directed system $\boldsymbol{D}(Y, \Sigma) = \{Y_\sigma, \psi_\sigma^\rho, \Sigma\}$ is called the *constant directed system* of the space Y on the set Σ and we have that the limit $\varinjlim\{Y_\sigma, \psi_\sigma^\rho, \Sigma\}$ is homeomorphic to Y. Therefore, the fibrewise sum $\bigoplus\{f_\sigma, \{Y_\sigma, \psi_\sigma^\rho, \Sigma\}\}$ is the projection of the fibrewise coproduct $\coprod_Y\{X_\sigma : \sigma \in \Sigma\}$ onto the base space Y as defined by I.M.James (see [8]). If one identifies the space X_σ with the space $i^\sigma X_\sigma \subset \bigoplus\{X_\sigma : \sigma \in \Sigma\}$, then the fibrewise sum $\bigoplus\{f_\sigma, \{Y_\sigma, \psi_\sigma^\rho, \Sigma\}\}$ is the combination of the maps f_σ which is denoted by $\triangledown\{f_\sigma : \sigma \in \Sigma\}$ (see [6]).

9. Direct systems

In this section we define the notion of direct systems in the category $\mathcal{MAP}$ which coincides with the standard notion of direct systems of maps whose limit is called the limit map.

Suppose we are given a collection of maps $f_\sigma : X_\sigma \to Y_\sigma$ for every $\sigma \in \Sigma$, where the indexing set Σ is directed by the relation $\leqslant$. We further suppose that we are given two direct systems $\boldsymbol{D}_T = \{X_\sigma, \phi_\sigma^\rho, \Sigma\}$ and $\boldsymbol{D}_B = \{Y_\sigma, \psi_\sigma^\rho, \Sigma\}$ with the property that the diagram

$$\begin{array}{ccc} X_\rho & \xrightarrow{\phi_\sigma^\rho} & X_\sigma \\ {\scriptstyle f_\rho}\downarrow & & \downarrow{\scriptstyle f_\sigma} \\ Y_\rho & \xrightarrow[\psi_\sigma^\rho]{} & Y_\sigma \end{array}$$

is commutative for every $\rho, \sigma \in \Sigma$ satisfying $\rho \leqslant \sigma$. In other words, $\{\phi_\sigma^\rho, \psi_\sigma^\rho\}$ is a morphism from f_ρ into f_σ for every $\rho, \sigma \in \Sigma$ satisfying $\rho \leqslant \sigma$. Consider the sum $f = \bigoplus\{f_\sigma : \sigma \in \Sigma\} : \bigoplus\{X_\sigma : \sigma \in \Sigma\} \to \bigoplus\{Y_\sigma : \sigma \in \Sigma\}$. Because of the above commutative diagram, the map f is relation preserving and so, passing to the quotient, yields a continuous map of $\varinjlim\boldsymbol{D}_T$ into $\varinjlim\boldsymbol{D}_B$ which is called the *limit map of the directed system* $\boldsymbol{D}_f = \{f_\sigma, \{\phi_\sigma^\rho, \psi_\sigma^\rho\}, \Sigma\}$ and is denoted by $\varinjlim\boldsymbol{D}_f$ or by $\varinjlim\{f_\sigma, \{\phi_\sigma^\rho, \psi_\sigma^\rho\}, \Sigma\}$. By definition, for every $x \in \varinjlim\boldsymbol{D}_T$, the image under the map $\varinjlim\boldsymbol{D}_f$ is the point $\psi^\sigma \circ f_\sigma(x_\sigma)$, where $x_\sigma \in X_\sigma$ is a representative of x.

We note that the map $\varinjlim \boldsymbol{D}_f$ is usually denoted by $\underrightarrow{\mathrm{Lim}} f_\sigma$ or by f^∞ and is called the *direct limit map induced by the map* $\{f_\sigma : \sigma \in \Sigma\}$ (see for example [5]).

The following result is known [5].

Proposition 9.1. *For the limit* $\varinjlim \boldsymbol{D}_f$ *of the dircted system* $\boldsymbol{D}_f = \{f_\sigma, \{\phi^\rho_\sigma, \psi^\rho_\sigma\}, \Sigma\}$ *the following hold:*

1. *If* f_σ *is a 1-1 map for every* $\sigma \in \Sigma$, *then so is* $\varinjlim \boldsymbol{D}_f$;
2. *If* f_σ *is an onto map for every* $\sigma \in \Sigma$, *then so is* $\varinjlim \boldsymbol{D}_f$;
3. *If* f_σ *is a quotient map for every* $\sigma \in \Sigma$, *then so is* $\varinjlim \boldsymbol{D}_f$;
4. *If* f_σ *is a homeomorphism for every* $\sigma \in \Sigma$, *then so is* $\varinjlim \boldsymbol{D}_f$.

Let ϕ^σ be the embedding of X_σ into the limit of $\boldsymbol{D}_T$ and let ψ^σ be the embedding of Y_σ into the limit of $\boldsymbol{D}_B$. Then the pair $\{\phi^\sigma, \psi^\sigma\}$ is called the *embedding of* f_σ *into the limit of* $\boldsymbol{D}_f$. By definition, the following diagram is commutative:

$$\begin{array}{ccccccc}
\varinjlim \boldsymbol{D}_T & \xleftarrow{\phi^\rho} & X_\rho & \xrightarrow{\phi^\rho_\sigma} & X_\sigma & \xrightarrow{\phi^\sigma} & \varinjlim \boldsymbol{D}_T \\
{\scriptstyle \varinjlim \boldsymbol{D}_f}\Big\downarrow & & {\scriptstyle f_\rho}\Big\downarrow & & \Big\downarrow{\scriptstyle f_\sigma} & & \Big\downarrow{\scriptstyle \varinjlim \boldsymbol{D}_f} \\
\varinjlim \boldsymbol{D}_B & \xleftarrow[\psi^\rho]{} & Y_\rho & \xrightarrow[\psi^\rho_\sigma]{} & Y_\sigma & \xrightarrow[\psi^\sigma]{} & \varinjlim \boldsymbol{D}_B
\end{array}$$

The following result is not difficult to prove.

Proposition 9.2. *For the limit* $\varinjlim \boldsymbol{D}_f$ *of the dircted system* $\boldsymbol{D}_f = \{f_\sigma, \{\phi^\rho_\sigma, \psi^\rho_\sigma\}, \Sigma\}$ *the following hold:*

1. *If* f_σ *is an open map for every* $\sigma \in \Sigma$ *and the projection* $pr_B : \bigoplus\{Y_\sigma : \sigma \in \Sigma\} \to \varinjlim \boldsymbol{D}_B$ *is open, then so is* $\varinjlim \boldsymbol{D}_f$;
2. *If* f_σ *is a closed map for every* $\sigma \in \Sigma$ *and the projection* $pr_B : \bigoplus\{Y_\sigma : \sigma \in \Sigma\} \to \varinjlim \boldsymbol{D}_B$ *is closed, then so is* $\varinjlim \boldsymbol{D}_f$.

Thus, by Proposition 8.1 and Corollary 8.2 we have the following result.

Proposition 9.3. *For the limit* $\varinjlim \boldsymbol{D}_f$ *of the dircted system* $\boldsymbol{D}_f = \{f_\sigma, \{\phi^\rho_\sigma, \psi^\rho_\sigma\}, \Sigma\}$ *the following hold:*

1. *If all the maps* $f_\sigma : X_\sigma \to Y_\sigma$ *are open and all the binding maps* ψ^ρ_σ *are open, then the limit map* $\varinjlim \boldsymbol{D}_f$ *is also open;*
2. *If all the maps* $f_\sigma : X_\sigma \to Y_\sigma$ *are closed, all the binding maps* ψ^ρ_σ *are closed,* Σ *has a maximal element* σ_0 *and the collection* $\{\psi^\rho_{\sigma_0} Y_\rho\}$ *is locally finite in* Y_{σ_0}, *then the limit map* $\varinjlim \boldsymbol{D}_f$ *is also closed;*
3. *If all the maps* $f_\sigma : X_\sigma \to Y_\sigma$ *and all the binding maps* $\psi^\rho_\sigma : Y_\rho \to Y_\sigma$ *are closed and* Σ *is a finite set, then the limit map* $\varinjlim \boldsymbol{D}_f$ *is also closed.*

Example 9.1. In the above context let Y be a topological space and let $Y_\sigma = Y$ for every $\sigma \in \Sigma$, where Σ is any non-empty directed set. We further take $\{Y_\sigma, \psi^\rho_\sigma, \Sigma\}$ to be the constant directed system $\boldsymbol{D}(Y, \Sigma)$ of the space Y on the set Σ. Thus we can identify the limit $\varinjlim\{Y_\sigma, \psi^\rho_\sigma, \Sigma\}$ with the space Y. In this case, the limit map $\varinjlim \boldsymbol{D}_f : \{X_\sigma, \phi^\rho_\sigma, \Sigma\} \to \boldsymbol{D}(Y, \Sigma) \cong Y$ is the *limit map induced by* $\{f_\sigma : \sigma \in \Sigma\}$ and is denoted by $\varinjlim f_\sigma$. This can be looked upon as the limit of a directed system in the category $\mathcal{TOP}_Y$.

10. Compact maps

The following definition was given by Vainstein [16].

Definition 10.1. A continuous map $f : X \to Y$ is called *compact* (otherwise called *perfect*) if it is a closed map and all the fibres $f^{-1}y$ are compact subsets of X.

The following result is known (see [13]).

Proposition 10.1. *A continuous map $f : X \to Y$ is compact if and only if for every point $y \in Y$ and every open (in X) cover $\mathcal{U}$ of $f^{-1}y$ there exists a neighborhood O of y such that $f^{-1}O$ is covered by a finite number of elements of $\mathcal{U}$.*

As is well known, the continuous image of a compact space is compact. An analogue of this statement is the following result.

Proposition 10.2. *Let the maps $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ be continuous and let $\{\lambda_T, \lambda_B\}$ be a {onto,compact}-morphism from f_1 into f_2. If the map f_1 is compact then so is the map f_2.*

Proof. For every point $y \in Y_2$, the fibre $f_2^{-1}y_2$ is compact as a continuous image of a compact space. Furthermore, if F is a closed subset of X_2, then $f_2(F) = f_2\lambda_T(\lambda_T)^{-1}(F) = \lambda_B f_1(\lambda_T)^{-1}(F)$ and so is closed in Y_2. Therefore, the map f_2 is compact. □

As in the case of the category $\mathcal{TOP}$ we have (see for example [13]:

Proposition 10.3. *A compact Hausdorff (resp. functionally Hausdorff) map $f : X \to Y$ is T_4 (resp. $T_{4\frac{1}{2}}$).*

Next we consider operations on compact maps. It is evident that a closed submap of a compact map is compact.

Let $\{f_\alpha : \alpha \in \mathcal{A}\}$ be a collection of continuous maps, where $f_\alpha : X_\alpha \to Y_\alpha$. The following result is well know (see [6]).

Proposition 10.4. *The Tychonoff product $f = \prod\{f_\alpha : \alpha \in \mathcal{A}\}$, where $X_\alpha \neq \emptyset$ for every $\alpha \in \mathcal{A}$, is compact if and only if all the maps f_α are compact.*

Now suppose we are given a collection of maps $f_\sigma : X_\sigma \to Y_\sigma$ for every $\sigma \in \Sigma$, where the indexing set Σ is directed by the relation $\leqslant$, and suppose we are given an inverse system $\{Y_\sigma, \lambda^\sigma_\rho, \Sigma\}$. Let $f = \prod\{f_\sigma : \sigma \in \Sigma\}$. From section 5 one can see that

$$p = f|_P : P \to \varprojlim\{Y_\sigma, \lambda^\sigma_\rho, \Sigma\}, \text{ and}$$

$$P = f^{-1}\left(\varprojlim\{Y_\sigma, \lambda^\sigma_\rho, \Sigma\}\right).$$

We therefore have the following result.

Proposition 10.5. *The fibrewise product* $p = \prod \left\{ f_\sigma, \{Y_\sigma, \lambda^\sigma_\rho, \Sigma\} \right\}$ *of compact maps* f_σ *with respect to the inverse system* $\{Y_\sigma, \lambda^\sigma_\rho, \Sigma\}$ *is a compact map.*

Likewise we have,

Proposition 10.6. *The limit map* $\lim\limits_{\leftarrow} \boldsymbol{S}_f$ *of the inverse system* $\boldsymbol{S}_f = \left\{ f_\sigma, \{\pi^\sigma_\rho, \lambda^\sigma_\rho\}, \Sigma \right\}$ *of compact maps* f_σ *is compact.*

For compact onto maps we have the following two results.

Proposition 10.7. *Let* $\boldsymbol{S}_f = \left\{ f_\sigma, \{\pi^\sigma_\rho, \lambda^\sigma_\rho\}, \Sigma \right\}$ *be an inverse system of compact onto maps. Then the limit map* $\lim\limits_{\leftarrow} \boldsymbol{S}_f$ *is also compact and onto.*

Proof. The fact that the limit map $\lim\limits_{\leftarrow} \boldsymbol{S}_f$ is compact follows from Proposition 10.6. We now show that it is also an onto map.

Let $y = \{y_\sigma\}$ be an arbitrary point of $\lim\limits_{\leftarrow} \boldsymbol{S}_B$. For every $\sigma \in \Sigma$, the set $Z_\sigma = f^{-1}_\sigma(y_\sigma)$ is a compact subset of X_σ. For any $\sigma, \rho \in \Sigma$ satisfying $\rho \leqslant \sigma$ we have

$$\pi^\sigma_\rho(Z_\sigma) = \pi^\sigma_\rho f^{-1}_\sigma(y_\sigma) \subset f^{-1}_\rho \lambda^\sigma_\rho(y_\sigma) = f^{-1}_\rho(y_\rho) = Z_\rho.$$

Therefore, $\boldsymbol{S'}_T = \left\{ Z_\sigma, \hat{\pi}^\sigma_\rho, \Sigma \right\}$, where $\hat{\pi}^\sigma_\rho(x_\sigma) = \pi^\sigma_\rho(x_\sigma)$ for $x_\sigma \in Z_\sigma$, is an inverse system of non-empty compact spaces. Thus, there exists at least one element $x' = \{x'_\sigma\} \in \lim\limits_{\leftarrow} \boldsymbol{S'}_T$ and it is not difficult to see that $\lim\limits_{\leftarrow} \boldsymbol{S}_f(x') = y$. □

Similarly one can prove that,

Proposition 10.8. *The fibrewise product* $p = \prod \left\{ f_\sigma, \{Y_\sigma, \lambda^\sigma_\rho, \Sigma\} \right\}$ *of compact onto maps* f_σ *with respect to the inverse system* $\{Y_\sigma, \lambda^\sigma_\rho, \Sigma\}$ *is a compact onto map.*

The following result is also known (see [6]).

Proposition 10.9. *The sum* $\bigoplus\{f_\alpha : \alpha \in \mathcal{A}\}$ *is a compact map if and only if all the maps* f_α *are compact.*

Finally we have the following results with respect to fibrewise sums and direct limits of compact maps.

Proposition 10.10. *Let* $\boldsymbol{D} = \{Y_\sigma, \psi^\rho_\sigma, \Sigma\}$ *be a directed system. The fibrewise sum* $f = \bigoplus \{f_\sigma, \{Y_\sigma, \psi^\rho_\sigma, \Sigma\}\}$ *is a compact map if all the maps* $f_\sigma : X_\sigma \to Y_\sigma$ *and all the binding maps* $\psi^\rho_\sigma : Y_\rho \to Y_\sigma$ *are compact and* Σ *is a finite set.*

Proof. The closedness of f was proved in Corollary 8.4. Since the directed set Σ is finite, there exists an element $\sigma_0 \in \Sigma$ such that $\sigma \leqslant \sigma_0$ for every $\sigma \in \Sigma$, that is σ_0 is a maximal element of Σ. Take any element $y \in \lim\limits_{\rightarrow} \boldsymbol{D}$. It is not difficult to see that $f^{-1}y = \bigcup\{(\psi^\sigma_{\sigma_0})^{-1}(f^{-1}_{\sigma_0} y_{\sigma_0}) : \sigma \leqslant \sigma_0\}$, where y_{σ_0} is the unique representative of y in Y_{σ_0}. Therefore the preimage $f^{-1}y$ is compact. □

Proposition 10.11. *Let* $\lim\limits_{\rightarrow} \boldsymbol{D}_f$ *be the limit of the dircted system* $\boldsymbol{D}_f = \{f_\sigma, \{\phi^\rho_\sigma, \psi^\rho_\sigma\}, \Sigma\}$. *If all the maps* $f_\sigma : X_\sigma \to Y_\sigma$ *and all the binding maps* $\psi^\rho_\sigma : Y_\rho \to Y_\sigma$ *are compact and* Σ *is a finite set, then the limit map* $\lim\limits_{\rightarrow} \boldsymbol{D}_f$ *is also compact.*

Proof. The closedness of the limit map follows from Proposition 9.3. A proof similar to the above shows that the projection $pr_B : \bigoplus\{Y_\sigma : \sigma \in \Sigma\} \to \underrightarrow{\lim}\boldsymbol{D}_B$ is a compact map. Therefore, by Proposition 10.9 we have that $pr_B \circ \bigoplus\{Y_\sigma : \sigma \in \Sigma\}$ is a compact map. But $pr_B \circ \bigoplus\{f_\sigma : \sigma \in \Sigma\} = \underrightarrow{\lim}\boldsymbol{D}_f \circ pr_T$, and since pr_T is onto we have that the limit map $\underrightarrow{\lim}\boldsymbol{D}_f$ is a compact map. □

For fibrewise sums we also have the following result on the finiteness of Σ.

Proposition 10.12. *Let $\boldsymbol{D} = \{Y_\sigma, \psi^\rho_\sigma, \Sigma\}$ be a directed system. If the fibrewise sum $f = \bigoplus\{f_\sigma, \{Y_\sigma, \psi^\rho_\sigma, \Sigma\}\}$ is a compact map, then the directed set Σ finite.*

Proof. Let the map $f : \bigoplus\{X_\sigma : \sigma \in \Sigma\} \to \underrightarrow{\lim}\boldsymbol{D}$ be compact. Take any element $y \in \underrightarrow{\lim}\boldsymbol{D}$. The preimage $f^{-1}y$ is compact and lies in $\bigoplus\{X_\sigma : \sigma \in \Sigma\}$ and so the set $\Sigma_y = \{\sigma : y \text{ has a representative in } Y_\sigma\}$ is finite, say $\Sigma_y = \{\sigma_1, \dots, \sigma_k\}$. It is not difficult to prove that there exists a $j \leqslant k$ satisfying

$$\sigma_i \leqslant \sigma_j \text{ for } i = 1, \dots, k. \tag{10.1}$$

Now assume that there exists $\sigma_0 \in \Sigma, \sigma_0 \neq \sigma_j$ such that either (a) σ_0 and σ_j are not $\leqslant$-related, or (b) $\sigma_j \leqslant \sigma_0$ but $\sigma_0 \nleqslant \sigma_j$. We show that both (a) and (b) lead to contradictions. Say (a) is true, then there exists some $\sigma \in \Sigma$ such that $\sigma_0 \leqslant \sigma$ and $\sigma_j \leqslant \sigma$. By definition we have $\sigma \neq \sigma_0$ and $\sigma \neq \sigma_j$. Consider some representative y_{σ_j} of y in Y_{σ_j}, then $\psi^{\sigma_j}_\sigma(y_{\sigma_j})$ is a representative of y in Y_σ. Therefore, $\sigma \in \Sigma_Y$ which means, using (10.1), that $\sigma \leqslant \sigma_j$ and consequently, that $\sigma_0 \leqslant \sigma_j$. This contradicts our assumption on the choice of σ_0. Similarly, if (b) is true, we have that $\psi^{\sigma_j}_{\sigma_0}(y_{\sigma_j})$ is a representative of y in Y_{σ_0}. Therefore, a similar argument to the one used for contradicting (a) proves that $\sigma_0 \leqslant \sigma_j$ which again leads to a contradiction.

From the above we have that $\sigma \leqslant \sigma_j$ for every $\sigma \in \Sigma$. Hence $\left(\psi^\sigma_{\sigma_j}\right)^{-1}(y_{\sigma_j})$ contains representatives of y in Y_σ and therefore there exists $i \leqslant k$ satisfying $\sigma = \sigma_i$. We have proved that the directed set Σ is a finite set. □

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