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INEQUALITIES ON INFINITE NETWORKS

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To the memory of Professor Dr. Werner Oettli

Abstract. Inequalities on networks have played important roles in the theory of netwoks. We study several famous inequalities on networks such as Wirtinger's inequality, Hardy's inequality, Poincaré-Sobolev's inequality and the strong isoperimetric inequality, etc. These inequalities are closely related to the smallest eigenvalue of weighted discrete Laplacian. We discuss some relations between these inequalities and the potential-theorerteic magnitude of the ideal boundary of an infinite network.

1. Problem Setting

Let X be a countable set of nodes, Y be a countable set of arcs and K be the node-arc incidence matrix. Assume that the graph $G := \{X, Y, K\}$ is locally finite and connected and has no self-loop. For a strictly positive real valued function r on Y, $N := \{G, r\}$ is called a network.

Let $L(X)$ be the set of all real valued functions on X, let $L^+(X)$ be the set of all non-negative $u \in L(X)$ and let $L_0(X)$ be the set of all $u \in L(X)$ with finite support. We denote by ε_A the characteristic function of the subset A of X and put $\varepsilon_x := \varepsilon_A$ in case $A = \{x\}$.

The discrete derivative du and the discrete Laplacian $\Delta u(x)$ of $u \in L(X)$ are defined by

$$\begin{aligned} du(y) &:= -r(y)^{-1} \sum\nolimits_{x\in X} K(x,y)u(x), \\ \Delta u(x) &:= \sum\nolimits_{y\in Y} K(x,y)[du(y)]. \end{aligned}$$

Denote by the total conductance at $x \in X$ by

$$c(x) := \sum\nolimits_{y\in Y} |K(x,y)| r(y)^{-1}.$$

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In case $r(y) = 1$ on Y, this quantity is equal to the number of the neighboring nodes of x and is called the degree of x. For $x, z \in X$ with $x \neq z$, the conductance $c(x, z)$ between x and z is defined by

$$c(x, z) := \sum\nolimits_{y \in Y} r(y)^{-1} |K(x, y)K(z, y)|.$$

We set $c(x, x) = 0$ for every $x \in X$. Notice that

$$\Delta u(x) = -c(x)u(x) + \sum\nolimits_{z \in X} c(x, z)u(z).$$

The mutual Dirichlet sum $D(u, v)$ of $u, v \in L(X)$ is defined by

$$D(u, v) := \sum\nolimits_{y \in Y} r(y)[du(y)][dv(y)]$$

if the sum on the right hand side converges. We call $D(u) := D(u, u)$ the Dirichlet sum of u and consider the following set of discrete Dirichlet functions:

$$D(N) := \{u \in L(X); D(u) < \infty\}.$$

Let m be a strictly positive real valued function on X and consider the following inner product:

$$< u, v >_m := \sum\nolimits_{x \in X} m(x)u(x)v(x)$$

if the sum on the right hand side converges. Let us put for simplicity

$$\|u\|_m := [< u, u >_m]^{1/2} \quad \text{and} \quad L_2(X; m) := \{u \in L(X); \|u\|_m < \infty\}.$$

For a nonempty subset B of X (Y resp.) and a function w on X (Y resp.), denote by $w(B)$ the sum of $w(\cdot)$ on B.

In this paper, we always assume that A_0 is a fixed nonempty finite subset of X such that $A_0 \neq X$.

We shall study the following conditions related to inequalities on the network N.

(C.1; m) There exists a constant $C_1 > 0$ such that

$$\|u\|_m^2 \leq C_1 D(u) \quad \text{for all} \quad u \in L_0(X).$$

(C.2; m, A_0) There exists a constant $C_2 > 0$ such that

$$\|u\|_m^2 \leq C_2 D(u) \quad \text{for all} \quad u \in L_0(X; A_0),$$

where

$$L_0(X; A_0) := \{u \in L_0(X); u = 0 \text{ on } A_0\}.$$

For simplicity, we introduce the following function

$$\chi_m(u) := \frac{D(u)}{\|u\|_m^2}$$

on $D(N)$ and consider the following values of extremum problems:

$$\begin{aligned} \lambda_m^{(1)} &:= \inf\{\chi_m(u); u \in L_0(X)\}, \\ \lambda_m^{(2)}(A_0) &:= \inf\{\chi_m(u); u \in L_0(X; A_0)\}. \end{aligned}$$

Notice that they are the best possible values of $1/C_1$ and $1/C_2$ respectively. Therefore Conditions (C.1; m) and (C.2; m, A_0) are equivalent to $\lambda_m^{(1)} > 0$ and $\lambda_m^{(2)}(A_0) > 0$ respectively.

Clearly $\lambda_m^{(1)} \leq \lambda_m^{(2)}(A_0)$. In case N is a finite network, we have

$$\lambda_m^{(1)} = 0 < \lambda_m^{(2)}(A_0).$$

We shall be concerned with the problem when does $\lambda_m^{(2)}(A_0) > 0$ imply $\lambda_m^{(1)} > 0$. We also give some suitable lower bounds for $\lambda_m^{(1)}$ and $\lambda_m^{(2)}(A_0)$ by using potential-theoretic method. Applications of our inequalities to the study of discrete potentials will be given in §5. We shall show in §6 some partial answers to the existence of optimal solutions for $\lambda_m^{(1)}$ and $\lambda_m^{(2)}(A_0)$.

Remark 1.1. Condition (C.1; m) implies the generalized strong isoperimetric inequality (GSI):

$$\|\varepsilon_A\|_m^2 \leq C_1 D(\varepsilon_A) \quad \text{or} \quad m(A) \leq C_1 r^{-1}(\partial A)$$

for every nonempty finite subset A of X. Here ∂A is the set of $y \in Y$ which connects A and $X \setminus A$ directly.

In case $m = 1$ on X and $r = 1$ on Y, Condition (C.1; m) is known as the Poincaré-Sobolev's inequality. In [1], it was proved that the Poincaré-Sobolev's inequality is equivalent to the following *isoperimetric inequality* under the condition that X is of bounded degree, i.e., $\sup\{c(x); x \in X\} < \infty$:

(SI) There exists a constant $C_4 > 0$ such that

$$|A| \leq C_4 |\partial A|$$

for every nonempty finite subset A of X. Here $|A|$ denotes the cardinality of A.

In case $m(x) = c(x)$ on X, it was proved in [17] that the generalized Poincaré-Sobolev's inequality (C.1; m) holds if and only if the generalized strong isoperimetric inequality (GSI) does.

2. Preliminaries

Lemma 2.1. *For every $u \in D(N)$, the following inequality holds:*

$$D(u) \leq 2 \sum\nolimits_{x \in X} c(x) u(x)^2$$

Proof. By definition, we have

$$\begin{aligned} D(u) &\leq 2 \sum\nolimits_{y \in Y} r(y)^{-1} \left[\sum\nolimits_{x \in X} K(x,y)^2 u(x)^2 \right] \\ &= 2 \sum\nolimits_{x \in X} u(x)^2 \left[\sum\nolimits_{x \in X} K(x,y)^2 r(y)^{-1} \right] \\ &= 2 \sum\nolimits_{x \in X} c(x) u(x)^2. \end{aligned}$$

□

We recall the following useful result(cf. Lemma 3 in [21]):

Lemma 2.2. *(Green'sformula) Let* $u \in L(X),\ f \in L_0(X)$*. Then*

$$D(u, f) = -\sum\nolimits_{x\in X}[\Delta u(x)]f(x) = -\sum\nolimits_{x\in X}[\Delta f(x)]u(x).$$

Let us further introduce the following set of functions:

$$D(N; A_0) := \{u \in D(N); u = 0 \text{ on } A_0\}.$$

Notice that $D(N; A_0)$ is a Hilbert space with the inner product $D(u, v)$. Denote by $D_0(N; A_0)$ the closure of $L_0(X; A_0)$ in $D(N; A_0)$. There exist the unique reproducing kernels $\tilde{g}_x^*$ and $\tilde{g}_x$ of $D_0(N; A_0)$ and $D(N; A_0)$ respectively, i.e.,

$$\begin{aligned} D(u, \tilde{g}_x^*) &= u(x) \quad \text{for all } u \in D_0(N; A_0), \\ D(u, \tilde{g}_x) &= u(x) \quad \text{for all } u \in D(N; A_0). \end{aligned}$$

We called $\tilde{g}_x$ the Kuramochi kernel of N with pole at x in [13].

We also notice that $D(N)$ is a Hilbert space with the inner product

$$((u, v))_D := D(u, v) + u(x_0)v(x_0),$$

where x_0 is a fixed node. We set $\|u\|_D = ((u, u))_D^{1/2}$. The set $D_0(N)$ of discrete Dirichlet potentials is defined as the closure of $L_0(X)$ in $D(N)$.

Recall that N is of parabolic type if the value

$$d(A, \infty) := \inf\{D(u); u \in L_0(X),\ u = 1 \text{ on } A\}$$

vanishes for some nonempty finite subset A of X (cf. [22]). We say that N is of hyperbolic type if it is not of parabolic type. Notice that N is of parabolic type if and only if $D(N; A_0) = D_0(N; A_0)$. In case N is of hyperbolic type, there exists a unique reproducing kernel g_x of $D_0(N)$, i.e.,

$$v(x) = D(v, g_x) \quad \text{for all } v \in D_0(N).$$

We call g_x the Green function of N with pole at x.

For every $f \in L^+(X)$, let us define potentials $\tilde{G}f,\ \tilde{G}^*f$ and Gf of f with respect to the above reproducing kernels $\tilde{g}_x, \tilde{g}_x^*$ and g_x. For example,

$$\tilde{G}^*f(x) := \sum\nolimits_{z\in X} \tilde{g}_x^*(z)f(z).$$

We see that $\tilde{G}f$ and $\tilde{G}^*f$ are superharmonic on $X \setminus A_0$ if they have finite values at some $x \in X \setminus A_0$. Similarly if $Gf(x) < \infty$ for some $x \in X$, then Gf is superharmonic on X (cf. [23]).

For mutually disjoint nonempty subsets A and B of X, let us consider two values of convex programs:

$$\begin{aligned} d_0(A, B) &:= \inf\{D(u); u \in L_0(X),\ u = 0 \text{ on } A,\ u = 1 \text{ on } B\}, \\ d(A, B) &:= \inf\{D(u); u \in L(X),\ u = 0 \text{ on } A,\ u = 1 \text{ on } B\}. \end{aligned}$$

Notice that $d(A, B) = d(B, A)$. We have by Theorem 2.1 in [14]

Lemma 2.3. *Let* $x \in X \setminus A_0$*. Then*

$$d_0(A_0, \{x\}) = 1/\tilde{g}_x^*(x) \quad \textit{and} \quad d(A_0, \{x\}) = 1/\tilde{g}_x(x).$$

We have by Theorem 2.2 in [16]

Lemma 2.4. *Let A be a finite subset of X. For any exhaustion $\{N_n\}$ $(N_n =< X_n, Y_n >)$ of N with $A \subset X_1$, $d(A, X \setminus X_n) \to d(A, \infty)$ as $n \to \infty$.*

We shall prove

Theorem 2.1. *If N is an infinite network which is of parabolic type, then $\{\tilde{g}_x(x); x \in X\}$ is unbounded.*

Proof. Suppose that $\{\tilde{g}_x(x); x \in X\}$ is bounded, i.e., there exists a constant $C > 0$ such that $0 < \tilde{g}_x(x) \leq C$ for all $x \in X \setminus A_0$. Let $\{N_n\}(N_n =< X_n, Y_n >)$ be an exhaustion of N with $A_0 \subset X_1$ and take a sequence $\{x_n\}$ of nodes such that $x_n \in X \setminus X_n$. Then

$$\frac{1}{C} \leq \frac{1}{\tilde{g}_{x_n}(x_n)} = d(A_0, \{x_n\}) \leq d(A_0, X \setminus X_n)$$

for every n. It follows from Lemma 2.4 that

$$d(A_0, \infty) \geq \frac{1}{C} > 0.$$

Thus N is of hyperbolic type. This is a contradiction. □

3. Relations between $\lambda_m^{(1)}$ and $\lambda_m^{(2)}(A_0)$

Theorem 3.1. *Let A_0 and A_0' be nonempty finite subsets of X such that $A_0' \subset A_0$. Then $\lambda_m^{(2)}(A_0') > 0$ if and only if $\lambda_m^{(2)}(A_0) > 0$.*

Proof. Since $A_0' \subset A_0$, we have $\lambda_m^{(2)}(A_0') \leq \lambda_m^{(2)}(A_0)$, so that the only if part holds. Assume that $\lambda_m^{(2)}(A_0') = 0$. There exists a sequence $\{f_n\}$ in $L_0(X; A_0')$ such that $\|f_n\|_m = 1$ and $D(f_n) \to 0$ as $n \to \infty$. Since $f_n(x) = 0$ on A_0', we see that $\{f_n\}$ converges pointwise to 0. Let $v_n = f_n - u_n$ with $u_n := f_n \varepsilon_{A_0}$. Then $v_n \in L_0(X; A_0)$ and

$$\|v_n\|_m^2 = 1 - \sum\nolimits_{x \in A_0} m(x) f_n(x)^2 \to 1$$

as $n \to \infty$. Since N is locally finite, Δf_n and Δu_n converge pointwise to 0. Here we remark

$$\Delta u_n(x) = \sum\nolimits_{z \in A_0} c(x, z) f_n(x).$$

Since A_0 is a finite set, we have by Lemma 2.2

$$\begin{aligned} D(v_n) &= D(f_n) - 2D(f_n, u_n) + D(u_n) \\ &= D(f_n) + 2\sum\nolimits_{x \in A_0} [\Delta f_n(x)] f_n(x) - \sum\nolimits_{x \in A_0} [\Delta u_n(x)] f_n(x) \to 0 \end{aligned}$$

as $n \to \infty$, and hence

$$\lambda_m^{(2)}(A_0) \leq \frac{D(v_n)}{\|v_n\|_m^2} \to 0.$$

□

Corollary 3.1. *If there exists a nonempty finite subset A_0 of X such that $\lambda_m^{(2)}(A_0) > 0$, then $\lambda_m^{(2)}(A) > 0$ for all nonempty finite subset A of X.*

Lemma 3.1. *Let $m_1, m_2 \in L^+(X)$ satisfy $0 < m_1 \leq m_2$. Then* (C.1; m_2) *and* (C.2; m_2, A_0) *imply* (C.1; m_1) *and* (C.2; m_1, A_0) *respectively.*

Theorem 3.2. *Assume that m is bounded. Then* (C.1; 1) *implies* (C.2; m, A_0).

Proof. There exists $M > 0$ such that $m(x) \leq M$ for all $x \in X$. If Condition (C.1; 1) holds, then there exists $C_1 > 0$ such that

$$\|u\|_m^2 \leq M\|u\|_1^2 \leq MC_1 D(u)$$

for every $u \in L_0(X)$. Since $L_0(X; A_0) \subset L_0(X)$, we have $\chi_m(u) \geq 1/(MC_1) > 0$ for all $u \in L_0(X; A_0)$, and hence $\lambda_m^{(2)}(A_0) \geq 1/(MC_1) > 0$. □

Theorem 3.3. *If N is of parabolic type, then $\lambda_m^{(1)} = 0$.*

Proof. Suppose that $\lambda_m^{(1)} > 0$ and let A be a nonempty finite subset of X. Since N is of parabolic type, there exists a sequence $\{f_n\}$ in $L_0(X)$ such that $f_n(x) = 1$ on A and $D(f_n) \to 0$ as $n \to \infty$. Then,

$$0 < m(A)\lambda_m^{(1)} \leq \lambda_m^{(1)}\|f_n\|_m^2 \leq D(f_n) \to 0$$

as $n \to \infty$. This is a contradiction. □

Theorem 3.4. *Assume that N is of hyperbolic type. Then Condition* (C.2; m, A_0) *is equivalent to Condition* (C.1; m).

Proof. Since $\lambda_m^{(1)} \leq \lambda_m^{(2)}(A_0)$ in general, we see that Condition (C.1; m) implies Condition (C.2; m, A_0). Assume that Condition (C.2; m, A_0) holds. By Theorem 3.1, we may assume that A_0 is a singleton $\{a\}$. For any $u \in L_0(X)$, let $f = u - u\varepsilon_a$. Then $f \in L_0(X; \{a\})$. Let $C_2 = 1/\lambda_m^{(2)}(\{a\})$. By our assumption, we have

$$\sum_{x \in X} m(x)f(x)^2 \leq C_2 D(f)$$

Since N is of hyperbolic type, we can find a constant $C_0 > 0$ depending only on a such that

$$|v(a)| \leq C_0[D(v)]^{1/2}$$

for all $v \in L_0(X)$ by Theorem 3.2 in [19]. We have

$$D(u\varepsilon_a) = u(a)^2 D(\varepsilon_a) \leq C_0^2 D(u)c(a),$$

so that

$$D(f) \leq 2(D(u) + D(u\varepsilon_a)) \leq 2(1 + C_0^2 c(a))D(u).$$

It follows that

$$\begin{aligned}
\|u\|_m^2 &= \sum_{x \in X} m(x)f(x)^2 + m(a)u(a)^2 \\
&\leq C_2 D(f) + m(a)D(u) \\
&\leq [2(1 + C_0^2 c(a))C_2 + m(a)C_0^2]D(u).
\end{aligned}$$

Thus we have

$$\lambda_m^{(1)} \geq \left(2(1 + C_0^2 c(a))C_2 + m(a)C_0^2\right)^{-1} > 0.$$

□

By Lemma 2.3, we have

Theorem 3.5. *For every $b \in X \setminus A_0$, the following inequality holds:*

$$\lambda_m^{(2)}(A_0) \le \frac{d_0(A_0, \{b\})}{m(b)} = \frac{1}{m(b)\tilde{g}_b^*(b)}.$$

Proof. For every $u \in L_0(X; A_0)$ with $u(b) = 1$, we have

$$\lambda_m^{(2)}(A_0) \le \chi_m(u) = \frac{D(u)}{\|u\|_m^2} \le \frac{D(u)}{m(b)},$$

which gives our first inequality. □

By Theorem 3.5, we have

Theorem 3.6. *If $\{m(x)\tilde{g}_x^*(x); x \in X\}$ is unbounded, then $\lambda_m^{(2)}(A_0) = 0$.*

Corollary 3.2. *Assume that $m(x) \ge 1$ on $X \setminus A_0$ and that N is an infinite network. If N is of parabolic type, then $\lambda_m^{(2)}(A_0) = 0$.*

Proof. Since $m(x) \ge 1$ on $X \setminus A_0$, we have by Theorem 3.5

$$\lambda_m^{(2)}(A_0) \le \frac{1}{\tilde{g}_x^*(x)} = \frac{1}{\tilde{g}_x(x)}$$

for any $x \in X \setminus A_0$. Since $\{\tilde{g}_x(x); x \in X\}$ is unbounded by Theorem 2.1, it follows that $\lambda_m^{(2)}(A_0) = 0$. □

We can not omit the assumption that N is an infinite network in Corollary 3.2, since $\lambda_m^{(2)}(A_0) > 0$ if N is a finite network.

4. Estimation of $\lambda_m^{(1)}$ and $\lambda_m^{(2)}(A_0)$

Hereafter we always assume that $X \setminus A_0$ is connected, i.e., any two nodes in $X \setminus A_0$ can be connected by a path whose nodes are in $X \setminus A_0$.

For a finite subnetwork $N' =< X', Y' >$ of N such that $A_0 \subset X'$ and $X' \setminus A_0$ is connected, we consider the following extremum problems:

$$\begin{aligned} \lambda_m^{(1)}(N') &:= \inf\{\chi_m(u); u \in L(X), u = 0 \text{ on } X \setminus X'\}, \\ \lambda_m^{(2)}(A_0; N') &:= \inf\{\chi_m(u); u \in D(N'; A_0)\}, \end{aligned}$$

where we set for simplicity

$$D(N'; A_0) := \{u \in L(X); u = 0 \text{ on } A_0 \cup (X \setminus X')\}.$$

As in [18], we have

Lemma 4.1. *Let $N' =< X', Y' >$ be a finite subnetwork of N with $A_0 \subset X'$. There exists a unique $u' \in D(N'; A_0)$ which has the following properties:*

(1) $\lambda_m^{(2)}(A_0; N') = \chi_m(u')$,

(2) $\Delta u'(x) = -\lambda_m^{(2)}(A_0; N')m(x)u'(x)$ *on* $X' \setminus A_0$.

(3) $u'(x) > 0$ *on* $X' \setminus A_0$ *and* $\|u'\|_m = 1$.

Lemma 4.2. *Let $N' =< X', Y' >$ be a finite subnetwork of N. There exists a unique $u' \in L(X)$ which has the following properties:*
(1) $u'(x) = 0$ *on* $X \setminus X'$,
(2) $\lambda_m^{(1)}(N') = \chi_m(u')$,
(3) $\Delta u'(x) = -\lambda_m^{(1)}(N')m(x)u'(x)$ *on* X',
(4) $u'(x) > 0$ *on* X' *and* $\|u'\|_m = 1$.

For each $x \in X' \setminus A_0$, there exists a unique function $\tilde{g}_x^{N'} \in D(N'; A_0)$ such that

$$u(x) = D(u, \tilde{g}_x^{N'}) \quad \text{for all } u \in D(N'; A_0),$$

since $D(N'; A_0)$ is a Hilbert space. We see that $\Delta\tilde{g}_x^{N'} = -\varepsilon_x$ on $X' \setminus A_0$ and $0 \le \tilde{g}_x^{N'}(z)$ on X.

Lemma 4.3. *Let $N' =< X', Y' >$ be a finite subnetwork of N with $A_0 \subset X'$ and let u' be the function obtained in Lemma 4.1. Then, for every $x \in X' \setminus A_0$*

$$u'(x) = \lambda_m^{(2)}(A_0; N') \sum\nolimits_{z \in X} \tilde{g}_x^{N'}(z)m(z)u'(z)$$

Proof. By the above observation, we have by Lemma 2.2

$$\begin{aligned} u'(x) &= D(u', \tilde{g}_x^{N'}) \\ &= -\sum\nolimits_{z \in X} \tilde{g}_x^{N'}(z)\Delta u'(z) \\ &= \lambda_m^{(2)}(A_0; N') \sum\nolimits_{z \in X} \tilde{g}_x^{N'}(z)m(z)u'(z). \end{aligned}$$

□

For every $f \in L^+(X)$, we put

$$\tilde{G}^{N'} f(x) := \sum\nolimits_{z \in X} \tilde{g}_x^{N'}(z)f(z).$$

Corollary 4.1. *The following relation holds:*

$$\frac{1}{\max\{\tilde{G}^{N'}m(x); x \in X'\}} \le \lambda_m^{(2)}(A_0; N') \le \frac{1}{\min\{\tilde{G}^{N'}m(x); x \in X'\}}.$$

Corollary 4.2. *The following relation holds:*

$$\frac{1}{m(X')\max\{\tilde{g}_x^{N'}; x \in X'\}} \le \lambda_m^{(2)}(A_0; N') \le \frac{1}{m(X')\min\{\tilde{g}_x^{N'}; x \in X'\}}.$$

As in Urakawa[20], we have

Lemma 4.4. *Let $N' =< X', Y' >$ be a finite subnetwork of N such that $A_0 \subset X'$ and assume that $X' \setminus A_0$ is connected. If $f \in D(N'; A_0)$ satisfies the condition that $f(x) > 0$ on $X' \setminus A_0$, then the following inequality holds:*

$$\min\{\frac{-\Delta f(x)}{m(x)f(x)}; x \in X' \setminus A_0\} \le \lambda_m^{(2)}(A_0; N') \le \max\{\frac{-\Delta f(x)}{m(x)f(x)}; x \in X' \setminus A_0\}.$$

Proof. Let u' be the function obtained in Lemma 4.1. Then

$$\begin{aligned} u'(x) &= 0 \ \text{ on } A_0 \cup (X \setminus X'),\ u'(x) > 0 \ \text{ on } X' \setminus A_0, \\ \Delta u'(x) &= -\lambda_m^{(2)}(A_0; N')m(x)u'(x) \ \text{ on } \ X' \setminus A_0. \end{aligned}$$

For $x \in X' \setminus A_0$, we have

$$\begin{aligned} \lambda_m^{(2)}(A_0; N') &= \frac{-\Delta u'(x)}{m(x)u'(x)} \\ &= \frac{-\Delta f(x)}{m(x)f(x)} + \frac{\psi(x)}{m(x)f(x)u'(x)}. \end{aligned}$$

Here $\psi(x) = u'(x)\Delta f(x) - f(x)\Delta u'(x)$. Since $f, u' \in L_0(X)$ and $f(x) = u'(x) = 0$ on $A_0 \cup (X \setminus X')$, we have by Lemma 2.2

$$\begin{aligned} \sum\nolimits_{x \in X' \setminus A_0} \psi(x) &= \sum\nolimits_{x \in X} \psi(x) \\ &= \sum\nolimits_{x \in X} u'(x)\Delta f(x) - \sum\nolimits_{x \in X} f(x)\Delta u'(x) \\ &= D(u', f) - D(f, u') = 0. \end{aligned}$$

Namely we see that either $\psi(x) = 0$ on X or $\psi(x)$ changes its sign on $X' \setminus A_0$. If $\psi(x) = 0$ on X, then

$$\frac{-\Delta f(x)}{m(x)f(x)} = \frac{-\Delta u'(x)}{m(x)u'(x)} = \lambda_m^{(2)}(A_0; N')$$

for every $x \in X' \setminus A_0$. Otherwise, there exists $a, b \in X' \setminus A_0$ such that $\psi(a) > 0$, $\psi(b) < 0$. Since $f(x)u'(x) > 0$ on $X' \setminus A_0$, it follows that

$$\frac{-\Delta f(a)}{m(a)f(a)} \leq \lambda_m^{(2)}(A_0; N') \leq \frac{-\Delta f(b)}{m(b)f(b)}.$$

□

Remark 4.1. Let us take $f = \tilde{G}^{N'}m$ in the this lemma. Then $\Delta f(x) = -m(x)$ on $X' \setminus A_0$, so that we have

$$\min\{\frac{1}{\tilde{G}^{N'}m(x)}; x \in X' \setminus A_0\} \leq \lambda_m^{(2)}(A_0; N') \leq \max\{\frac{1}{\tilde{G}^{N'}m(x)}; x \in X' \setminus A_0\}.$$

This is the same inequality as in Corollary 4.1.

Let $\{N_n\}(N_n =< X_n, Y_n >)$ be an exhaustion of N such that $A_0 \subset X_1$ and $X_1 \setminus A_0$ is connected. Then we have

Theorem 4.1. *The sequence $\{\lambda_m^{(1)}(N_n)\}$ converges to $\lambda_m^{(1)}$.*

Proof. We have

$$\lambda_m^{(1)} \leq \lambda_m^{(1)}(N_{n+1}) \leq \lambda_m^{(1)}(N_n).$$

For any $\varepsilon > 0$ we can find $u \in L_0(X)$ such that $\chi_m(u) < \lambda_m^{(1)} + \varepsilon$. There exists n_0 such that $u = 0$ on $X \setminus X_n$ for all $n \geq n_0$. Thus $\lambda_m^{(1)}(N_n) \leq \chi_m(u)$ for all $n \geq n_0$. Hence $\{\lambda_m^{(1)}(N_n)\}$ converges to $\lambda_m^{(1)}$. □

Similarly we can prove

Theorem 4.2. *The sequence* $\{\lambda_m^{(2)}(A_0; N_n)\}$ *converges to* $\lambda_m^{(2)}(A_0)$.

Let $\tilde{g}_x^{*(n)} := \tilde{g}_x^{N'}$ and $\tilde{G}^{*(n)} f(x) := \tilde{G}^{N'} f(x)$ with $N' = N_n$. We have by the minimum principle that $\tilde{g}_x^{*(n)} \leq \tilde{g}_x^{*(n+1)} \leq \tilde{g}_x^*$ on X. Notice that

$$\lim_{n\to\infty} \tilde{g}_x^{*(n)}(z) = \tilde{g}_x^*(z)$$

for each $z \in X$.

Theorem 4.3. *If the* $\tilde{G}^*$*-potential* $\tilde{G}^* m$ *of* m *is bounded on* X*, then*

$$\lambda_m^{(2)}(A_0) \geq \frac{1}{\sup\{\tilde{G}^* m(x); x \in X\}} > 0.$$

Proof. Let $\{N_n\}(N_n =< X_n, Y_n >)$ be an exhaustion of N with $A_0 \subset X_1$. Since $\tilde{G}^{*(n)} m(x) \leq \tilde{G}^* m(x)$ and $\gamma := \sup\{\tilde{G}^* m(x); x \in X\} < \infty$ by our assumption, we have by Corollary 4.1

$$\lambda_m^{(2)}(A_0; N_n) \geq \frac{1}{\gamma} > 0$$

for all n, and our inequality follows from Theorem 4.2. □

Corollary 4.3. *Assume that* N *is of parabolic type and let* $\tilde{m}$ *be a strictly positive real valued function on* X *such that* $\tilde{m}(X) < \infty$*. Let* $m(x) := 1$ *on* A_0 *and* $m(x) := \tilde{m}(x)/\tilde{g}_x^*(x)$ *for* $x \in X \setminus A_0$*. Then* $\lambda_m(A_0) > 0$.

Proof. Since $\tilde{g}_x^*(z) \leq \tilde{g}_x^*(x)$, we have

$$\tilde{G}^* m(x) = \sum\nolimits_{z \in X \setminus A_0} \frac{\tilde{g}_x^*(z)\tilde{m}(z)}{\tilde{g}_z^*(z)} \leq \tilde{m}(X) < \infty$$

for all $x \in X \setminus A_0$. □

Similarly we have by Corollary 4.2

Theorem 4.4. *If* $\{\tilde{g}_x^*(x); x \in X\}$ *is bounded and if* $m(X) < \infty$*, then*

$$\lambda_m^{(2)}(A_0) \geq \frac{1}{m(X) \sup\{\tilde{g}_x^*(x); x \in X\}} > 0.$$

By taking the Green function $g_x^{(n)}$ of N_n in place of $\tilde{g}_x^{*(n)}$, we can prove

Theorem 4.5. *If the Green potential* Gm *of* m *is bounded on* X*, then* (C.1; m) *holds and*

$$\lambda_m^{(1)} \geq \frac{1}{\sup\{Gm(x); x \in X\}} > 0.$$

Theorem 4.6. *If* $\{g_x(x); x \in X\}$ *is bounded and if* $m(X) < \infty$*, then*

$$\lambda_m^{(1)} \geq \frac{1}{m(X) \sup\{g_x(x); x \in X\}} > 0.$$

5. Dirichlet potentials

As applications of our inequalities, we shall study the relations between $D_0(N)$ and $L_2(X; m)$.

Theorem 5.1. *Assume that Condition* (C.1; m) *holds. Then* $D_0(N) \subset L_2(X; m)$ *and there exists* $C_1 > 0$ *such that*

$$\|u\|_m^2 \leq C_1 D(u)$$

for every $u \in D_0(N)$.

Proof. Let $u \in D_0(N)$. There exists a sequence $\{f_n\}$ in $L_0(X)$ such that $\|u - f_n\|_D \to 0$ as $n \to \infty$. It follows that $D(f_n) \to D(u)$ as $n \to \infty$ and $\{f_n\}$ converges pointwise to u. By Condition (C.1; m), there exists $C_1 > 0$ such that

$$\sum\nolimits_{x \in X} m(x) f_n(x)^2 \leq C_1 D(f_n).$$

By Fatou's lemma,

$$\sum\nolimits_{x \in X} m(x) u(x)^2 \leq \liminf_{n \to \infty} \sum\nolimits_{x \in X} m(x) f_n(x)^2.$$

Thus we have

$$\sum\nolimits_{x \in X} m(x) u(x)^2 \leq C_1 D(u),$$

so that $u \in L_2(X; m)$. □

Similarly we have

Theorem 5.2. *If Condition* (C.2; m, A_0) *holds, then* $D_0(N; A_0) \subset L_2(X; m)$ *and* $\|u\|_m^2 \leq \lambda_m^{(2)}(A_0) D(u)$ *for every* $u \in D_0(N; A_0)$.

Lemma 5.1. *Assume that there exists a constant* $C > 0$ *such that* $c(x) \leq C m(x)$ *on* X. *Then* $L_2(X; m) \subset D_0(N)$.

Proof. Let $u \in L_2(X; m)$ and take and an exhaustion $\{N_n\} (N_n =< X_n, Y_n >)$ of N. Define f_n by $f_n(x) := u(x)$ for $x \in X_n$ and $f_n(x) := 0$ for $x \in X \setminus X_n$. We have by Lemma 2.1

$$\begin{aligned} D(u - f_n) &\leq 2 \sum\nolimits_{x \in X} c(x)(u(x) - f_n(x))^2 \\ &\leq 2C \sum\nolimits_{x \in X \setminus X_n} m(x) u(x)^2 \to 0, \end{aligned}$$

as $n \to \infty$. Since $f_n \in L_0(X)$ and $\{f_n\}$ converges pointwise to u, we see that $\|u - f_n\|_D \to 0$ as $n \to \infty$, and therefore $u \in D_0(N)$. □

By Theorem 5.1 and Lemma 5.1, we have

Theorem 5.3. *Assume that Condition* (C.1; m) *holds and that there exists a constant* $C > 0$ *such that* $c(x) \leq C m(x)$ *on* X. *Then* $D_0(N) = L_2(X; m)$.

Corollary 5.1. *Assume that* $m(x) = 1$ *on* X, $r(y) = 1$ *on* Y *and* $\sup\{c(x); x \in X\} < \infty$. *Then* $D_0(N) = L_2(X; m)$.

Corollary 5.2. *Assume that N is of hyperbolic type. If Condition (C.1; m) holds, then $g_x \in L_2(X;m)$ for all $x \in X$ and $\{m(x)g_x(x); x \in X\}$ is bounded.*

Proof. Since $g_x \in D_0(N)$, we see by Theorem 5.1 that $g_x \in L_2(X;m)$ for all $x \in X$ and there exists $C_1 > 0$ such that

$$m(x)g_x(x)^2 \leq \sum\nolimits_{z\in X} m(x)g_x(z)^2 \leq C_1 D(g_x) = C_1 g_x(x).$$

Therefore $\{m(x)g_x(x); x \in X\}$ is bounded. □

Denote by $\tilde{D}_0(N)$ the closure of $L_0(X)$ in $D(N)$ with respect to the norm:

$$|u|_D^2 := D(u) + \|u\|_m^2.$$

Since $\|u\|_D \leq |u|_D$, we see that $\tilde{D}_0(N) \subset D_0(N)$.

Theorem 5.4. *If Condition (C.1; m) holds, then $\tilde{D}_0(N) = D_0(N)$.*

Proof. Let $u \in D_0(N)$. There exists a sequence $\{f_n\}$ in $L_0(X)$ such that $\|u - f_n\|_D \to 0$ as $n \to \infty$. By Theorem 5.1, there exists $C_1 > 0$ such that

$$\|u - f_n\|_m^2 \leq C_1 D(u - f_n) \leq C_1 \|u - f_n\|_D^2$$

for all n. Therefore $\|f_n - u\|_m \to 0$ as $n \to \infty$, and hence $|u - f_n|_D \to 0$ as $n \to \infty$. Namely $u \in \tilde{D}_0(N)$. □

Remark 5.1. *We see easily that $\lambda_m^{(1)} = \inf\{\chi_m(u); u \in \tilde{D}_0(N)\}$. If Condition (C.1; m) holds, then*

$$\lambda_m^{(1)} = \inf\{\chi_m(u); u \in D_0(N)\}.$$

Similarly, if Condition (C.2; m, A_0) holds, then

$$\lambda_m^{(2)}(A_0) = \inf\{\chi_m(u); u \in D_0(N; A_0)\}.$$

6. Existence of an optimal solution

First we shall give a characterization of $\lambda_m^{(2)}(A_0)$.

Theorem 6.1. *Let $\Lambda(A_0)$ be the set of $\lambda > 0$ for which there exists $u \in L(X)$ satisfying the following condition:*
(E) $\Delta u + \lambda m u = 0$ *on* $X \setminus A_0, u = 0$ *on* A_0 *and* $u > 0$ *on* $X \setminus A_0$.
Then $\sup \Lambda(A_0) \leq \lambda_m^{(2)}(A_0)$.

Proof. Let $\lambda \in \Lambda(A_0)$ and u be a function which satisfies condition (E). Consider an exhaustion $\{N_n\}(N_n =< X_n, Y_n >)$ be of N such that $A_0 \subset X_1$. There exists $v_n \in L(X)$ such that $\Delta v_n + \lambda_m^{(2)}(A_0; N_n) m v_n = 0$ on $X_n \setminus A_0$, $v_n = 0$ on $A_0 \cup (X \setminus X_n)$ and $v_n \geq 0$ on $X_n \setminus A_0$. Put

$$P := (\lambda - \lambda_m^{(2)}(A_0; N_n)) \sum\nolimits_{x \in X_n \setminus A_0} m(x)u(x)v_n(x).$$

Since $\Delta u + \lambda m u = 0$ on $X_n \setminus A_0$, we have by Lemma 2.2

$$\begin{aligned} P &= -\sum\nolimits_{x \in X_n} v_n(x)[\Delta u(x)] + \sum\nolimits_{x \in X_n} u(x)[\Delta v_n(x)] \\ &= -\sum\nolimits_{x \in X} v_n(x)[\Delta u(x)] + \sum\nolimits_{x \in X} u(x)[\Delta v_n(x)] \\ &\quad -\sum\nolimits_{x \in X \setminus X_n} u(x)[\Delta v_n(x)] \\ &= D(v_n, u) - D(u, v_n) - \sum\nolimits_{x \in X \setminus X_n} u(x)[\Delta v_n(x)] \\ &= -\sum\nolimits_{x \in X \setminus X_n} u(x)[\Delta v_n(x)]. \end{aligned}$$

For each boundary node x of $X_n \setminus A_0$, i.e., $x \notin X_n \setminus A_0$ and there exists $y \in Y_n$ such that $K(x, y) \neq 0$, we have

$$\Delta v_n(x) = \sum\nolimits_{z \in X_n \setminus A_0} c(x, z) v_n(z) \geq 0.$$

Therefore $P \leq 0$. Since $u(x)v_n(x) \geq 0$ on $X_n \setminus A_0$, we obtain $\lambda \leq \lambda_m^{(2)}(A_0; N_n)$. Our assertion follows from Theorem 4.1. □

Similarly we have

Theorem 6.2. *Let Λ be the set of $\lambda > 0$ for which there exists $u \in L(X)$ such that $\Delta u + \lambda m u = 0$ on X and $u > 0$ on X. Then* $\sup \Lambda \leq \lambda_m^{(1)}$.

This result was proved in [6] in case $r = 1$ and $m = 1$.

Now we shall be concerned with the existence of an optimal solution for $\lambda_m^{(1)}$ and $\lambda_m^{(2)}(A_0)$ in $D_0(N)$ and $D_0(N; A_0)$ respectively (cf. Remark 5.1).

Theorem 6.3. *If Condition* (C.1; m) *holds, then there exists a nonconstant $u^* \in L(X)$ such that $u^*(x) > 0$ on X and*

$$\Delta u^*(x) = -\lambda_m^{(1)} m(x) u^*(x) \quad \textit{on} \ X.$$

Proof. Let $\{N_n\}(N_n =< X_n, Y_n >)$ be an exhaustion of N and let u_n^* be the function determined in Lemma 4.2. Take $x_0 \in X_1$ and put $v_n(x) := u_n^*(x)/u_n^*(x_0)$. Then $v_n(x_0) = 1$, $\chi_m(v_n) = \chi_m(u_n^*) = \lambda_m^{(1)}(N_n)$, $v_n(x) > 0$ on X_n and $\Delta v_n(x) = -\lambda_m^{(1)}(N_n) m(x) v_n(x)$ on X_n. For any $x \in X$, $x \neq x_0$, there exists n_0 such that $x \in X_n$ for all $n \geq n_0$. Since v_n is superharmonic on X_n, we see by Harnak's inequality (cf. Theorem 2.3 in [24]) that there exists a constant $\alpha(x_0, x) > 0$ (depending only on x_0 and x) such that

$$v_n(x) \leq \alpha(x_0, x) v_n(x_0) = \alpha(x_0, x)$$

for all $n \geq n_0$. Namely $\{v_n(x)\}$ is bounded for every $x \in X$. By using the diagonal process, we may assume that $\{v_n\}$ converges pointwise to a function $u^* \in L(X)$. Clearly $u^* \in L^+(X)$ and $u^*(x_0) = 1$. It follows from Theorem 4.1 that

$$\Delta u^*(x) = -\lambda_m^{(1)} m(x) u^*(x)$$

for all $x \in X$. Since $\Delta u^*(x) \le 0$ on X, we see by the minimum principle that $u^*(x) > 0$ on X. By the relation $\Delta u^*(x_0) = -\lambda_m^{(1)} m(x_0) u^*(x_0) < 0$, we see that u^* is nonconstant. □

Corollary 6.1. *If* $\lambda_m^{(1)} > 0$, *then* $\max \Lambda = \lambda_m^{(1)}$.

Similarly to Theorem 6.3, we have

Theorem 6.4. *If Condition* (C.2; m, A_0) *holds, then there exists* $\tilde{u} \in L(X)$ *such that* $\tilde{u}(x) = 0$ *on* A_0, $\tilde{u}(x) > 0$ *on* $X \setminus A_0$ *and*

$$\Delta \tilde{u}(x) = -\lambda_m^{(2)}(A_0) m(x) \tilde{u}(x) \quad on \quad X.$$

Corollary 6.2. *If* $\lambda_m^{(2)}(A_0) > 0$, *then* $\max \Lambda(A_0) = \lambda_m^{(2)}(A_0)$.

Theorem 6.5. *Assume that* $\lambda_m^{(1)} > 0$ *and that* $u^* \in D_0(N)$ *satisfies the difference equation:*

$$\Delta u^*(x) = -\lambda_m^{(1)} m(x) u^*(x) \quad on \quad X.$$

Then $\chi_m(u^*) = \lambda_m^{(1)}$.

Proof. There exists a sequence $\{f_n\}$ in $L_0(X)$ such that $\|u^* - f_n\|_D \to 0$ as $n \to \infty$.

$$\lambda_m^{(1)} \|u^* - f_n\|_m^2 \le D(u^* - f_n) \to 0$$

as $n \to \infty$, so that $\{f_n\}$ converges weakly to u^* both in $L_2(X; m)$ and $D_0(N)$. By our assumption, we have $D(u^*, f_n) = \lambda_m^{(1)} < u^*, f_n >_m$, and hence $D(u^*) = \lambda_m^{(1)} \|u^*\|_m^2$. □

Theorem 6.6. *Assume that the Poincaré-Sobolev inequality, or Condition* (C.1; 1)*) holds. If* $m(X) < \infty$, *then* $\lambda_m^{(2)}(A_0) \in \Lambda(A_0)$ *and there exists* $\tilde{u} \in D_0(N; A_0)$ *which satisfies* $\lambda_m^{(2)}(A_0) = \chi_m(\tilde{u})$, $\tilde{u}(x) > 0$ *on* $X \setminus A_0$, *and*

$$\Delta \tilde{u}(x) = -\lambda_m^{(2)}(A_0) m(x) \tilde{u}(x) \text{ on } X \setminus A_0.$$

Proof. By Theorem 3.2, $\lambda_m^{(2)}(A_0) > 0$. Let $\{N_n\} (N_n =< X_n, Y_n >)$ be an exhaustion of N such that $A_0 \subset X_1$ and let u_n be the function determined in Lemma 4.1. By the Poincaré-Sobolev inequality, there exists $C_1 > 0$ such that

$$\sum\nolimits_{x \in X} u_n(x)^2 \le C_1 D(u_n)$$

for all n. Since $\lambda_m^{(2)}(A_0; N_n) = D(u_n)$ and $\{\lambda_m^{(2)}(A_0; N_n)\}$ converges to $\lambda_m^{(2)}(A_0)$ by Theorem 4.1, we see that $\{D(u_n)\}$ is bounded. It follows that there exists $M' > 0$ such that $|u_n(x)| \le M'$ on X for all n. By using the diagonal process, we may assume that $\{u_n\}$ converges pointwise to a function $\tilde{u} \in L(X)$. It follows that $\tilde{u} \in L^+(X)$, $|\tilde{u}(x)| \le M'$ on X and

$$\Delta \tilde{u}(x) = -\lambda_m^{(2)}(A_0) m(x) \tilde{u}(x)$$

on $X \setminus A_0$. We see by Lemma 2.2 that

$$D(\tilde{u}, u_n) = \lambda_m^{(2)}(A_0) < \tilde{u}, u_n >_m .$$

Since $\|u_n\|_m = 1$, $\{u_n(x)\}$ is uniformly bounded and $m(X) < \infty$, it follows from the Lebegue's dominated convergence theorem that

$$\lim_{n\to\infty} < u_n, \tilde{u} >_m = \lim_{n\to\infty} \sum\nolimits_{x\in X} m(x)u_n(x)\tilde{u}(x) = \|\tilde{u}\|_m^2 = 1.$$

Since $\{D(u_n)\}$ is bounded and $u_n \in L_0(X)$ converges pointwise to $\tilde{u}(x)$, we see that $\{u_n\}$ converges weakly to $\tilde{u} \in D_0(N)$, so that $D(u_n, \tilde{u}) \to D(\tilde{u})$ as $n \to \infty$. Thus we have $D(\tilde{u}) = \lambda_m^{(2)}(A_0)$. By the minimum principle, we see that $\tilde{u} > 0$ on $X \setminus A_0$. □

Remark 6.1. Theorem 6.6 does not hold in general if we replace the condition $\lambda_m^{(1)} > 0$ by $\lambda_m^{(2)}(A_0) > 0$. This was shown in [11].

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