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CONTINUOUS DEPENDENCE ON AGING AND BIRTH FUNCTIONS FOR SIZE-STRUCTURED POPULATION MODELS OF GENERAL TYPE

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Abstract. We are concerned with size-structured population models having aging and birth functions of general type. We show the continuous dependence of the solution on the aging and birth functions by showing the exact estimate.

1. Introduction

We are concerned with size structured population models having the aging and birth functions of general type. In our model, we treat the growth rate depending on the individual's size and time. In this framework, N. Kato and H. Torikata [2] have shown the results on local existence and continuous dependence on the initial data. It is important to know that the solution of the system is stable with respect to small purterbations of given data. This is a brief note on the continuous dependence of the solution on the aging and birth functions as well as the initial data. Actually, we show the exact estimate which generalizes [2, Theorem 2.2] and give a generalization of the result given by G. Webb [4, Theorem 2.6].

Our model is motivated by the growth of plants in forests or plantations. In case of plants, it is natural to consider the growth rate depending on the individual's size and time. For, the growth of plants depends on their circumstances, which certainly change with time, and the size is important to catch the sunlight. The model is formulated as the following initial boundary value problem in which some nonlocal terms are involved:

$$\text{(SDP)} \qquad \begin{cases} u_t + (V(x,t)u)_x = G(u(\cdot,t))(x), & x \in [0,l),\ 0 \le t \le T, \\ V(0,t)u(0,t) = C(t) + F(u(\cdot,t)), & 0 \le t \le T, \\ u(x,0) = \nu(x), & x \in [0,l), \end{cases}$$

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where the unknown function $u(x,t)$ represents the density of population with respect to the size x at time t and hence $\int_{x_1}^{x_2} u(x,t)dx$ is the population with size between x_1 and x_2 at time t. $l \in (0,\infty]$ is the maximum size, F corresponds to the birth process and G corresponds to the death and migration processes. Usually, F and G involve some nonlocal terms as in the example of Gurtin-MacCamy type (see below). The function V is the growth rate and C describes the inflow of zero-size individuals (i.e. newborns) from an external source.

For the related works, we refer to A. Calsina and J. Saldaña [1] where they have treated the nonlinear growth rate but restricted the birth and aging functions to the Gurtin-MacCamy type:

$$\begin{aligned} F(u(\cdot,t)) &= \int_0^l \beta(x,P(t))u(x,t)dx, \\ G(u(\cdot,t))(x) &= -m(x,P(t))u(x,t), \end{aligned}$$

where $P(t) := \int_0^l u(x,t)dx$ is the total population at time t. On the other hand G. Webb [4] developed the theory of age-dependent population dynamics having aging and birth functions of general type. Note that the particular case $V(x,t) \equiv 1$ is nothing but the age-depandent model.

As mentioned above, our purpose is to show the continuous dependence of the solution on given data. The result here is not sufficiently general because we fix the growth rate. The dependence on the growth rate is important but possesses an intrinsic difficulty. We will publish the full general result elsewhere.

This paper is organized as follows. In Section 2, we review some existence results obtained in [2] and [3]. Our main result is stated and proved in Section 3.

2. Existence results (Review)

In this section we state our general assumptions for the models and review some results obtained by N. Kato and H. Torikata [2] and N. Kato [3].

Let $L^1 := L^1(0,l;\mathbb{R}^N)$ be the Banach space of Lebesgue integrable functions from $[0,l)$ to $\mathbb{R}^N$ with usual norm $\|f\|_{L^1} := \int_0^l |f(x)|dx$ for $f \in L^1$, where $|\cdot|$ denotes the norm of $\mathbb{R}^N$ defined by $|x| := \sum_{i=1}^N |x_i|$ for $x = (x_1,\cdots,x_N) \in \mathbb{R}^N$. For $T > 0$, let $L_T := C([0,T];L^1)$ be the Banach space of continuous L^1-valued functions on $[0,T]$ with the supremum norm $\|u\|_{L_T} := \sup_{0\le t\le T} \|u(t)\|_{L^1}$ for $u \in L_T$. It is known (cf. [4, Lemma 2.1]) that $u \in L_T$ is identified with the elements in $L^1((0,l)\times(0,T);\mathbb{R}^N)$ by $u(t)(x) = u(x,t)$ for $0 \le t \le T$, *a.e.* $x \in (0,l)$.

We make the following general hypotheses for (SDP).

(F) $F : L^1 \to \mathbb{R}^N$ is locally Lipschitz, i.e.,there is an increasing function $c_F : [0,\infty) \to [0,\infty)$ such that

$$|F(\phi_1) - F(\phi_2)| \le c_F(r)\|\phi_1 - \phi_2\|_{L^1}$$

for all ϕ_1, $\phi_2 \in L^1$ with $\|\phi_1\|_{L^1}$, $\|\phi_2\|_{L^1} \le r$.

(G) $G : L^1 \to L^1$ is locally Lipschitz, i.e., there is an increasing function $c_G : [0,\infty) \to [0,\infty)$ such that

$$\|G(\phi_1) - G(\phi_2)\|_{L^1} \le c_G(r)\|\phi_1 - \phi_2\|_{L^1}$$

for all $\phi_1,\ \phi_2 \in L^1$ with $\|\phi_1\|_{L^1},\ \|\phi_2\|_{L^1} \le r$.

(V) $V : [0,l) \times [0,T] \to (0,\infty)$ is a bounded function with upper bound $\bar{V}$. $V(x,t)$ is Lipschitz continuous with respect to x uniformly for t, i.e., there is a constant $L_V > 0$ such that

$$|V(x_1,t) - V(x_2,t)| \le L_V|x_1 - x_2|, \quad \forall x_1,\ x_2 \in [0,l),\ t \in [0,T].$$

For each $x \in [0,l)$, $V(x,t)$ is continuous in t. Further, in case $l < \infty$, we assume $V(l,\cdot) = 0$.

(C) $C : [0,T] \to \mathbb{R}^N$ is a continuous function.

The characteristic curve through $(x_0,t_0) \in [0,l) \times [0,T]$ is defined by the solution of the differential equation

$$\begin{cases} x'(t) = V(x(t),t), & t \in [t_0,T] \\ x(t_0) = x_0 \in [0,l). \end{cases} \tag{2.1}$$

We denote the solution by $\varphi(t;t_0,x_0)$. Note that the Lipschitz continuity of V assumed in (V) guarantees the existence of the unique solution $x(t) = \varphi(t;t_0,x_0)$ of (2.1) on $[t_0,T]$.

Let $z(t) := \varphi(t;0,0)$ denote the characteristic curve through the origin $(0,0)$ in (x,t)-plane. This curve is the trajectory in the (x,t)-plane of the newborns at $t = 0$. For $(x_0,t_0) \in [0,l) \times [0,T]$ satisfying $x_0 < z(t_0)$, we define the initial time $\tau := \tau(t_0,x_0) \in [0,t_0]$ implicitly by the relation

$$\varphi(t;\tau,0) = x, \ \text{or equivalently, } \varphi(\tau;t,x) = 0. \tag{2.2}$$

For each $(x_0,t_0) \in [0,l) \times [0,T]$, let $\tau^*(t_0,x_0) := \tau(t_0,x_0)$ for $x_0 < z(t_0)$ and $\tau^*(t_0,x_0) := 0$ for $x_0 \ge z(t_0)$. Note that the solution $x(t) = \varphi(t;t_0,x_0)$ of (2.1) can be extended on $[\tau^*(t_0,x_0),T]$ and satisfies the integral equation

$$\varphi(t;t_0,x_0) = x_0 + \int_{t_0}^{t} V(\varphi(\sigma;t_0,x_0),\sigma)d\sigma \quad \text{for } t \in [\tau^*(t_0,x_0),T].$$

Notice that $x(t) = \varphi(t;t_0,x_0)$ satisfies $0 \le x(t) < l$ for every $t \in [\tau^*(t_0,x_0),T]$ provided that $x_0 \in [0,l)$. In case $l < \infty$, if $x_0 = l$, then $x(t) \equiv l$.

Along the characteristic curves, we seek the solution. Observe that if $u(x,t)$ satisfies (SDP), then $u(\varphi(s;t,x),s)$ is differentiable $a.e.$ $s \in (\tau^*(t,x),T)$ and satisfies

$$\frac{d}{ds}u(\varphi(s;t,x),s) = G(u(\cdot,s))(\varphi(s;t,x)) - V_x(\varphi(s;t,x),s)u(\varphi(s;t,x),s).$$

By integrating this relation over $(\tau^*(t,x),T)$, we come to the following

Definition 2.1. We call $u \in L_T$ a solution of (SDP) on $[0,T]$ if u satisfies

$$u(x,t) = \begin{cases} \frac{C(\tau)+F(u(\cdot,\tau))}{V(0,\tau)} + \int_\tau^t \tilde{G}_s(u(\cdot,s))(\varphi(s;\tau,0))ds & a.e.\ x \in (0,z(t)), \\ \nu(\varphi(0;t,x)) + \int_0^t \tilde{G}_s(u(\cdot,s))(\varphi(s;t,x))ds & a.e.\ x \in (z(t),l), \end{cases}$$

where $\tau := \tau(t,x)$ is given by (2.2) and $\tilde{G}_t$ is defined by

$$\tilde{G}_t(\phi)(x) := G(\phi)(x) - V_x(x,t)\phi(x), \quad \forall t \in [0,T],\ a.e.\ x \in (0,l)$$

for $\phi \in L^1$.

Remark. Note that if $V \equiv 1$ (i.e. the age-dependent case), then the above definition becomes exactly the same as [4, (1.49)].

The following local existence result is obtained by N. Kato and H. Torikata [2, Theorems 2.1].

Theorem 2.1. (Local existence) *Let* (F), (G), (V), *and* (C) *hold. Then for each $r > 0$, there exists some $T > 0$ depending on r such that for the initial data $\nu \in L^1$ satisfying $\|\nu\|_{L^1} \le r$, there exists the uniqne solution $u \in L_T$ of* (SDP) *on* $[0,T]$.

Let $L^1_+ := L^1(0,l;\mathbb{R}^N_+)$, where $\mathbb{R}^N_+$ is the usual positive cone in $\mathbb{R}^N$. The global existence of positive solution given by N. Kato [3] is stated as follows.

Theorem 2.2. (Positive global existence) *In addition to* (F), (G) *and* (V), *we assume the following hypotheses:*

- (F_+) $F(L^1_+) \subset \mathbb{R}^N_+$.
- (G_+) *There is an increasing function $c_+ : [0,\infty) \to [0,\infty)$ such that $r > 0$, $\phi \in L^1_+$ and $\|\phi\|_{L^1} \le r$ imply $G(\phi) + c_+(r)\phi \in L^1_+$.*
- (C_+) *$C : [0,T] \to \mathbb{R}^N_+$ is a continuous function.*

Assume further that there exists some $\omega \in \mathbb{R}$ satisfying

$$\sum_{i=1}^N \Big[F(\phi)_i + \int_0^l G(\phi)_i(x)dx\Big] \le \omega \sum_{i=1}^N \int_0^l \phi_i(x)dx, \quad \forall \phi \in L^1_+.$$

Then for each $\nu \in L^1_+$ and $T > 0$, there exists a unique solution $u \in C([0,T];L^1_+)$ of (SDP) *on* $[0,T]$ *and the following estimate holds:*

$$\|u(\cdot,t)\|_{L^1} \le e^{(\omega+L_V)t}\|\nu\|_{L^1} + \int_0^t e^{(\omega+L_V)(t-s)}|C(s)|ds, \quad t \in [0,T].$$

3. Continuous dependence on birth and aging functions

Our interest here lies in whether the solution depends continuously on the birth and aging functions as well as the initial data. For that purpose, let $n \in \mathbb{N} \cup \{0\}$ and consider the following approximating problem:

$$(\mathrm{SDP}_n) \qquad \begin{cases} \partial_t u_n + \partial_x (V(x,t)u_n) = G_n(u_n(\cdot,t))(x), \quad x \in [0,l),\ 0 \le t \le T, \\ V(0,t)u_n(0,t) = C_n(t) + F_n(u_n(\cdot,t)), \quad 0 \le t \le T, \\ u_n(x,0) = \nu_n(x), \quad x \in [0,l). \end{cases}$$

Corresponding to the conditions (F), (G) and (C), we impose the following hypotheses on F_n, G_n and C_n:

(F_n) $F_n : L^1 \to \mathbb{R}^N$ and there is an increasing function $c_F : [0,\infty) \to [0,\infty)$ (independent of n) such that

$$|F_n(\phi_1) - F_n(\phi_2)| \le c_F(r)\|\phi_1 - \phi_2\|_{L^1}$$

for all ϕ_1, $\phi_2 \in L^1$ with $\|\phi_1\|_{L^1}$, $\|\phi_2\|_{L^1} \le r$.

(G_n) $G_n : L^1 \to L^1$ and there is an increasing function $c_G : [0,\infty) \to [0,\infty)$ (independent of n) such that

$$\|G_n(\phi_1) - G_n(\phi_2)\|_{L^1} \le c_G(r)\|\phi_1 - \phi_2\|_{L^1}$$

for all ϕ_1, $\phi_2 \in L^1$ with $\|\phi_1\|_{L^1}$, $\|\phi_2\|_{L^1} \le r$.

(C_n) $C_n : [0,T] \to \mathbb{R}^N$ are continuous functions.

By the local existence result reviewed in Section 2, under the hypotheses (F_n), (G_n), (C_n) and (V), for each $n \in \mathbb{N} \cup \{0\}$ and for each initial data $\nu_n \in L^1$, there exist some $T_n > 0$ and $u_n \in L_{T_n}$ such that u_n is a (local) solution of (SDP_n) on $[0,T_n]$ (in the sense of Definition 2.1) :

$$(3.1) \quad u_n(x,t) = \begin{cases} \frac{C_n(\tau)+F_n(u_n(\cdot,\tau))}{V(0,\tau)} + \int_\tau^t \tilde{G}_{s,n}(u_n(\cdot,s))(\varphi(s;\tau,0))ds & a.e.\ x \in (0,z(t)), \\ \nu_n(\varphi(0;t,x)) + \int_0^t \tilde{G}_{s,n}(u_n(\cdot,s))(\varphi(s;t,x))ds & a.e.\ x \in (z(t),l), \end{cases}$$

where $\tau := \tau(t,x)$ is given by (2.2) and $\tilde{G}_{t,n}$ is defined by

$$\tilde{G}_{t,n}(\phi)(x) := G_n(\phi)(x) - V_x(x,t)\phi(x), \quad t \in [0,T],\ a.e.\ x \in (0,l)$$

for $\phi \in L^1$.

Our main result is stated as follows:

Theorem 3.1. *Let $n \in \mathbb{N}\cup\{0\}$ and assume that* (F_n), (G_n), (C_n) *and* (V) *hold. Let $u_n \in L_{T_n}$ be the local solutions of* (SDP_n) *on* $[0,T_n]$. *Suppose that there exist some*

$T > 0$ and $r > 0$ such that $0 < T \leq \inf_{n\in\mathbb{N}\cup\{0\}} T_n$ and $\sup_{n\in\mathbb{N}\cup\{0\}} \|u_n\|_{L_T} \leq r$. Then we have the following estimate:

$$\|u_n(\cdot,t) - u_0(\cdot,t)\|_{L^1} \leq \exp[(c_F(r) + c_G(r) + 2L_V)t]L(n), \forall t \in [0,T], \tag{3.2}$$

where

$$\begin{aligned} L(n) := &\int_0^T |C_n(s) - C_0(s)|ds + \int_0^T |F_n(u_0(\cdot,s)) - F_0(u_0(\cdot,s))|ds \\ &+ \int_0^T \|G_n(u_0(\cdot,s)) - G_0(u_0(\cdot,s))\|_{L^1} ds + \|\nu_n - \nu_0\|_{L^1}. \end{aligned} \tag{3.3}$$

In particular, assume further the following approximating conditions:

$$\begin{aligned} &\lim_{n\to\infty} F_n(\phi) = F_0(\phi) \text{ in } \mathbb{R}^N, \quad \forall\phi \in L^1, \\ &\lim_{n\to\infty} G_n(\phi) = G_0(\phi) \text{ in } L^1, \quad \forall\phi \in L^1, \\ &\lim_{n\to\infty} C_n = C_0 \text{ in } L^1(0,T;\mathbb{R}^N), \\ &\lim_{n\to\infty} \nu_n = \nu_0 \text{ in } L^1. \end{aligned} \tag{3.4}$$

Then we conclude that $\lim_{n\to\infty} \|u_n - u_0\|_{L_T} = 0$.

Remark. Note that from the proof of Theorem 2.1 in Section 2 (see [2]), we find that the local existing time T_n can be chosen independently of n provided that the assumptions (F_n), (G_n), (C_n) and (3.4) hold. Also, in view of Theorem 2.2 in Section 2, the assumption of existence of $T > 0$ and $r > 0$ as above is satisfied in many natural settings.

Proof of Theorem 3.1. Firstly, we have

$$\begin{aligned} &\|u_n(\cdot,t) - u_0(\cdot,t)\|_{L^1} \\ &\leq \int_0^{z(t)} \left|\frac{C_n(\tau) + F_n(u_n(\cdot,\tau))}{V(0,\tau)} - \frac{C_0(\tau) + F_0(u_0(\cdot,\tau))}{V(0,\tau)}\right| dx \\ &\quad + \int_0^{z(t)} \int_\tau^t |\tilde{G}_{s,n}(u_n(\cdot,s))(\varphi(s;\tau,0)) - \tilde{G}_{s,0}(u_0(\cdot,s))(\varphi(s;\tau,0))| ds dx \\ &\quad + \int_{z(t)}^l |\nu_n(\varphi(0;t,x)) - \nu_0(\varphi(0;t,x))| dx \\ &\quad + \int_{z(t)}^l \int_0^t |\tilde{G}_{s,n}(u_n(\cdot,s))(\varphi(s;t,x)) ds - \tilde{G}_{s,0}(u_0(\cdot,s))(\varphi(s;t,x))| ds dx \\ &=: L_1 + L_2 + L_3 + L_4. \end{aligned}$$

For L_1, we have the following estimate:

$$
\begin{aligned}
L_1 \le & \int_0^{z(t)} \frac{1}{V(0,\tau)} |C_n(\tau) - C_0(\tau)| dx \\
& + \int_0^{z(t)} \frac{1}{V(0,\tau)} |F_n(u_n(\cdot,\tau)) - F_n(u_0(\cdot,\tau))| dx \\
& + \int_0^{z(t)} \frac{1}{V(0,\tau)} |F_n(u_0(\cdot,\tau)) - F_0(u_0(\cdot,\tau))| dx \\
\le & \int_0^{T} e^{L_V(t-s)} |C_n(s) - C_0(s)| ds \\
& + c_F(r) \int_0^{t} e^{L_V(t-s)} \|u_n(\cdot,s) - u_0(\cdot,s)\|_{L^1} ds \\
& + \int_0^{T} e^{L_V(t-s)} |F_n(u_0(\cdot,s)) - F_0(u_0(\cdot,s))| ds.
\end{aligned} \tag{3.5}
$$

Here, we have used the change of variable $s = \tau = \tau(t,x)$ (cf. [2, Lemma 3.4 (i)]).

Next, by Fubini's theorem and by changing variable $\xi = \varphi(s;\tau,0) = \varphi(s;t,x)$ (cf. [2, Lemma 3.4 (ii)]), we find that

$$
\begin{aligned}
L_2 + L_4 &= \int_0^t \int_{\tau_t^{-1}(s)}^{l} |\tilde{G}_{s,n}(u_n(\cdot,s))(\varphi(s;t,x)) - \tilde{G}_{s,0}(u_0(\cdot,s))(\varphi(s;t,x))| dx ds \\
&\le \int_0^t e^{L_V(t-s)} \int_0^l |\tilde{G}_{s,n}(u_n(\cdot,s))(\xi) - \tilde{G}_{s,0}(u_0(\cdot,s))(\xi)| d\xi ds,
\end{aligned}
$$

where $\tau_t^{-1}(s)$ is the inverse of $x \mapsto \tau(t,x)$. See [2, Lemma 3.3]. On the other hand, it follows from (V) and (G_n) that

$$
\begin{aligned}
& \|\tilde{G}_{s,n}(u_n(\cdot,s)) - \tilde{G}_{s,0}(u_0(\cdot,s))\|_{L^1} \\
& \le \|G_n(u_n(\cdot,s)) - G_n(u_0(\cdot,s))\|_{L^1} + \|G_n(u_0(\cdot,s)) - G_0(u_0(\cdot,s))\|_{L^1} \\
& \quad + \|V_x(\cdot,s)u_n(\cdot,s) - V_x(\cdot,s)u_0(\cdot,s)\|_{L^1} \\
& \le (c_G(r) + L_V)\|u_n(\cdot,s) - u_0(\cdot,s)\|_{L^1} + \|G_n(u_0(\cdot,s)) - G_0(u_0(\cdot,s))\|_{L^1}.
\end{aligned}
$$

Hence $L_2 + L_4$ is estimated as follows:

$$
\begin{aligned}
L_2 + L_4 \le & (c_G(r) + L_V) \int_0^t e^{L_V(t-s)} \|u_n(\cdot,s) - u_0(\cdot,s)\|_{L^1} ds \\
& + \int_0^T e^{L_V(t-s)} \|G_n(u_0(\cdot,s)) - G_0(u_0(\cdot,s))\|_{L^1} ds.
\end{aligned} \tag{3.6}
$$

Finally, by changing variable $\xi = \varphi(0;t,x)$ (cf. [2, Lemma 3.4]), it is easily seen that

$$
L_3 \le \int_{z(t)}^{l} |\nu_n(\varphi(0;t,x)) - \nu_0(\varphi(0;t,x))| dx \le e^{L_V t} \int_0^l |\nu_n(\xi) - \nu_0(\xi)| d\xi. \tag{3.7}
$$

Consequently, combining (3.5), (3.6) and (3.7), we obtain

$$\begin{aligned} &e^{-L_V t}\|u_n(\cdot,t)-u_0(\cdot,t)\|_{L^1} \\ &\le L(n)+(c_F(r)+c_G(r)+L_V)\int_0^t e^{-L_V s}\|u_n(\cdot,s)-u_0(\cdot,s)\|_{L^1}ds, \end{aligned}$$

where $L(n)$ is given by (3.3). Therefore, by Gronwall's lemma, the desired inequality (3.2) follows. Finally, the approximating conditions in (3.4) assure that $L(n) \to 0$ as $n \to \infty$ and the proof is complete. $\square$

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