PSEUDOCONSISTENT LOGIC AND TENSE LOGIC

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Dedicated to Professor Maretsugu Yamasaki for his 60th birthday

1. Introduction

In our usual logic, we do not infer arbitrary proposition from a contradictory one. Also in executing programs, there is a state that a proposition $A$ holds in some program and in another there is a state in which $A$ does not hold. To explain these situations, recently, the logic called paraconsistent is proposed and investigated. ([1, 2, 3] etc.) Since the logic has two kinds of negation operators, there are cases such that both $A$ and not $A$ are theorems and hence it is difficult to obtain the concept of truth. To the contrary, De Glas has proposed in [4] a pseudoconsistent logic (PCL) in which $A \land \sim A \rightarrow \bot$ is not a theorem but so $\sim (A \land \sim A)$ is. He also gave the axiomatization of PCL and proved the completeness theorem by two kinds of models, PC-models and I-models. These models are based on PC-algebras and partially ordered sets, respectively.

But there is an important question which is not referred: Is the logic PCL decidable?

In the present paper we prove the decidability of PCL according to the following steps:

1. PCL is characterized by the the class of pre-ordered sets instead of that of partially ordered sets, that is $\vdash_{PCL} A \iff A : PO-valid$;
2. TL is characterized by the class of some kinds of Kripke-type models, that is, $\vdash_{TL} A \iff A : TL-valid$;
3. PCL can be embedded into a certain tense logic (TL), that is, for some map $\xi$, $A : PO-valid \iff \xi(A) : TL-valid$;
4. TL is decidable and hence so PCL is.

2. Pseudoconsistent logic and its semantics

First of all we define (propositional) pseudoconsistent logic according to De Glas [4]. In the following we simply write PCL. The logic has the language as follows:

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\[Key\text{ words and phrases. pseudoconsistent logic, tense logic.}\]
• $p_0, p_1, p_2, \cdots$ : denumerable propositional variables
• $\bot$ : constant
• $\sim, \land, \lor, \rightarrow$ : logical symbols

We put $\Pi_0 = \{p_0, p_1, p_2, \cdots\} \cup \{\bot\}$ and denote by $\Pi$ the set of all formulas and by $A, B, C, \cdots$ formulas.

$PCL$ is the following axiomatic system:

Axioms
1. $\bot \rightarrow A$
2. $A \rightarrow (B \rightarrow A)$
3. $(A \rightarrow (B \rightarrow C)) \rightarrow (A \rightarrow C)$
4. $A \land B \rightarrow A \land A \rightarrow A$
5. $A \rightarrow (B \rightarrow (A \land B))$
6. $A \rightarrow A \lor B \rightarrow A \lor A$
7. $(A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow (A \lor B \rightarrow C))$
8. $(A \rightarrow B) \rightarrow (\sim A \lor B)$
9. $A \lor \sim A$
10. $\sim \sim A \rightarrow A$
11. $\sim (A \land \sim A)$

Rules of inference
1. $B$ is inferred from $A$ and $A \rightarrow B$
2. $\sim A \rightarrow B$ is inferred from $A \rightarrow B$

If $A$ is a provable in $PCL$, then we write $\vdash_{PCL} A$. It is proved in [4] that

Proposition 1. 1. $\vdash_{PCL} A \land (A \rightarrow B) \rightarrow B$
2. $\vdash_{PCL} (\sim A \rightarrow B) \rightarrow (A \lor B)$
3. if $\vdash_{PCL} A \rightarrow B$ and $\vdash_{PCL} A \rightarrow (B \rightarrow C)$ then $\vdash_{PCL} A \rightarrow C$
4. if $\vdash_{PCL} A \rightarrow B$ and $\vdash_{PCL} B \rightarrow C$ then $\vdash_{PCL} A \rightarrow C$
5. if $\vdash_{PCL} \sim A \rightarrow B$ then $\vdash_{PCL} \sim B \rightarrow A$
6. if $\vdash_{PCL} A \rightarrow B$ then $\vdash_{PCL} \sim B \rightarrow \sim A$
7. if $\vdash_{PCL} A \rightarrow B$ then $\vdash_{PCL} \sim A \lor B$
8. if $\vdash_{PCL} A \rightarrow B$ and $\vdash_{PCL} A \rightarrow \sim B$ then $\vdash_{PCL} \sim A$
9. if $\vdash_{PCL} A \rightarrow B$ and $\vdash_{PCL} A \rightarrow B$ then $\vdash_{PCL} B$ for any $B$
10. if $\vdash_{PCL} A \rightarrow B$ and $\vdash_{PCL} B$ for any $B$
11. if $\vdash_{PCL} A$ then $\vdash_{PCL} \sim A$.

In order to develop the algebraic semantics for the logic $PCL$ we define a $PC$-algebra.

We call a structure $(L, \land, \lor, \rightarrow, \sim, 0, 1)$ a $PC$-algebra when it satisfies the conditions:
1. $(L, \leq, \land, \lor)$ is a complete distributive lattice
2. $x \rightarrow y = \lor\{z : x \land z \leq y\}$
3. $\sim x = \land\{y : x \lor y = 1\}$

It is clear that any Boolean algebra is a $PC$-algebra. We note that if we define $\sim x$ as the element $\lor\{z : x \land z = 0\}$ then the algebra is called a $Heyting$ algebra.
So the difference of PC-algebra and Heyting one is only the definition of \( \neg x \). We have the following results of PC-algebra in [4].

**Proposition 2.**
1. \( x \leq y \iff x \rightarrow y \)
2. \( x \land \neg x \geq 0 \)
3. \( \neg(x \land y) = \neg x \lor \neg y \)
4. \( \neg(x \lor y) = \neg x \land \neg y \)
5. \( \neg \neg x \leq x \)
6. \( \neg x \leq y \iff x \lor y = 1 \)
7. \( x = 1 \iff \neg \neg x = 1 \)
8. if \( \neg x = 0 \) then \( x = 1 \)
9. if \( \neg x = 1 \) then \( x \geq 0 \)
10. \( \neg y \rightarrow x = 1 \implies \neg x \rightarrow y = 1 \)
11. \( \neg y \rightarrow \neg x = 1 \implies x \rightarrow y = 1 \)
12. \( x \rightarrow 1 = 1 \)
13. \( 0 \rightarrow x = 1 \)
14. \( x \rightarrow (y \rightarrow x) = 1 \)
15. \( x \rightarrow (y \rightarrow x \land y) = 1 \)
16. \( x \rightarrow x \lor y = 1 \)

We have a well-known example of the PC-algebra. Let \( X \) be a topological space and \( \Omega \) be the set of closed subspaces. For any element \( A, B \in \Omega \) if we define
1. \( A \land B = A \cap B \)
2. \( A \lor B = A \cup B \)
3. \( A \rightarrow B = \bigcup \{ C \in \Omega : A \cap C \subseteq B \} \)
4. \( \neg A = \text{closure of } (X - A) \)

then \( (\Omega, \land, \lor, \rightarrow, \neg, \phi, X) \) is the PC-algebra. Moreover it is proved in [4] that any PC-algebra \( A \) can be embedded into a topological PC-algebra \( \Omega \).

We define a global semantics. A mapping \( v : \Pi_0 \rightarrow A \) is called a *valuation* on a PC-algebra \( A \). The valuation \( v \) can be extended to the set \( \Pi \) of formulas recursively:
- \( v(\bot) = 0 \)
- \( v(\neg A) = \neg v(A) \)
- \( v(A \land B) = v(A) \land v(B) \)
- \( v(A \lor B) = v(A) \lor v(B) \)
- \( v(A \rightarrow B) = v(A) \rightarrow v(B) \)

A formula \( A \) is called an \( A \)-valid if \( v(A) = 1 \) for every valuation \( v : \Pi \rightarrow A \). We say that \( A \) is **PC-valid** when it is \( A \)-valid for any PC-algebra \( A \). It is proved in [4] the completeness theorem of PCL.

**Theorem 1.** \( \vdash_{PCL} A \) if and only if \( A \) is PC-valid.

Another semantics called local semantics is given in [4]. This is a Kripke-like style and is based on the partially ordered sets (poset). Let \( (I, \leq) \) be a poset. A subset \( S \subseteq I \) is said to be *anti-hereditary* if whenever \( i \in S \) and \( j \leq i \) then \( j \in S \).
We denote by $I^*$ the set of all anti-hereditary subsets of $I$. A PCL-model on $I$ is a structure $\mathcal{I} = (I, \leq, \varphi)$, where $\varphi : \Pi_0 \to I^*$ is a mapping which is called an $I$-valuation in [4]. Of course the map $\varphi$ can be extended to one with the domain $\Pi$ the set of all formulas. By $\mathcal{I} \models_i A$ or simply $\models_i A$, we mean that a formula $A$ is true in the model $\mathcal{I}$ at a state $i \in I$, which is defined as follows:

- $\models_i p$ iff $i \in \varphi(p)$, where $p \in \Pi_0$;
- $\not\models_i \bot$, that is, it is not the case $\models_i \bot$;
- $\models_i \neg A$ iff there exists $j$ such that $i \leq j$ and not $\models_j A$;
- $\models_i A \land B$ iff $\models_i A$ and $\models_i B$;
- $\models_i A \lor B$ iff $\models_i A$ or $\models_i B$;
- $\models_i A \to B$ iff for any $j \leq i$ if $\models_j A$ then $\models_j B$.

A formula $A$ is called true in $\mathcal{I}$, denoted $\mathcal{I} \models A$, if $\mathcal{I} \models_i A$ for every $i \in I$. Also $A$ is said to be $\mathcal{I}$-valid, denoted $\mathcal{I} \models A$, when $A$ is true in the model $\mathcal{I} = (I, \leq)$. For the local semantics it is proved in [4]

**Proposition 3.** $\vdash_{PCL} A$ iff $A$ is $\mathcal{I}$-valid for any poset $\mathcal{I}$.

We note that in the proof of the above we do not use the condition of asymmetry of the relation. This implies that it can be proved for the class of pre-ordered sets. That is,

**Theorem 2.** $\vdash_{PCL} A$ iff $A$ is $\mathcal{I}$-valid for any pre-ordered set $\mathcal{I}$.

3. Tense logic $\mathcal{L}$ and its semantics

In this section we define a certain tense logic ($\mathcal{L}$) and show the completeness theorem by the class of Kripke-type models. The logic has the following language:

- $p_0, p_1, p_2, \cdots$ : denumerable propositional variables
- $H, G$ : unary tense operators
- $\neg, \land, \lor, \to$ : logical symbols

$\mathcal{L}$-formulas are defined as usual, especially, we denote $\neg G \neg A$ and $\neg H \neg A$ by $FA$ and $PA$, respectively. We put $\Phi_0$ the set of all propositional variables, that is, $\Phi_0 = \{p_0, p_1, p_2, \cdots\}$. $\Phi$ the set of all $\mathcal{L}$-formulas, and $A, B, C, \cdots$ the $\mathcal{L}$-formulas. $\mathcal{L}$ is the axiomatic system which is defined as the smallest tense logic $K_t$ with the axioms expressing reflexivity and transitivity, that is,

**Axioms**

1. every tautology
2. $G(A \to B) \to (GA \to GB)$
3. $H(A \to B) \to (HA \to HB)$
4. $A \to HFA$, $A \to GPA$
5. $GA \to A$, $HA \to A$
6. $GA \to GGA$, $HA \to HHA$

As is well-known in the theory of modal (or tense) logic, axiom 4 expresses the reflexivity and axiom 5 transitivity.

**Rules of inference**
1. $B$ inferred from $A$ and $A \rightarrow B$
2. $GA$ is inferred from $A$
3. $HA$ is inferred from $A$

Now we define a semantics called a Kripke-model. Let $< W, R >$ be a structure where (i) $W$ is a non-empty set (ii) $R$ is a reflexive and transitive relation on $W$, that is, $W$ is the pre-ordered set with the relation $R$. We write $xRy$ instead of $(x, y) \in R$. Hence the condition (ii) is expressed by

- $xRx$ for every $x \in W$
- $xRy, yRz \Rightarrow xRz$ for any $x, y, z \in W$

A function $V : \Phi_0 \rightarrow 2^W$ is also called a valuation on the structure $< W, R >$. The valuation $V$ is extended to $V^*$ which domain is the set $\Phi$ of all the TL-formulas as follows:

1. $V^*(p) = V(p)$ if $p \in \Phi_0$
2. $V^*(\neg A) = \{x | x \notin V^*(A)\}$
3. $V^*(A \land B) = V^*(A) \cap V^*(B)$
4. $V^*(A \lor B) = V^*(A) \cup V^*(B)$
5. $V^*(A \rightarrow B) = \{x | x \in V^*(A) \Rightarrow V^*(B)\}$
6. $V^*(GA) = \{x | \forall y(xRy \Rightarrow y \in V^*(A))\}$
7. $V^*(HA) = \{x | \forall y(yRx \Rightarrow y \in V^*(A))\}$

For the sake of simplicity we use the same symbol $V$ as the extended valuation. A formula $A \in \Phi$ is said to be true at $x$ in a model $M = < W, R, V >$ when $x \in V(A)$ and denoted by $M | x = A$. We say that a formula $A$ is TL-valid if $A$ is true at each element $x \in W$ in every model $M = < W, R, V >$, that is, $M | x = A$ for every model $M = < W, R, V >$ and $x \in W$. For the semantics we can show the soundness theorem of the logic TL.

**Theorem 3.** $\vdash_{TL} A \Rightarrow A$ is TL-valid.

**Proof.** By induction on the length of the proof. $\Box$

In order to prove the converse (Completeness Theorem), we define a special model called the canonical model. For any set of formulas $\Gamma \subseteq \Phi$, we say that $\Gamma$ is inconsistent if there exist some formulas $A_i \in \Gamma$ such that $\vdash_{TL} \neg(A_1 \land \cdots \land A_n)$ and that consistent otherwise. We can show that every consistent set of formulas has a maximal one, that is,

**Proposition 4.** If $\Gamma$ is a consistent set, then there exists a maximal consistent set $\Gamma^*$ containing $\Gamma$.

**Proof.** We define a sequence $\{ \Gamma_n \}$ of subsets of formulas as follows:

$$\Gamma_0 = \Gamma$$

$$\Gamma_{n+1} = \begin{cases} 
\Gamma_n \cup \{A_n\} & \text{if } \Gamma_n \cup \{A_n\} \text{ is consistent} \\
\Gamma_n \cup \{\neg A_n\} & \text{otherwise}
\end{cases}$$

It easy to show that each $\Gamma_n$ is consistent and so $\Gamma^* = \bigcup_n \Gamma_n$ is. It is also clear that $\Gamma^*$ is the maximal consistent set containing $\Gamma$. $\Box$
Concerning to any maximal consistent set we have the results.

**Proposition 5.** For every maximal consistent set $\Delta$, we have

1. $\vdash_{TL} A \implies A \in \Delta$
2. $A \notin \Delta \iff \neg A \in \Delta$
3. $A \land B \in \Delta \iff A \in \Delta \land B \in \Delta$
4. $A \lor B \in \Delta \iff A \in \Delta \lor B \in \Delta$
5. $A \rightarrow B \in \Delta \iff \text{if } A \in \Delta \text{ then } B \in \Delta$.

**Proof.** We only show the case of (1). Suppose that $\vdash_{TL} A$ and $A \notin \Delta$. Since $\Delta \cup \{A\}$ is inconsistent by maximality of $\Delta$, there exist some formulas $B_i \in \Delta$ such that $\vdash_{TL} \neg(B_1 \land \cdots \land B_n \land A)$. This yields to $\vdash_{TL} A \rightarrow \neg(B_1 \land \cdots \land B_n)$. It follows from $\vdash_{TL} A$ that $\vdash_{TL} \neg(B_1 \land \cdots \land B_n)$ and hence that $\Delta$ is inconsistent. But this is a contradiction. Hence if $\vdash_{TL} A$ then $A \in \Delta$.

For any maximal consistent sets $x, y$, we have

**Lemma 1.** For every formula $A \in \Psi$, the two conditions

1. If $GA \in x$ then $A \in y$.
2. If $HA \in y$ then $A \in x$.

are equivalent to each other.

**Proof.** Suppose that (1). If $HA \in y$ but $A \notin x$, since $x$ is the maximal consistent set, then $\neg A \in x$. From $\vdash_{TL} \neg A \Rightarrow GPA$, we have $GP\neg A \in x$ and hence $P\neg A \in y$ by (1). This means that $\neg HA \in y$. This contradicts to the fact $y$ being consistent.

The converse can be proved similarly.

Now we are ready to define a canonical model. Let $W_{TL}$ be the set of all maximal consistent sets of $TL$. For every $x, y \in W_{TL}$, we define a relation $R_{TL}$ on $W_{TL}$ as

$$xR_{TL}y \iff \forall A \in \Phi (GA \in x \implies A \in y).$$

Lastly, a valuation $V_{TL}$ is defined by $V_{TL}(p) = \{x \in W_{TL} | p \in x\}$. In this case we call $< W_{TL}, R_{TL}, V_{TL} >$ the canonical model of $TL$. The model plays an important role in the theory of modal or tense logics. We note from the above $xR_{TL}y$ if and only if $\forall A \in \Phi (HA \in y \implies A \in x)$.

**Lemma 2.** $< W_{TL}, R_{TL}, V_{TL} >$ is indeed our model, that is, $R_{TL}$ is reflective and transitive.

**Proof.** First of all we shall prove that $R_{TL}$ is reflexive. Suppose that $GA \in x$ for any $x \in W_{TL}$. Since $\vdash_{TL} GA \rightarrow A$, we have $GA \rightarrow A \in x$ and thus $A \in x$. This implies that $xR_{TL}x$. Secondly, suppose that $xR_{TL}y, yR_{TL}z$, and $GA \in x$. Since $\vdash_{TL} GA \rightarrow GGA$, we have $GGA \in x$. It follows from $xR_{TL}y$ that $GA \in y$. We also obtain $A \in z$ from $yR_{TL}z$. This means that $R_{TL}$ is transitive. Thus $< W_{TL}, R_{TL}, V_{TL} >$ is certainly our considered model.

As to the canonical model we have the fundamental theorem to be able to show the completeness.
Theorem 4. For every formula \( A \in \Phi \) and maximal consistent set \( x \in W_{TL} \),
\[
x \in V_{TL}(A) \iff M_{TL} \models x A.
\]

Proof. We shall prove the theorem by induction on the construction of a formula.

If \( A \) is a propositional variable \( p \in \Phi_0 \), then it is evident from definition that
\( x \in V_{TL}(p) \) iff \( p \in x \).

If \( A \) is a formula of the form \( \neg B \), then we have \( x \in V_{TL}(\neg B) \) iff \( x \notin V_{TL}(B) \)
if \( M_{TL} \not\models x B \) by induction hypothesis (IH) iff \( M_{TL} \models \neg B \).

It is similar to the cases of \( B \land C \), \( B \lor C \), and of \( B \rightarrow C \).

Let \( A \) be a formula of the form \( GB \). Suppose that \( x \in V_{TL}(GB) \) and \( xR_{TLy} \)
for every \( y \in \Phi \). By definition of \( V_{TL} \) we have \( y \in V_{TL} \). It follows from IH that
\( M_{TL} \models y B \) for every \( y \) such that \( xR_{TLy} \). Thus \( M_{TL} \models x GB \) by definition.

Conversely, we assume that \( M_{TL} \models x GB \). If \( x \notin V_{TL}(GB) \), then there exists
\( y \in \Phi \) such that \( xR_{TLy} \) and \( y \notin V_{TL}(B) \) by definition of \( V_{TL} \). It follows from
IH that \( M_{TL} \not\models y B \) for some \( y \) such that \( xR_{TLy} \). This implies \( M_{TL} \not\models x GB \)
and hence a contradiction. Thus we have \( x \in V_{TL}(GB) \) iff \( M_{TL} \models x GB \).

We can also prove the theorem in the case of \( HB \) similarly. \( \square \)

From the above we can show the completeness theorem of \( TL \).

Theorem 5. If \( A \) is \( TL \)-valid then \( \models_{TL} A \). Hence
\[
\models_{TL} A \iff A : TL{-valid}.
\]

Proof. Suppose that \( \not\models_{TL} A \). Since \( \{\neg A\} \) is a consistent set, there exists a maxi-
mal consistent set \( x \in W_{TL} \) such that \( \{\neg A\} \subseteq x \). It follows from the above that
\( M_{TL} \not\models x A \). This means that \( A \) is not \( TL \)-valid. \( \square \)

4. Decidability of \( TL \)

In this section we shall prove the decidability of the logic \( TL \) using the filtration
method. The method is familiar to the theory of modal logics. Let \( \Psi \subseteq \Phi \) be a
subset of \( TL \)-formulas which is closed under subformulas, that is, if \( A \in \Psi \) and
\( B \) is a subformula of \( A \) then \( B \in \Psi \). For every \( TL \)-model \( M =< W, R, V > \), we
define a relation \( \equiv \) on \( W \) as follows: For any \( x, y \in W \),
\[
x \equiv y \iff \forall A \in \Psi (M \models x A \iff M \models y A).
\]

It is easy to show that the relation is the equivalence one, so we omit the proof.

Proposition 6. \( \equiv \) is the equivalence relation.

We define a filtration model \( M^* =< W^*, R^*, V^* > \) of \( M =< W, R, V > \)
through \( \Psi \). We put \( W^* = \{ \overline{x} | x \in W \} \) and \( \overline{x} = \{ y \in W | x \equiv y \} \). For each
\( [x], [y] \in W^* \), a binary relation \( R^* \) is define as
\[
[x]R^*[y] \iff \forall GA \in \Psi (M \models x GA \implies M \models y GA \land A) \land \forall HB \in \Psi (M \models y HB \implies M \models x HB \land B).
\]

The valuation \( V^* \) is also defined by
\[
V^*(p) = \begin{cases} 
\{ \overline{x} | M \models x p \} & \text{if } p \in \Phi_0 \\
W^* & \text{otherwise}
\end{cases}
\]
The following is proved from the definition of $R^*$. 

**Lemma 3.** If $xRy$ in $M$, then $[x]R^*[y]$ in $M^*$.

*Proof.* Suppose $xRy$. For every formula of the form $GA \in \Psi$, if $M \models_x GA$, then $M \models_y A$ by $xRy$. On the other hand, since $M \models_x GA \to GGA$, we have $M \models_x GGA$ and hence $M \models_y GA$. This implies that $M \models_y GA \land A$. In case of $HB \in \Psi$, we assume $M \models_y HB$. It follows from $xRy$ that $M \models_x B$. Since $M \models_y HB \to HHB$, we have $M \models_x HHB$, $M \models_x HB$ and thus $M \models_x HB \land B$. Hence these mean that $[x]R^*[y]$. \hfill \square

According to the definition, we can show that the filtration model is indeed a $TL$-model.

**Lemma 4.** $M^* = < W^*, R^*, V^* >$ is the $TL$-model.

*Proof.* We only show that $R^*$ is a reflexive and transitive relation on $W^*$. Firstly suppose that $[x] \in W^*$. For every formula $GA \in \Psi$, if $M \models_x GA$, since $M \models_x GA \to A$, we have $M \models_x A$ and hence $M \models_x GA \land A$. It is similar for any formula of the form $HB \in \Psi$. This implies that $[x]R^*[x]$ and that $R^*$ is reflexive.

Secondly, suppose that $[x]R^*[y]$ and $[y]R^*[z]$. For any formula $GA \in \Psi$, if $M \models_x GA$, since $M \models_y GA \land A$ by supposition, then we have $M \models_y GA$. It follows that $M \models_z GA \land A$. This means that $R^*$ is transitive. \hfill \square

For the filtration model $M^*$ of $M$ through $\Psi$, we establish the fundamental theorem.

**Theorem 6.** For every $A \in \Psi$ and $x \in W$,

$M^* \models_{[x]} A \iff M \models_x A$

*Proof.* By induction on $A$.

- If $p \in \Psi_0$ then $M^* \models_{[x]} p$ iff $p \in V^*([x])$ iff $M \models_x p$.
- For the formula of the form $B \land C \in \Psi$, since $B, C \in \Psi$, we have $M^* \models_{[x]} B \land C$ iff $M^* \models_{[x]} B$ and $M^* \models_{[x]} C$ iff $M \models_x B$ and $M \models_x C$ iff $M \models_x B \land C$.
- For the case of $GB \in \Psi$, suppose that $M \models_x GB$. There exists an element $y \in W$ such that $xRy$ and $M \not\models_y B$. Since $[x]R^*[y]$ and $M^* \not\models_{[y]} B$ by induction hypothesis (IH), we have $M^* \not\models_{[x]} GB$. Conversely, if $M^* \not\models_{[x]} GB$ then there exists $y \in W$ such that $[x]R^*[y]$ and $M^* \not\models_{[y]} B$. It follows from IH that $M \not\models_y B$ and hence $M \not\models_y GB \land B$. Since $[x]R^*[y]$, this means that $M \not\models_x GB$.
- The other cases are proved similarly. \hfill \square

Now we show the decidability of $TL$. Let $\not\models_{TL} A$ and $\Psi$ be the set of subformulas of $A$. It is clear that $\Psi$ is finite and closed under subformulas. By completeness theorem of $TL$, there exists a $TL$-model $M = < W, R, V >$ and $x \in W$ such that $M \not\models_x A$. By the above we can construct the filtration model $M^*$ of $M$ through $\Psi$. Since $\Psi$ is the finite set, $M^*$ is also a finite $TL$-model. For that model we have $M^* \not\models_{[x]} A$. This means that if $A$ is not a $TL$-theorem then it is not $TL$-valid in some finite model. Since the logic $TL$ is finitely axiomatized, we can prove

**Theorem 7.** $TL$ is decidable.
5. Embedding of PCL into TL

Let $\xi$ be a map from the set $\Pi$ of all PCL-formulas to the set $\Phi$ of all TL-formulas as follows:

1. $\xi(p) = Hp$, where $p \in \Pi_0 - \{\bot\} = \Phi_0$
2. $\xi(\bot) = p \land \neg p$ for some fixed $p \in \Phi_0$
3. $\xi(\sim A) = F \neg \xi(A)$
4. $\xi(A \land B) = \xi(A) \land \xi(B)$
5. $\xi(A \rightarrow B) = H(\xi(A) \rightarrow \xi(B))$

We can show that PCL can be embedded into TL in the sense that $A$ is PCL-valid iff $\xi(A)$ is TL-valid for every formula $A \in \Pi$.

**Theorem 8.** For every formula $A \in \Pi$,

$A$ is PCL-valid $\iff \xi(A)$ is TL-valid.

**Proof.** If $\xi(A)$ is not TL-valid, then there exists a TL-model $M =< W, R, V >$ such that $M \not\models_x \xi(A)$ for some element $x \in W$. We note that the relation $R$ is reflexive and transitive, that is, $< W, R >$ is a pre-ordered set. We construct a PCL-model $I =< I, \leq, \varphi >$ as follows:

- $I = W$
- $\leq = R$, that is, $x \leq y$ if and only if $xRy$
- $\varphi(p) = \{x \in W | \forall y(yRx \implies y \in V(p))\}$

We note that $\varphi(p)$ is anti-hereditary. For if $x \in \varphi(p)$ and $y \leq x$, then we have $yRx$ for every $z$ such that $yRz$. By transitivity, $zRx$. This yields to $z \in V(p)$. Hence $\varphi(p)$ is anti-hereditary and $I$ is indeed the PCL-model. For that model we can show that $I \models_x \alpha$ iff $M \models_x \alpha$ for any formula $\alpha$ in $\Pi$. Since $M \not\models_x \xi(A)$, we obtain that $I \not\models_x A$ for the PCL-model $I$. Thus $A$ is not PCL-valid unless $\xi(A)$ is TL-valid.

Conversely, assume that $A$ is not PCL-valid. There exists a PCL-model $I =< I, \leq, \varphi >$ such that $I \not\models_x A$ for some $x \in I$. From that model we can construct a TL-model $M =< W, R, V >$, where

1. $W = I$
2. $R = \leq$, that is, $xRy$ is defined by $x \leq y$
3. $V(p) = \varphi(p)$.

It is obvious that $M$ is the TL-model. It is also easy to show that $I \models_u \alpha$ iff $M \models_u \xi(\alpha)$ for every formula $\alpha \in \Pi$ and $u \in W$. Since $I \not\models_x A$, we have $M \not\models_x \xi(A)$. This means that $\xi(A)$ is not TL-valid.

Therefore we can prove the theorem completely.

From the above we can obtain the main result of our paper.

**Theorem 9.** PCL is decidable.
Proof.

\[ \forall_{PCL} A \iff A : \text{not } PCL \text{ - valid} \]
\[ \iff \xi(A) : \text{not } TL \text{ - valid} \]
\[ \iff \exists M : \text{finite } TL \text{ - model}, \exists x; M \not\models_x \xi(A) \]
\[ \iff \exists I : \text{finite } PCL \text{ - model}, \exists x; I \not\models_x A \]

Since \( PCL \) is the finitely axiomatizable logic, it follows from the above that it is decidable.

References


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