Mem. Fac. Sci. Eng. Shimane Univ.
Series B: Mathematical Science
33 (2000), pp.11–19

SOME QUESTIONS ON TRANSFINITE DIMENSIONS

Yasunao Hattori

Dedicated to Professor Maretsugu Yamasaki for his 60th birthday

Abstract. In the present note, we summarize open questions arising from recent research in transfinite dimensions.

1. Introduction

In the present note, we shall consider some open questions on transfinite dimensions. Some of them are already asked by the author in the literatures.

We denote trInd (trind) by large (small) transfinite inductive dimension which is a natural transfinite extension of large (small) inductive dimension Ind (ind). A normal space X is called *strongly countable-dimensional* if X is the union of countably many closed subsets X_n, $n = 1, 2, \ldots$, with $\dim X_n < \infty$. For each ordinal number α, we write $\alpha = \lambda(\alpha) + n(\alpha)$, where $\lambda(\alpha)$ is a limit ordinal and $n(\alpha)$ is a finite ordinal. For a normal space X and a non-negative integer n, we put

$$P_n(X) = \bigcup\{U : U \text{ is an open set of } X \text{ such that } \dim \overline{U} \leq n\}.$$

Let X be a normal space and α be either an ordinal number or the integer -1. The *strong small transfinite dimension* sind of X is defined as follows ([B]):

(i) $\operatorname{sind} X = -1$ if and only if $X = \emptyset$.

(ii) $\operatorname{sind} X \leq \alpha$ if X is expressed in the form $X = \bigcup\{P_\xi : \xi < \alpha\}$, where $P_\xi = P_{n(\xi)}(X \setminus \bigcup\{P_\eta : \eta < \lambda(\xi)\})$.

If $\operatorname{sind} X \leq \alpha$ for some α, we say that X *has strong small transfinite dimension.*

1991 *Mathematics Subject Classification.* Primary 54F45; secondary 54C10, 54E35.

Key words and phrases. Large transfinite dimension, small transfinite dimension, strong small transfinite dimension, order dimension, strongly countable-dimensional space, finitistic space, K-approximation, metrizable space.

Recall from [H4] that a normal space X *has strong large transfinite dimension* if X has both large transfinite dimension and strong small transfinite dimension. The following characterization of spaces that have strong large transfinite dimension is useful.

Proposition A [H3, Propositions 2.2 and 2.3]**.** *Let X be a metrizable space. Then X has strong large transfinite dimension if and only if X is finitistic and strongly countable-dimensional.*

The concept of finitistic spaces was introduced by Swan [Sw] for working in fixed point theory and is applied to the theory of transformation groups by using the cohomological structures (cf. [AP]). For a family $\mathcal{U}$ of a space X the order $\operatorname{ord}\mathcal{U}$ of $\mathcal{U}$ is defined as follows: $\operatorname{ord}_x\mathcal{U} = |\{U \in \mathcal{U} : x \in U\}|$ for each $x \in X$ and $\operatorname{ord}\mathcal{U} = \sup\{\operatorname{ord}_x\mathcal{U} : x \in X\}$. We say a family $\mathcal{U}$ has *finite order* if $\operatorname{ord}\mathcal{U} = n$ for some natural number n. A space X is said to be *finitistic* if every open cover of X has an open refinement with finite order. We notice that finitistic spaces are also called *boundedly metacompact spaces* (cf. [FMS]). It is clear that all compact spaces and all finite dimensional paracompact spaces are finitistic spaces. More precisely, we have a useful characterization of finitistic spaces.

Proposition B [H2]**,** [DMS]**.** *A paracompact space X is finitistic if and only if there is a compact subspace C of X such that $\dim F < \infty$ for every closed subspace F with $F \cap C = \emptyset$.*

The dimension-theoretic properties of finitistic spaces are investigated by several authors (cf. [DP], [DS], [DT], [DMS] [H2] and [H6]).

We denote the set of natural numbers by $\mathbb{N}$. We refer the reader to [E] and [N] for basic results in dimension theory.

2. Questions arising from characterizations by means of K-approximations

In [DMS], Dydak-Mishra-Shukla introduced a concept of a K-approximation of a mapping to a metric simplicial complex and characterized n-dimensional spaces and finitistic spaces in terms of K-approximations. Let X be a space, K a metric simplicial complex and $f : X \to K$ a continuous mapping. A mapping $g : X \to K$ is said to be a *K-approximation of f* if for each simplex $\sigma \in K$ and each $x \in X$, $f(x) \in \sigma$ implies $g(x) \in \sigma$. A K-approximation $g : X \to K$ of f is called an *n-dimensional K-approximation* if $g(X) \subset K^{(n)}$ and a *finite dimensional K-approximation* if $g(X) \subset K^{(m)}$ for some natural number m, where $K^{(m)}$ denotes the m-skeleton of K. We have the following characterizations of infinite-dimensional spaces by means of K-approximations in [H5]. For a space

X we denote

$$\mathcal{D}(X) = \{D : D \text{ is a closed discrete subset of } X\}.$$

Theorem 2.1 [H5, Theorem]**.** *For a metrizable space X the following are equivalent.*

(a) *X has strong large transfinite dimension.*
(b) *There is a function $\varphi : \mathcal{D}(X) \to \omega$ such that for every metric simplicial complex K and every continuous mapping $f : X \to K$ there is a K-approximation g of f such that $g(D) \subset K^{(\varphi(D))}$ for each $D \in \mathcal{D}(X)$.*
(c) *For every integer $m \geq -1$ there is a function $\psi : \mathcal{D}(X) \to \omega$ such that for every metric simplicial complex K and every continuous mapping $f : X \to K$ there is a finite dimensional K-approximation g of f such that $g(D) \subset K^{(\psi(D))}$ for each $D \in \mathcal{D}(X)$ and $g|f^{-1}(K^{(m)}) = f|f^{-1}(K^{(m)})$.*

Theorem 2.2 [H5, Corollary]**.** *For a paracompact space X the following are equivalent.*

(a) *X is a strongly countable-dimensional space.*
(b) *There is a function $\varphi : X \to \omega$ such that for every metric simplicial complex K and every continuous mapping $f : X \to K$ there is a K-approximation g of f such that $g(x) \in K^{(\varphi(x))}$ for each $x \in X$.*
(c) *For every integer $m \geq -1$ there is a function $\psi : X \to \omega$ such that for every metric simplicial complex K and every continuous mapping $f : X \to K$ there is a K-approximation g of f such that $g(x) \in K^{(\psi(x))}$ for each $x \in X$ and $g|f^{-1}(K^{(m)}) = f|f^{-1}(K^{(m)})$.*

Concerning the Theorems 2.1 and 2.2, we can ask the following.

Question 2.1. Are the conditions (a) and (b) in Theorem 2.1 equivalent for paracompact spaces?

We have a simple answer the question, i.e., the implication (b) $\Rightarrow$ (a) does not hold. In fact, for each $m, n \in \mathbb{N}$ with $m \leq n$, Vopěnka [Vo] constructed a compact space $X_{m,n}$ such that $\dim X_{m,n} = m$ and $\operatorname{Ind} X_{m,n} = n$. Let X be the topological sum $\bigoplus_{n=1}^{\infty} X_{1,n}$ of $X_{1,n}$, $n \in \mathbb{N}$. Then X does not have large transfinite dimension (and hence X does not satisfy (a)). Since $\dim X = 1$, it follows from [DMS] that for every metric simplicial complex K and every continuous mapping $f : X \to K$ there is a 1-dimensional K-approximation g of f. Hence X satisfies the condition (b).

Now, we consider the following condition which is weaker than (a).

(a') X is a strongly countable-dimensional space satisfying the following condition (K) (cf. [P]):

(K) There is a compact subspace C of X such that $\dim F < \infty$ for every closed subspace F of X with $F \cap C = \emptyset$.

We consider the relations between (a), (b) in Theorem 2.1 and (a') for normal (paracompact) spaces.

In [E, §7.3], Engelking reformulated the class of spaces that have strong small transfinite dimension by use of a new dimension function transfinite dimensional kernel trker. He called a space that has transfinite dimensional kernel as a *shallow space*. One should notice that a normal space X is a shallow space if and only if X has strong small transfinite dimension and $\operatorname{sind} X = \operatorname{trker} X$ if $\operatorname{sind} X$ is a limit ordinal and $\operatorname{sind} X = \operatorname{trker} X + 1$ otherwise.

We shall consider four implications separately.

I. (a) $\Rightarrow$ (a').

Fact 2.1 ([E, Theorem 7.1.23])**.** *If a weakly paracompact, strongly hereditarily normal space X has large transfinite dimension* $\operatorname{Ind} X$*, then X satisfies the condition (K).*

Fact 2.2 ([E, Theorem 7.3.13]; [H1, Theorem 1.2] for metrizable spaces). *If a weakly paracompact perfectly normal shallow space X, then X is a strongly countable-dimensional space***.**

We can ask the following.

Question 2.2. Can we drop the perfectness in Fact 2.2? I.e., is a weakly paracompact normal shallow space a strongly countable-dimensional?

We have a partial answer the question.

Theorem 2.3. *Let X be a hereditarily weakly paracompact and hereditarily normal space. If X is a shallow space, then X is a strongly countable-dimensional space.*

Proof. We show by the transfinite induction on $\operatorname{sind} X = \alpha$.
Case 1. Suppose that α is a limit ordinal number. We notice that X is expressed in the form $X = \bigcup\{P_\xi : \xi < \alpha\}$, where $P_\xi = P_{n(\xi)}(X \setminus \bigcup\{P_\eta : \eta < \lambda(\xi)\})$. We put $G_\xi = \bigcup\{P_\eta : \eta < \xi\}$ for $\xi < \alpha$. Then $\{G_\xi : \xi < \alpha\}$ is an open covering of X and $\operatorname{sind} G_\xi \le \xi < \alpha$. By the inductive assumption, G_ξ is strongly countable-dimensional for each $\xi < \alpha$. Hence it follows from [E, Theorem 5.2.17] that X is strongly countable-dimensional.

Case 2. Suppose that $\alpha = \beta + 1$. Let $Y = X \setminus \bigcup\{P_\xi : \xi < \lambda(\alpha)\})$. Then $\operatorname{sind} Y \leq \beta < \alpha$. By the inductive assumption, Y is strongly countable-dimensional. Hence there is a countable cover $\{F_1, F_2, \dots\}$ of Y by finite dimensional closed sets. Since P_α is a closed set of X such that $X = Y \cup P_\alpha$ and $\dim P_\alpha = n(\alpha) < \infty$. We put $E_i = F_i \cup P_\alpha$ for each $i \in \mathbb{N}$. Then it follows that E_i closed in X and $\dim E_i \leq \max\{\dim F_i, \dim P_\alpha\}$. Hence X is strongly countable-dimensional. □

Corollary 2.1. *Let X be a hereditarily weakly paracompact, strongly hereditarily normal space. Then the implication (a) $\Rightarrow$ (a') holds.*

Question 2.2'. Do Theorem 2.3 and the Corollary 2.1 hold for weakly paracompact normal spaces?

II. (a') $\Rightarrow$ (a).

It is known that a normal space X is a shallow space if and only if every non-empty closed subspace F of X contains a non-empty normal open subspace U of F such that $\dim U < \infty$ ([E, Problem 7.3.A]). Hence, by the Baire category theorem, every normal Čech-complete, strongly countable-dimensional space is a shallow space. This implies the following.

Proposition 2.1. *Let X be a normal space satisfying the condition (K). If X is a strongly countable-dimensional space, then X is a shallow space.*

Proof. Let C be a compact subspace of X such that $\dim F < \infty$ for every closed subspace F of X with $F \cap C = \emptyset$. Then C is a compact strongly countable-dimensional space. Hence C is shallow and hence X is a shallow space by [E, Problem 7.3.H]. □

As we mentioned above, there is a paracompact space X such that $\dim X = 1$, but X does not have large transfinite dimension. This example leads the Ind-version of the condition (a'). A normal space X is *strongly countable-dimensional with respect to* Ind (shortly *s.c.d.*-Ind) if X is a union of countably many closed subspaces X_n, $n \in \mathbb{N}$, such that $\operatorname{Ind} X_n < \infty$ for each $n \in \mathbb{N}$. Further, we introduce a notion similar to the condition (K).

(K-Ind) There is a compact subspace C of X such that $\operatorname{Ind} F < \infty$ for every closed subspace F of X with $F \cap C = \emptyset$.

We consider the following.

(a") X is an s.c.d.-Ind space satisfying the condition (K-Ind).

Then we have

Proposition 2.2. *Let X be a hereditarily normal space. If X is an s.c.d.-*Ind *space satisfying the condition (*K-Ind*), then X has large transfinite dimension* $\operatorname{Ind} X$.

Proof. Let C be a compact subspace of X such that $\operatorname{Ind} F < \infty$ for every closed subspace F of X with $F \cap C = \emptyset$. Since C is an s.c.d.-Ind compact space, by [F, Theroems 1, 3], C has large transfinite dimension. Then it follows from [E, Lemma 7.1.24] that X has large transfinite dimension and $\operatorname{Ind} X \leq \omega_0 + \operatorname{Ind} C$. □

Question 2.3. Does Proposition 2.2 hold for normal spaces?

Question 2.4 [F, Problem 3]**.** Does every compact space which can be represented as the union of countably many subspaces which all have large transfinite dimension have itself large transfinite dimension?

We notice that if Question 2.4 has an affirmative answer, then Question 2.3 does.

III. (a') $\Rightarrow$ (b).

By the proof of Theorem 2.1 (see [H5]), we have the following.

Proposition 2.3. *Let X be a strongly countable-dimensional paracompact space. If there is a compact subspace C of X such that C has a countable character and* $\operatorname{Ind} F < \infty$ *for every closed subspace F of X with $F \cap C = \emptyset$, then X satisfies the condition (b).*

We do not know the proposition above holds for every strongly countable-dimensional paracompact space satisfying the condition (K).

IV. (b) $\Rightarrow$ (a').

The proof of (b) $\Rightarrow$ (a) of Theorem 2.1 works well for paracompact spaces and it shows that the condition (b) implies the condition (a') ([H5]). (The metrizability is used for the equivalence between (a) and (a') in Theorem 2.1). Hence the implication (b) $\Rightarrow$ (a') holds for every paracompact space X.

We turn our attention to finitistic spaces. In [H6], we proved that there is a universal space $L(\tau)$ for the class of finitistic metrizable spaces with the weight $\leq \tau$ and asked if $\{h : h : X \to L(\tau) \text{ is a homeomorphic embedding}\}$ is dense in the function space $C(X, L(\tau))$ ([H6, Question 1]). Quite recently, Kulesza answers the question negatively. The following question asked in the same paper still remains open.

Question 2.5 [H6, Question 2]**.** Let X be a metrizable (or paracompact) space such that $\{f \in C(X, I^\omega) : \overline{f(X)} \setminus f(X)$ is countable-dimensional$\}$ is residual in $C(X, I^\omega)$. Is X finitistic?

3. Questions on order dimension

In this section, we shall consider a transfinite dimension defined by an order of closed mappings. In 1955, K. Morita [M] proved a fundamental theorem on the dimension and closed mappings in metrizable spaces : *For a metrizable space X $\dim X \leq n$ if and only if there are a metrizable space Z with $\dim Z = 0$ and a closed mapping f of Z onto X such that every fiber of f contains at most $n+1$ points.* The theorem has many applications to infinite dimensional spaces. We begin with some definitions.

Definition 3.1 [B2]**.** Let L be a set, $\operatorname{Fin} L$ the collection of all non-empty finite subsets of L and M a subset of $\operatorname{Fin} L$. Let $\alpha > 0$ be an ordinal number. For $\sigma \in \{\emptyset\} \cup \operatorname{Fin} L$, we put

$$M^\sigma = \{\tau \in \operatorname{Fin} L : \sigma \cup \tau \in M \text{ and } \sigma \cap \tau = \emptyset\}.$$

If $\sigma = \{a\}$, we write $M^\sigma = M^a$. Then the *order* $\operatorname{Ord} M$ of M is inductively defined as follows:

(1) $\operatorname{Ord} M = 0$ if $M = \emptyset$.
(2) $\operatorname{Ord} M \leq \alpha$ if $\operatorname{Ord} M^a < \alpha$ for each $a \in L$.

If $\operatorname{Ord} M \leq \alpha$ for some ordinal number α, we say that $\operatorname{Ord} M$ *exists* (or M *has* $\operatorname{Ord} M$).

We notice that $\operatorname{Ord} M \leq n$ iff $|\sigma| \leq n$ for each $\sigma \in M$, where $n < \omega$.

By use of this order, F. G. Arenas ([A]) defined an order of mappings as follows.

Definition 3.2 [A]**.** Let X and Y be topological spaces and $f : X \to Y$ a mapping. Let $\mathcal{T}(X)$ be the topology of X. We put

$$O(f) = \{\tau = \{U_1, \dots, U_n\} \in \operatorname{Fin}\mathcal{T}(X) \ : \\ \overline{U_i} \cap \overline{U_j} = \emptyset \text{ for } i \neq j \text{ and } \bigcap_{i=1}^n f(\overline{U_i}) \neq \emptyset\}.$$

Then the *order* $\operatorname{Ord} f$ of f is defined as $\operatorname{Ord} f = \operatorname{Ord} O(f)$.

By use of the transfinite order of mappings, Arenas extended the covering dimension to a transfinite dimension as follows.

Definition 3.3 [A]**.** Let X be a Tychonoff space and α an ordinal number. Then X *has the order dimension* $\operatorname{O-dim} X \leq \alpha$ if and only if there are a strongly zero-dimensional space Z and a perfect mapping f of Z onto X such that $\operatorname{Ord} f \leq \alpha + 1$. We say that X *has the order dimension* $\operatorname{O-dim} X$ (or $\operatorname{O-dim} X$ *exists*) if $\operatorname{O-dim} X \leq \alpha$ for some ordinal number α.

The following is a characterization of spaces that have large transfinite dimension in terms of finite-to-one closed continuous mappings.

Theorem 3.1 [HY, Theorem 2.7]**.** *A metrizable space X has the order dimension $\operatorname{O-dim} X$ if and only if X has large transfinite dimension $\operatorname{Ind} X$.*

Connects with the theorem, we can ask the following.

Question 3.1. Does the theorem hold for every paracompact space?

A normal space X is called *countable-dimensional* if X is the union of countably many subspaces X_n, $n = 1, 2, \ldots$, with $\dim X_n < \infty$.

Question 3.2 [HY, Question 2.9]**.** Let X be a compact space having the order dimension $\operatorname{O-dim} X$. Is X countable-dimensional?

Remark. In [A], Arenas mentioned that there is a compact space X having the order dimension $\operatorname{O-dim} X$, but X is not countable-dimensional. However, his proof is not correct (see [HY, Remark 2.8]). Hence the question above still seems to be open.

We further the investigation of the relation between the order dimension and large transfinite dimension.

Theorem 3.2 [HY, Theroem 3.5]**.** *Let X be a metrizable space having the order dimension $\operatorname{O-dim} X$. Then the inequality $\operatorname{Ind} X \leq \operatorname{O-dim} X$ holds.*

Concerning the result above, we can ask the following:

Question 3.3 [HY, Question 3.6]**.** Let X be a metrizable space having large transfinite dimension $\operatorname{Ind} X$. Does the inequality $\operatorname{O-dim} X \leq \operatorname{Ind} X$ hold?

We notice that $\operatorname{Ind} S^\alpha = \operatorname{O-dim} S^\alpha = \alpha$ holds for every ordinal number $\alpha < \omega_1$, where S^α denotes the Smirnov's compactum ([HY, Theorem 3.8]).

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Department of Mathematics, Shimane University, Matsue, 690-8504 JAPAN
e-mail: hattori@math.shimane-u.ac.jp