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SOME ASPECTS ON CIRCLES AND HELICES IN A COMPLEX PROJECTIVE SPACE

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Abstract. In this note we study helices in a complex projective space. We characterize complex projective spaces among Hermitian symmetric spaces by the property that all holomorphic circles are closed. We also give examples of helices with multiple points.

1. Introduction.

A smooth curve $\gamma = \gamma(s)$ parametrized by its arclength s is called a *helix* of order d if there exist orthonormal frame fields $\{V_1 = \dot\gamma = \frac{d\gamma}{ds}, \cdots, V_d\}$ along γ and positive numbers $\kappa_1, \cdots, \kappa_{d-1}$ which satisfy the following system of ordinary equations

$$\nabla_{\dot\gamma} V_j(s) = -\kappa_{j-1} V_{j-1}(s) + \kappa_j V_{j+1}(s), \qquad j = 1, \cdots, d, \tag{1.1}$$

where $V_0 \equiv V_{d+1} \equiv 0$ and $\nabla_{\dot\gamma}$ denotes the covariant differentiation along γ. The numbers κ_j $(j = 1, \cdots, d-1)$ and the orthonormal frame $\{V_1, \cdots, V_d\}$ are called the *curvatures* and the *Frenet frame* of γ, respectively. A helix of order 2 is usually called a *circle*. That is, for a positive constant k, a circle of curvature k satisfies the following equations with a unit vector field $Y = Y(s)$ along γ:

$$\begin{cases} \nabla_{\dot\gamma}\dot\gamma = kY \\ \nabla_{\dot\gamma} Y = -k\dot\gamma. \end{cases} \tag{1.2}$$

In a Euclidean space $\mathbb{R}^3$, helices of order 3 are ordinary helices and circles are nothing but circles in the sense of Euclidean geometry.

In the recent works [2], [3], [4] and [8], K. Mashimo, K. Tojo and the authors studied circles in details in a Riemannian symmetric space of rank one. We call a smooth curve γ parametrized by its arclength s *closed* if there exists s_0 $(\neq 0)$

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with $\gamma(s+s_0)=\gamma(s)$ for every s. The minimal positive s_0 is called the length of this closed curve. A curve is said to be open if it is not closed. In these papers they showed that in symmetric spaces of rank one every circle is a simple curve, and that for arbitrary positive k there exist many open circles and many closed circles with curvature k.

In the first half of this paper we provide a characterization of a complex projective space $\mathbb{C}P^n(c)$ of constant holomorphic sectional curvature c in the class of compact Hermitian symmetric spaces in terms of some particular circles. In a Kähler manifold $(M, \langle\ ,\ \rangle, J)$, circles γ are naturally classified by the index $\langle\dot\gamma, JY\rangle$, which is constant along γ and is called the *complex torsion* of γ. In classes of circles in a Kähler manifold, the most important one is the class of holomorphic circles. A *holomorphic circle* is a circle satisfying the condition that $\dot\gamma$ and Y span a holomorphic plane, that is, $Y = J\dot\gamma$ or $Y=-J\dot\gamma$. For a holomorphic circle γ, the equations (1.2) reduce to

$$\nabla_{\dot\gamma}\dot\gamma = kJ\dot\gamma \quad\text{or}\quad \nabla_{\dot\gamma}\dot\gamma=-kJ\dot\gamma. \tag{1.3}$$

We can interpret such circles in terms of physics (see [1]). Our attempt is to study Kähler manifolds from the viewpoint of real Riemannian geometry.

Theorem 1. *Let M be an n-dimensional compact Hermitian symmetric space. Suppose that for some $k>0$, every holomorphic circle of curvature k in M is closed. Then M is complex analytically isometric to a complex projective space $\mathbb{C}P^n$ with Fubini-Study metric of constant holomorphic sectional curvature.*

In a real space form, that is a sphere, a Euclidean space or a hyperbolic space, every helix is a simple curve, that is a curve which does not have multiple points. But the aspect of helices is not the same in a complex projective space. In the latter half of this paper we give examples of closed helices of order 4 with multiple points in a complex projective space.

2. Characterization of a complex projective space.

In order to prove Theorem 1 we investigate holomorphic circles in $M = \mathbb{C}P^1(c_1)\times\mathbb{C}P^1(c_2) (= S^2(c_1)\times S^2(c_2))$ which is the simplest example of a Hermitiam symmetric space of rank 2. Here we denote by $S^n(c)$ an n-dimensional standard sphere of curvature c. The following is a key for Theorem 1.

Proposition 1. *Let $\gamma=\gamma(s)$ be a holomorphic circle of curvature k in $M = \mathbb{C}P^1(c_1)\times\mathbb{C}P^1(c_2)$ with the initial unit vector $\dot\gamma(0)=(X_1,X_2)\in T\mathbb{C}P^1(c_1)\times T\mathbb{C}P^1(c_2)$. Then the following hold:*

(1) *When $X_1=0$ (resp. $X_2=0$), the circle γ is a simple closed curve with length $\frac{2\pi}{\sqrt{k^2+c_2}}$ (resp. $\frac{2\pi}{\sqrt{k^2+c_1}}$).*
(2) *When $X_1\neq 0$ and $X_2\neq 0$, we find*

(i) *If* $\sqrt{\frac{k^2+c_2\|X_2\|^2}{k^2+c_1\|X_1\|^2}}$ *is rational, then the circle* γ *is a simple closed curve. Its length is the least common multiple of* $\frac{2\pi}{\sqrt{k^2+c_1\|X_1\|^2}}$ *and* $\frac{2\pi}{\sqrt{k^2+c_2\|X_2\|^2}}$.

(ii) *If* $\sqrt{\frac{k^2+c_2\|X_2\|^2}{k^2+c_1\|X_1\|^2}}$ *is irrational, then the circle* γ *is a simple open curve.*

Proof.
(1) Since in $\mathbb{C}P^1(c) = S^2(c)$ circles of curvature k are small circles with length $\frac{2\pi}{\sqrt{k^2+c}}$, the assertion is obvious (see [11] or [3]).
(2) Let γ be a holomorphic circle in M. We just treat the case $\nabla_{\dot\gamma}\dot\gamma = kJ\dot\gamma$. In this case we find $\nabla_{\dot\gamma_i}\dot\gamma_i(s) = kJ\dot\gamma_i(s)$ $(i = 1, 2)$, where $\gamma(s) = (\gamma_1(s), \gamma_2(s)) \in \mathbb{C}P^1(c_1) \times \mathbb{C}P^1(c_2)$. We first get that $\|\dot\gamma_i(s)\| = \|X_i\|$ for every s $(i = 1, 2)$. In fact, we see

$$\begin{aligned}\nabla_{\dot\gamma_i}\langle\dot\gamma_i(s), \dot\gamma_i(s)\rangle &= 2\langle\nabla_{\dot\gamma_i}\dot\gamma_i(s), \dot\gamma_i(s)\rangle\\ &= 2k\langle J\dot\gamma_i(s), \dot\gamma_i(s)\rangle = 0.\end{aligned}$$

We here set $\sigma_i(s) = \gamma_i(\frac{s}{\|X_i\|})$ so that $\|\dot\sigma_i\| = 1$ $(i = 1, 2)$. We then have

$$\begin{aligned}\nabla_{\dot\sigma_i}\dot\sigma_i(s) &= \frac{1}{\|X_i\|^2}\nabla_{\dot\gamma_i}\dot\gamma_i\left(\frac{s}{\|X_i\|}\right)\\ &= \frac{k}{\|X_i\|^2}J\dot\gamma_i\left(\frac{s}{\|X_i\|}\right) = \frac{k}{\|X_i\|}J\dot\sigma_i(s).\end{aligned}$$

This implies that the curve $\sigma_i = \sigma_i(s)$ is a holomorphic circle of curvature $\frac{k}{\|X_i\|}$ in $CP^1(c_i)$. Hence, the curve σ_i is a closed curve with length $\frac{2\pi}{\sqrt{\frac{k^2}{\|X_i\|^2}+c_i}}$ $(i = 1, 2)$.

Take a point $x = (x_1, x_2)$ on the circle γ in M. If x moves along γ with velocity 1, then the point x_i moves along the curve σ_i with velocity $\|X_i\|$. Hence the circle γ is closed if and only if the ratio $\frac{2\pi}{\|X_1\|\sqrt{\frac{k^2}{\|X_1\|^2}+c_1}} : \frac{2\pi}{\|X_2\|\sqrt{\frac{k^2}{\|X_2\|^2}+c_2}} = \sqrt{\frac{k^2+c_2\|X_2\|^2}{k^2+c_1\|X_1\|^2}}$ is rational. Thus we get the conclusion. □

As an immediate consequence of Proposition 1 we find the following:

Corollary. *If a Kähler manifold* M *admits* $\mathbb{C}P^1 \times \mathbb{C}P^1$ *as a totally geodesic Kähler submanifold, then for arbitrary positive* k *there exist closed holomorphic circles and open holomorphic circles of curvature* k *in* M.

We are now in a position to prove Theorem 1.

Proof of Theorem 1. It is known that any compact Hermitian symmetric space of rank $d(\geq 2)$ admits $\mathbb{C}P^1 \times \mathbb{C}P^1$ as a totally geodesic Kähler submanifold (see, [10]). Corollary tells us that our manifold M satisfying the hypothesis of Theorem

1 is a compact Hermitian symmetric space of rank one, namely M is complex analytically isometric to $\mathbb{C}P^n$. Every holomorphic circle of curvature $k(>0)$ in $\mathbb{C}P^n(c)$ is a closed curve with length $\frac{2\pi}{\sqrt{k^2+c}}$ (see, [3]). $\square$

Motivated by Theorem 1, we pose the following problems:

Problem 1. *Let M be a simply connected compact Kähler homogeneous manifold (i.e. M is a Kähler C-space). Suppose for some $k>0$ every holomorphic circle of curvature k in M is closed. Then, is M complex analytically isometric to $\mathbb{C}P^n$ with Fubini-Study metric of constant holomorphic sectional curvature ?*

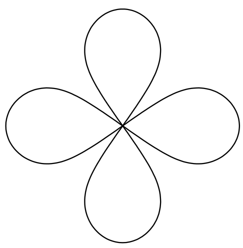
Problem 2. *Let M be a compact Kähler manifold and k be a positive number. Suppose that all holomorphic circles in M with curvature k are closed with the same length. Under what conditions can we conclude that M is a complex projective space?*

These are in some sense complex versions of the L.W. Green's theorem on closed geodesics (see [7] and [6]).

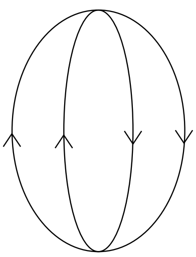
3. Helices with multiple points in a complex projective space.

In this section we study how to get examples of closed helices with multiple points in a complex projective space. We shall call a point $x=\gamma(s_*)\in M$ of a closed curve γ an *$n-$recurrent point* of γ if there exist n positive numbers $t_1<t_2<\cdots<t_n$ satisfying the following three conditions;

(i) $x=\gamma(s_*)=\gamma(s_*+t_1)=\cdots=\gamma(s_*+t_n)$.
(ii) t_n is the length of the closed curve γ.
(iii) For any $t\in(0,t_n]$, if $t\neq t_i$ $(i=1,\cdots,n)$, then $\gamma(s_*+t)\neq x$.



one 4-recurrent point



two 2-recurrent points

Note that a closed curve γ is simple if and only if each point on γ is 1-recurrent. When a closed curve γ has an n-recurrent point $x=\gamma(s_*)$, we call $t_{i-1,i}=t_i-t_{i-1}$ the *recurrent time* from the point $\gamma(s_*+t_{i-1})$ to the point $\gamma(s_*+t_i)$ $(i=1,\cdots,n)$, where $t_0=0$. An n-recurrent point $(n\geq 2)$ is also called a multiple point.

In order to get our examples of closed helices with multiple points in a complex projective space, we study a circle of curvature k in a flat torus $T^2=S^1(r_1)\times S^1(r_2)$. Here $S^1(r)$ denotes a circle (i.e. a 1-dimensional sphere) of radius r. We denote by $\tilde{\gamma}$ a covering circle of γ in $\mathbb{R}^2$. Note that the circle $\tilde{\gamma}$ is a circle of radius $1/k$ in the sense of Euclidean Geometry. This implies that every circle γ of curvature k in T^2 is a closed curve of length $2\pi/k$.

Here we study multiple points of circles γ in a flat torus $T^2 = S^1(r_1) \times S^1(r_2)$. We may set without loss of generality the following initial conditions on a covering circle $\tilde{\gamma}$ in $\mathbb{R}^2$:

$$\tilde{\gamma}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} (\in \mathbb{R}^2),\ X_0 = \dot{\tilde{\gamma}}(0) = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} (\in U_{\tilde{\gamma}(0)}\mathbb{R}^2 = S^1(1)).$$

The principal normal unit vector then satisfies $Y_0 = J\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ or $Y_0 = -J\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, where J denotes the natural complex structure on $\mathbb{R}^2 = \mathbb{C}$ which is given by $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. In the following, we only discuss the case that $Y_0 = J\dot{\gamma}(0)$. When $Y_0 = -J\dot{\gamma}(0)$, the circle $\sigma(s) = \gamma(-s)$ satisfies $Y_0 = J\dot{\sigma}(0)$.

A direct computation yields that the circle $\tilde{\gamma}(s) = \begin{pmatrix} \tilde{\gamma}_1(s) \\ \tilde{\gamma}_2(s) \end{pmatrix}$ is expressed as

$$\begin{cases} \tilde{\gamma}_1(s) = \dfrac{1}{k}(v_2 \cdot \cos\ ks\ +\ v_1 \cdot \sin\ ks) - \dfrac{v_2}{k}, \\ \tilde{\gamma}_2(s) = \dfrac{1}{k}(-v_1 \cdot \cos\ ks\ +\ v_2 \cdot \sin\ ks) + \dfrac{v_1}{k}. \end{cases} \tag{3.1}$$

We here prove the following:

Proposition 2. *Every circle γ of curvature k in a flat torus $T^2 = S^1(r_1)\times S^1(r_2)$ is a closed curve of length $2\pi/k$. Moreover, the following hold*:

(i) *When $k > \frac{1}{\pi\cdot min\{r_1,r_2\}}$, γ is a simple curve.*

(ii) *When $k \leq \frac{1}{\pi\cdot min\{r_1,r_2\}}$, γ has a multiple point.*

Proof. For a given circle γ we denote by $\tilde{\gamma}$ its covering circle in $\mathbb{R}^2$.
(i) In this case, since the circle $\tilde{\gamma}$ lies in the interior of a fundamental region $\mathfrak{F} = \{(x_1, x_2) \mid 0 \leq x_1 < 2\pi r_1,\ 0 \leq x_2 < 2\pi r_2\}$ of $T^2 = S^1(r_1) \times S^1(r_2)$, the assertion (i) is obvious.
(ii) By using the expression (3.1) we shall show that the circle γ has a multiple point. We set $\gamma(t_0) = \gamma(0)$, which implies that $\tilde{\gamma}_1(t_0) = 2\pi r_1 p$ and $\tilde{\gamma}_2(t_0) = 2\pi r_2 q$ for some integers p and q. These, together with the equations (3.1), yield

$$\begin{cases} \cos\ kt_0 = 2\pi k(r_1v_2p - r_2v_1q) +\ 1, \\ \sin\ kt_0 = 2\pi k(r_1v_1p + r_2v_2q). \end{cases} \tag{3.2}$$

Hence, when $pq \neq 0$, the equations (3.2) give

$$\pi k = \frac{-r_1v_2p + r_2v_1q}{r_1^2p^2 + r_2^2q^2}. \tag{3.3}$$

We here note the following: If we take another pair (p', q') of integers which is different from the pair (p, q) and satisfies (3.3), then we find by an easy computation that $t'_0 \neq t_0$, where t'_0 is defined in the equations (3.2) by replacing (p, q) by (p', q'). Therefore the point $\gamma(0)$ is 1-recurrent if and only if there does not exist a pair of integers $(p, q)(\neq (0, 0))$ satisfying (3.3). Hence our discussion shows that the circle γ is simple if and only if for any $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} (\in S^1(1))$ there does not exist a pair of integers $(p, q)(\neq (0, 0))$ satisfying (3.3).

Due to this criterion we can conclude that our circle is not simple. In fact, when $r_1 \leq r_2$, we set $v_2 = -\pi k r_1 (\geq -1)$ and $v_1 = \pm\sqrt{1 - (\pi k r_1)^2}$. Then the pair $(p, q) = (1, 0)$ is a solution of the equation (3.3). And also, when $r_1 > r_2$, we set $v_1 = \pi k r_2 (< 1)$ and $v_2 = \pm\sqrt{1 - (\pi k r_2)^2}$. Then the pair $(p, q) = (0, 1)$ is a solution of the equation (3.3). $\square$

By virtue of the discussion in the proof of Proposition 2 we can establish the following:

Proposition 3. *Let γ be a circle of curvature k in a flat torus $T^2 = S^1(r_1) \times S^1(r_2)$. A point $\gamma(s_*)(\in T^2)$ is an n-recurrent point of γ if and only if there exist $(n-1)$ pairs of integers $(p, q)(\neq (0, 0))$ satisfying the following equation ;*

$$\pi k = \frac{-r_1 v_2 p + r_2 v_1 q}{r_1^2 p^2 + r_2^2 q^2}, \tag{3.4}$$

where $(v_1, v_2) \in \mathbb{R}^2$ denotes the tangent vector $\dot{\tilde{\gamma}}(s_*)$ of a covering circle $\tilde{\gamma}$ in R^2 for γ.

From now, by using Proposition 3 we give a class of helices with multiple points in a complex projective plane $\mathbb{C}P^2$ with the aid of a well-known isometric parallel imbedding of a 2-dimensional flat torus into $\mathbb{C}P^2$ (see [9] for detail). We consider a Riemann surface $N = (S^1 \times S^1)/\varphi$. Here by representing the first component by $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ and the second component by $S^1 = \{(a_1, a_2) \in \mathbb{R}^2 \mid (a_1)^2 + (a_2)^2 = 1\}$, we define the identification φ by $\varphi\left((e^{i\theta}, (a_1, a_2))\right) = (-e^{i\theta}, (-a_1, -a_2))$. The Riemannian metric on N is given by $\langle A + \xi, B + \eta \rangle = \frac{2}{9}\langle A, B \rangle_{S^1} + \frac{2}{3}\langle \xi, \eta \rangle_{S^1}$ for tangent vectors $A, B \in TS^1$ of the first component and tangent vectors $\xi, \eta \in TS^1$ of the second component, where $\langle\ ,\ \rangle_{S^1}$ denotes the canonical metric of S^1. We define an isometric imbedding of N into $\mathbb{C}P^2(4)$ by

$$f(e^{i\theta}, (a_1, a_2)) = \pi \left(\frac{1}{3}(e^{-\frac{2i\theta}{3}} + 2a_1 e^{\frac{i\theta}{3}}), \frac{\sqrt{2}}{3}(e^{-\frac{2i\theta}{3}} - a_1 e^{\frac{i\theta}{3}}), \frac{2}{\sqrt{6}} i a_2 e^{\frac{i\theta}{3}} \right), \tag{3.5}$$

where $\pi : S^5(1) \longrightarrow \mathbb{C}P^2(4)$ is the Hopf fibration. This isometric imbedding f is parallel, that is f has the parallel second fundamental form σ_f, and totally real.

We now study images of circles in N under this isometric imbedding. As we see in [9], the imbedding f maps each geodesic on N to a circle of curvature $\frac{1}{\sqrt{2}}$ in $\mathbb{C}P^2(4)$. This circle does not have multiple points, but it is not necessarily closed in $\mathbb{C}P^2(4)$. By direct calculations we obtain the following (for details, see [5]).

Fact. *For a circle γ of curvature $k(>0)$ on N, the curve $f\circ\gamma$ is a helix of order* 4 *in $\mathbb{C}P^2(4)$. More precisely,*

(1) *when $k=\frac{1}{2}$, it is a helix of proper order* 3 *with curvatures* $\kappa_1=\frac{\sqrt{3}}{2}$, $\kappa_2=\sqrt{\frac{3}{2}}$,

(2) *when $k\neq\frac{1}{2}$, it is a helix of proper order* 4 *with curvatures* $\kappa_1=\sqrt{k^2+\frac{1}{2}}$, $\kappa_2=\frac{3k}{\sqrt{2k^2+1}}$, $\kappa_3=\frac{|4k^2-1|}{\sqrt{2(2k^2+1)}}$.

Moreover the helix $f\circ\gamma$ is closed of length $2\pi/k$ and has a multiple point if and only if $k\leqq\frac{3}{\sqrt{2}\pi}$. The number of multiple points is greater than 2.

Combining Fact and Proposition 3, we obtain the following examples.

Examples. (1) When $k=\frac{3}{\sqrt{2}\pi}$, the helix $f\circ\gamma$ has three 2-recurrent points with the same recurrent time $\frac{\sqrt{2}}{3}\pi^2$. Every point on this helix except these is simple.
(2) When $k=\frac{3\sqrt{6}}{4\pi}$, the helix $f\circ\gamma$ has two 3-recurrent points with the same reccurent time $\frac{4\sqrt{6}}{27}\pi^2$. Every point on this helix except these is simple.
(3) When $k=\frac{3\sqrt{2}}{4\pi}$ the helix $f\circ\gamma$ has a 6-recurrent point with the same reccurent time $\frac{2\sqrt{2}}{9}\pi^2$. Every point on this helix except these is simple.
(4) When $k=\frac{\sqrt{6}}{2\pi}$ the helix $f\circ\gamma$ has eight 2-recurrent points.

4. Remark.

As an immediate consequence of Proposition 2 we establish the following:

Theorem 2. *If a Riemannian manifold M admits a flat torus $T^2=S^1(r_1)\times S^1(r_2)$ as a totally geodesic submanifold, then for any $k\leq 1/(\pi\cdot\min\{r_1,r_2\})$ we have a closed circle of curvature k with multiple point. Hence, in particular any compact symmetric space M of rank $d(\geq 2)$ has many closed circles with multiple points.*

The authors have an impression that holomorphic circles in Kähler geometry may play a similar role of geodesics in real Riemannian geometry.

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