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HARDY'S INEQUALITY ON FINITE NETWORKS

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Abstract. The smallest eigenvalue of a weighted discrete Laplacian is closely related to a generalized Hardy's inequality on networks. We shall estimate the smallest eigenvalue by using a discrete Kuramochi potential with some numerical experiments.

1. PROBLEM SETTING

Let X be a finite set of nodes, Y be a finite set of arcs and K be the node-arc incidence matrix. Assume that the graph $G := \{X, Y, K\}$ is connected and has no self-loop. For every two nodes $a, b \in X$, denote by $\rho(a, b)$ the geodesic distance between a and b, i.e., the minimum number of arcs in the paths between a and b.

For a strictly positive real-valued function r, $N := \{G, r\}$ is called a network. Denote by $L(X)$ the set of all real valued functions on X, by $L^+(X)$ the set of all nonnegative $u \in L(X)$.

For $u \in L(X)$, the discrete derivative du, the discrete Laplacian $\Delta u(x)$ and the Dirichlet sum $D(u)$ of u on N are defined by

$$\begin{aligned} du(y) &:= -r(y)^{-1} \sum_{x \in X} K(x, y) u(x), \\ \Delta u(x) &:= \sum_{y \in Y} K(x, y)[du(y)], \\ D(u) &:= \sum_{y \in Y} r(y)[du(y)]^2. \end{aligned}$$

The mutual Dirichlet sum $D(u, v)$ of $u, v \in L(X)$ is defined by

$$D(u, v) := \sum_{y \in Y} r(y)[du(y)][dv(y)].$$

Let A_0 be a nonempty subset of X such that $X \setminus A_0$ is connected and let $m \in L(X)$ satisfy $m(x) = 0$ on A_0 and $m(x) > 0$ on $X \setminus A_0$.

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A generalized Hardy's inequality is to find the best possible constant $C_m > 0$ such that

$$\sum\nolimits_{x\in X} m(x)u(x)^2 \le C_m D(u)$$

for all $u \in L(X)$ such that $u(x) = 0$ on A_0.

By special choices of N, A_0 and m, we obtain Wirtinger's inequality and Hardy's inequality in [2] and [3]. We shall show that $1/C_m$ is equal to the smallest eigenvalue of an eigenvalue problem. We shall also give an esimation of this value by using a discrete Kuramochi potential studied in [4] and [5].

2. Minimum eigenvalue

Let us put

$$L(X; A_0) := \{u \in L(X); u = 0 \text{ on } A_0\}.$$

For simplicity, let us put

$$\begin{aligned} ((u, v))_m &:= \sum\nolimits_{x\in X} m(x)u(x)v(x), \\ \|u\|_m &:= [((u, u))_m]^{1/2}, \\ \chi_m(u) &:= \frac{D(u)}{\|u\|_m^2}. \end{aligned}$$

We shall consider the extremum problem (H_m):

$$\begin{aligned} \text{Find} \quad \lambda_m &:= \inf\{\chi_m(u); u \in L(X; A_0)\} \\ &= \inf\{D(u); u \in L(X; A_0), \|u\|_m = 1\}. \end{aligned}$$

Proposition 2.1. *There exists an optimal solution $\tilde{u}$ of problem (H_m), i.e., $\lambda_m = D(\tilde{u}), \tilde{u} \in L(X; A_0)$ and $\|\tilde{u}\|_m = 1$.*

Proof. Let $\{v_k\}$ be a sequence in $L(X; A_0)$ such that $\chi_m(v_k) \to \lambda_m$ as $k \to \infty$. Put $u_k = v_k/\|v_k\|_m$. Then $\|u_k\|_m = 1$ and

$$\chi_m(u_k) = D(u_k) = D(v_k)/\|v_k\|_m^2 = \chi_m(v_k).$$

Since $\{u_k(x)\}$ is bounded for each $x \in X$, we may assume that $\{u_k\}$ converges pointwise to a function $\tilde{u} \in L(X; A_0)$. We have $\|\tilde{u}\|_m = 1$ and

$$\lim_{k\to\infty} D(u_k) = D(\tilde{u}),$$

so that $\chi_m(\tilde{u}) = \lambda_m$. □

Denote by $S(\lambda_m)$ be the set of all optimal solutions of problem (H_m), i.e.,

$$S(\lambda_m) := \{u \in L(X; A_0); \chi_m(u) = \lambda_m\}.$$

Consider the following eigenvalue problem of finding a number μ and a nonzero function $u \in L(X; A_0)$ which satisfy

$$(Eig) \qquad \Delta u(x) = -\mu m(x)u(x) \text{ on } X \setminus A_0.$$

Denote by $E_m(\Delta)$ the set of all μ satisfying (Eig) and by $EV_m(\mu)$ the set of nonzero functions u satisfying (Eig) with $\mu \in E_m(\Delta)$.

For every $u \in EV_m(\mu)$, we have

$$\begin{aligned} D(u) &= -\sum\nolimits_{x\in X}[\Delta u(x)]u(x) \\ &= \mu\sum\nolimits_{x\in X} m(x)u(x)^2 = \mu\|u\|_m^2. \end{aligned}$$

Since $D(u)$ is positive definite on the set $L(X; A_0)$, we see that $E_m(\Delta)$ consists of positive real numbers.

By the above observation, we have

Proposition 2.2. $\lambda_m = \min\{\mu; \mu \in E_m(\Delta)\}$.

Lemma 2.1. *$S(\lambda_m) = EV_m(\lambda_m)$.*

Proof. By the above observation, it suffices to show $S(\lambda_m) \subset EV_m(\lambda_m)$. Let $u \in S(\lambda_m)$. Denote by $\varepsilon_x \in L(X)$ the characteristic function of the set $\{x\}$. For any real number t and $x \in X \setminus A_0$, we have

$$\lambda_m = \chi_m(u) \le \chi_m(u + t\varepsilon_x),$$

or

$$\lambda_m\|u + t\varepsilon_x\|_m^2 \le D(u + t\varepsilon_x).$$

Noting the relation

$$D(u + t\varepsilon_x) = D(u) + 2tD(\tilde{u}, \varepsilon_x) + t^2 D(\varepsilon_x),$$
$$\|u + t\varepsilon_x\|_m^2 = \|u\|_m^2 + 2t((u, \varepsilon_x))_m + t^2\|\varepsilon_x\|_m^2,$$

we obtain

$$D(u, \varepsilon_x) = \lambda_m((u, \varepsilon_x))_m.$$

Since $D(u, \varepsilon_x) = -\Delta u(x)$ and $((u, \varepsilon_x))_m = m(x)u(x)$, we conclude that $u \in EV_m(\lambda_m)$. □

Lemma 2.2. *Assume that $u \in S(\lambda_m)$. Then $|u| \in S(\lambda_m)$ and $u(x_1)u(x_2) \ge 0$ for every $x_1, x_2 \in X \setminus A_0$ with $\rho(x_1, x_2) = 1$.*

Proof. Let $v = |u|$. Then $v \in L(X; A_0)$ and $D(v) \le D(u)$ holds (cf.[9]). Since $\|v\|_m = \|u\|_m$, we have

$$\lambda_m \le \chi_m(v) \le \chi_m(u) = \lambda_m,$$

and hence $v \in S(\lambda_m)$. Suppose that there exist $x_1, x_2 \in X \setminus A_0$ such that $\rho(x_1, x_2) = 1$ and $u(x_1)u(x_2) < 0$. Let $y' \in Y$ be an arc whose endpoints are x_1 and x_2. Then

$$\begin{aligned} |dv(y')| &= r(y')^{-1}|v(x_1) - v(x_2)| \\ &< r(y')^{-1}|u(x_1) - u(x_2)| = |du(y')|, \end{aligned}$$

so that $D(v) < D(u)$. Thus $\lambda_m = \chi_m(v) < \chi_m(u) = \lambda_m$. This is a contradiction. □

Corollary 2.1. *If $u \in S(\lambda_m)$, then either $u = |u|$ or $u = -|u|$.*

Lemma 2.3. *If $u \in S(\lambda_m)$ is non-negative, then $u(x) > 0$ on $X \setminus A_0$.*

Proof. Let $u \in S(\lambda_m)$ be nonnegative. By Lemma 2.1,

$$\Delta u(x) = -\lambda_m m(x)u(x) \leq 0 \quad \text{on } X \setminus A_0.$$

Namely u is superharmonic on $X \setminus A_0$. By the minimum principle (cf. [9]), we have $u(x) > 0$ on $X \setminus A_0$. □

Corollary 2.2. *If $u \in S(\lambda_m)$, then either $\Delta u(x) < 0$ on $X \setminus A_0$ or $\Delta u(x) > 0$ on $X \setminus A_0$.*

Lemma 2.4. *The dimension of $EV_m(\lambda_m)$ is one. Namely, if $u_1, u_2 \in EV_m(\lambda_m)$, then u_1 and u_2 are proportional.*

Proof. Assume that there exist $u_1, u_2 \in EV_m(\lambda_m)$ such that they are not proportional. Choose numbers α and β such that $|\alpha|+|\beta| > 0$ and $\alpha u_1(x_1)+\beta u_2(x_1) = 0$ for some $x_1 \in X \setminus A_0$. Let $u = \alpha u_1 + \beta u_2$. Then $u \neq 0$, since u_1 and u_2 are not proportional. We have

$$\begin{aligned} \Delta u(x) &= \alpha \Delta u_1(x) + \beta \Delta u_2(x) \\ &= -\lambda_m m(x)u_1(x) - \lambda_m m(x)u_2(x) \\ &= -\lambda_m m(x)u(x). \end{aligned}$$

Namely $u \in EV_m(\lambda_m) = S(\lambda_m)$. We have

$$\Delta u(x_1) = \lambda_m m(x)u(x_1) = 0.$$

This contadicts Corollary 2.2. □

Summing up these results, we obtain

Theorem 2.1. *There exists a unique $\tilde{u} \in L(X; A_0)$ such that*
(1) $\lambda_m = D(\tilde{u})$ *and* $\|\tilde{u}\|_m = 1$;
(2) $\tilde{u}(x) > 0$ *on* $X \setminus A_0$;
(3) $\Delta\tilde{u}(x) = -\lambda_m m(x)\tilde{u}(x)$ *on* $X \setminus A_0$.

3. Estimation of λ_m

Let us put

$$D(N; A_0) := \{u \in L(X; A_0); D(u) < \infty\}.$$

Since N is a finite network, we see that $D(N; A_0) = L(X; A_0)$. Notice that $D(N; A_0)$ is a Hilbert space with the inner product $D(u, v)$ (cf. [9]).

The Kuramochi function $\tilde{g}_x$ of N with pole at $x \in X \setminus A_0$ is defined by the *reproducing property*:

$$u(x) = D(u, \tilde{g}_x) \quad \text{for all } u \in D(N; A_0)$$

(cf. [4]). For each nonempty subset B of $X \setminus A_0$, let us put

$$d(A_0, B) := \inf\{D(u); u \in D(N; A_0),\ u = 1 \text{ on } B\}.$$

We have

Lemma 3.1. *$\tilde{g}_x$ has the following properties:*
(1) $\tilde{g}_x(z) = 0$ *on* A_0;
(2) $0 \le \tilde{g}_x \le \tilde{g}_x(x)$ *on* X;
(3) $\Delta\tilde{g}_x(z) = -\varepsilon_x(z)$ *on* $X \setminus A_0$.
(4) $d(A_0, \{x\}) = 1/\tilde{g}_x(x)$.

Now we shall estimate the value of λ_m. Our idea is to use the discrete Kuramochi function studied in [4] and [5]. A similar idea can be founded in [8] to estimate Lyapunov's inequality.

The Kuramochi potential $\tilde{G}m(x)$ of m is defined by

$$\tilde{G}m(x) := \sum_{z\in X} \tilde{g}_x(z)m(z).$$

Lemma 3.2. *Let $\tilde{u}$ be as in Theorem 2.1. Then*

$$\tilde{u}(x) = \lambda_m \sum_{z\in X} m(z)[\tilde{u}(z)]\tilde{g}_x(z).$$

Proof. By the reproducing property of the Kuramochi function and Lemma 3.1, we have

$$\begin{aligned}\tilde{u}(x) &= D(\tilde{u}, \tilde{g}_x) \\ &= -\sum_{z\in X}[\Delta\tilde{u}(z)]\tilde{g}_x(z) \\ &= \lambda_m \sum_{z\in X} m(z)[\tilde{u}(z)]\tilde{g}_x(z).\end{aligned}$$

□

Theorem 3.1. *The following estimation holds:*

$$\min\{\tilde{G}m(x); x \in X \setminus A_0\} \le \frac{1}{\lambda_m} \le \max\{\tilde{G}m(x); x \in X \setminus A_0\}.$$

Proof. Let $\tilde{u}$ be as in Theorem 2.1. There exists $b \in X \setminus A_0$ such that $\tilde{u}(b) = \max\{\tilde{u}(x); x \in X\}$. Then we have by Lemma 3.2

$$\begin{aligned}\tilde{u}(b) &= \lambda_m \sum_{z\in X} m(z)[\tilde{u}(z)]\tilde{g}_b(z) \\ &\le \lambda_m\tilde{u}(b) \sum_{z\in X} \tilde{g}_b(z)m(z) \\ &= \lambda_m\tilde{u}(b)\tilde{G}m(b) \\ &\le \lambda_m\tilde{u}(b)\max\{\tilde{G}m(x); x \in X \setminus A_0\}.\end{aligned}$$

We can prove the right hand side inequality similarly. □

Theorem 3.2. *Let $m(X) := \sum_{x\in X} m(x)$. Then the following estimation holds:*

$$\min\{d(A_0, \{x\}); x \in X \setminus A_0\} \le m(X)\lambda_m \le \max\{d(A_0, \{x\}); x \in X \setminus A_0\}.$$

Proof. Let $\tilde{u}$ be as in Theorem 2.1. There exists $b \in X \setminus A_0$ such that $\tilde{u}(b) = \max\{\tilde{u}(x); x \in X\}$. Then we have by Lemma 3.2

$$\begin{aligned} \tilde{u}(b) &= \lambda_m \sum\nolimits_{z\in X} m(z)[\tilde{u}(z)]\tilde{g}_b(z) \\ &\le \lambda_m \tilde{u}(b) \max\{\tilde{g}_x(x); x \in X \setminus A_0\} m(X) \\ &= \lambda_m \tilde{u}(b) \max\{1/d(A_0, \{x\}); x \in X \setminus A_0\}. \end{aligned}$$

□

4. Classical Hardy's inequality

In this section, we consider the following special finite network $N = \{X, Y, K, r\}$ defined by:

$$X = \{x_0, x_1, \cdots, x_n\}, Y = \{y_1, y_2, \cdots, y_n\}$$
$$K(x_i, y_i) = 1,\ K(x_{i-1}, y_i) = -1 \text{ for } i = 1, 2, \cdots, n$$

and $K(x, y) = 0$ for any other pair.
Notice that the graph $\{X, Y, K\}$ is a subgraph of the one-dimesional lattice domain $\mathbf{Z}$. For simplicity, we set

$$u_k := u(x_k), r_k := r(y_k), w_k = r_k^{-1}(u_k - u_{k-1}).$$

Then $\Delta u(x_k) = w_{k+1} - w_k$ for $1 \le k \le n-1$, $\Delta u(x_0) = w_1$ and $\Delta u(x_n) = -w_n$. Furthermore

$$D(u) = \sum_{k=1}^{n} r_k^{-1}(u_k - u_{k-1})^2 = \sum_{k=1} r_k w_k^2.$$

We shall prove

Theorem 4.1. *Let $A_0 := \{x_0\}$ and put $R_k = \sum_{j=1}^{k} r_j$. Then*

$$\sum_{k=1}^{\infty} r_k \left(\frac{u_k}{R_k}\right)^2 \le 4D(u)$$

for every $u \in L(X; A_0)$.

Proof. Let us put $v_k := u_k - u_{k-1}$ and $\alpha_k := u_k/R_k$. Then

$$\begin{aligned} r_k\alpha_k^2 - 2\alpha_k v_k &= r_k\alpha_k^2 - 2\alpha_k(\alpha_k R_k - \alpha_{k-1}R_{k-1}) \\ &= (r_k - 2R_k)\alpha_k^2 + 2R_{k-1}\alpha_k\alpha_{k-1} \\ &\le (r_k - 2R_k)\alpha_k^2 + R_{k-1}(\alpha_k^2 + \alpha_{k-1}^2) \\ &= -R_k\alpha_k^2 + R_{k-1}\alpha_{k-1}^2. \end{aligned}$$

Since $u_0 = 0$, we have

$$\sum_{k=1}^{n} (r_k\alpha_k^2 - 2\alpha_k v_k) \le \sum_{k=1}^{n} (-R_k\alpha_k^2 + R_{k-1}\alpha_{k-1}^2) = -R_n\alpha_n^2 \le 0.$$

Therefore we have

$$\begin{aligned}\sum_{k=1}^{n} r_k\alpha_k^2 &\le 2\sum_{k=1}^{n}\alpha_k v_k \\ &\le 2[\sum_{k=1}^{n} r_k\alpha_k^2]^{1/2}[\sum_{k=1}^{n} r_k^{-1}v_k^2]^{1/2},\end{aligned}$$

so that

$$\sum_{k=1}^{n} r_k\alpha_k^2 \le 4\sum_{k=1}^{n} r_k^{-1}v_k^2 = 4D(u).$$

□

Corollary 4.1. *Let* $A_0 = \{x_0\}$ *and* $m_k = m(x_k) := \frac{r_k}{R_k^2}$*. Then* $\lambda_m \ge 1/4$*.*

Corollary 4.2. *Assume that* $A_0 = \{x_0\}$ *and* $r_k = 1$ *for all* k*. Then*

$$\sum_{k=1}^{n}(\frac{u_k}{\rho(x_0,x_k)})^2 \le 4\sum_{k=1}^{n}(u_k - u_{k-1})^2$$

for all u_k $(k = 0, 1, \cdots, n)$ *with* $u_0 = 0$*.*

Notice that $\rho(x_0, x_k) = R_k$ and $\rho(x_0, x_k) = k$ in Corollary 4.2, this inequality can be found in [2], page 239. We may expect that Corollary 4.2 would also holds in the general case. However it is not true as shown by Table 4 in the next section.

Hereafter in this section we always assume that $A_0 = \{x_0\}$ and $m(x_k) := r_k/R_k^2$. In order to obtain the value λ_m, we calculate the minimum eigenvalue of (Eig) numerically:

$$\begin{aligned}-2u_1 + u_2 &= \mu m_1 u_1 \\ -2u_k + u_{k+1} + u_{k-1} &= \mu m_k u_k \quad \text{for } 2 \le k \le n-1 \\ -u_n + u_{n-1} &= \mu m_n u_n\end{aligned}$$

In order to study λ_m as a function of the size n of N, we denote it by $\lambda(n) := \lambda_m(n)$. Some numerical experiments are given in the next section.

In the present case, the Kuramochi function is given by

$$\tilde{g}_{x_k}(x_j) = \begin{cases} R_j & \text{for } \; 0 \le j \le k \\ R_k & \text{for } \; k < j \le n \end{cases}$$

We estimate λ_m by using the Kuramochi potential $\tilde{G}m$:

$$\tilde{G}m(x_k) = \sum_{j=1}^{k}\frac{r_j}{R_j} + R_k\sum_{j=k+1}^{n}\frac{r_j}{R_j^2}.$$

It is easily seen that

$$\mu^*(n) := \max\{\tilde{G}m(x); x \in X \setminus A_0\} = \sum_{k=1}^{n} \frac{r_k}{R_k}$$

$$\mu_*(n) := \min\{\tilde{G}m(x); x \in X \setminus A_0\} = \sum_{k=1}^{n} \frac{r_k}{R_k^2}.$$

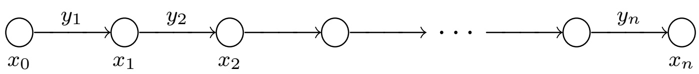
By Theorem 3.1, we have

$$\frac{1}{\mu^*(n)} \leq \lambda(n) \leq \frac{1}{\mu_*(n)}$$

Some numerical experiments for these quantities are also given in the next section.

5. Numerical experiments

Let $G = \{X, Y, K\}$ be the same graph as in Section 4. The graph can be drawn as follows:



We take $A_0 = \{x_0\}$ and $m(x_k) := r_k/R_k^2$ except in Table 4.

Table 1: The case where $r_k = 1$ for all k.

n	$\lambda(n)$	$1/\mu^*(n)$	$1/\mu_*(n)$
10	0.502934	0.341417	0.645258
100	0.376383	0.192776	0.611627
1000	0.318182	0.133592	0.608297

Table 2: The case where $r_k = 1/k$ for all k.

n	$1/\mu^*(n)$	$1/\mu_*(n)$
30	0.439971	0.625684
100	0.394713	0.604038
$10,000$	0.344817	0.583205

Calculus of the minimum eigenvalue:

n	$\lambda(n)$	Software
30	0.553865	*Mathematica*
100	0.518052	*Mathematica*
$10,000$	0.4564519	*Matlab*
$100,000$	0.4412748	*Matlab*

Table 3: The case where $r_k = 2^{1-k}$ for all k.

n	$1/\mu^*(n)$	$1/\mu_*(n)$	Software
15	0.622407	0.729114	*Mathematica*
20	0.622396	0.728854	*Mathematica*
28	0.622396	0.728854	*Mathematica*
29	0.622396	0.728854	*Mathematica*

Calculus by Mathematica shows that

$$1/\mu^*(n) = 0.622396 \quad for \quad n \geq 19$$
$$1/\mu_*(n) = 0.728854 \quad for \quad n \geq 19$$

Calculus of the minimum eigenvalue:

n	$\lambda(n)$	Software	
5	0.708196	*Mathematica*	
15	0.697629	*Mathematica*	
17	0.697622	*Mathematica*	
18	0.697625	*Mathematica*	increases
20	0.69765	*Mathematica*	increases
28	0.666465	*Mathematica*	decreases
29	0	*Mathematica*	absurd
29	0.697618	*Matlab*	

Calculus by Mathematica shows that $\lambda(n)$ becomes strange if $n \geq 18$.

Finally we change $m(x)$ slightly and estimate $\lambda_m(n)$ in this case.

Table 4: We choose $m(x_k) = \dfrac{1}{R_k^2}$ and $r_k = 2^{1-k}$. Then we obtain:

n	$\lambda_m(n)$	$1/\mu^*(n)$	$1/\mu_*(n)$
30	0.0663717	0.0632777	0.116446
100	1.68955?	0.0196387	0.0383323

Calculus by Mathematica shows that $\lambda_m(n)$ becomes strange if $n \geq 51$.
We remark that

$$\mu_*(n) = \tilde{G}m(x_1) = \sum_{k=1}^{n} \frac{1}{R_k^2} \to \infty$$

as $n \to \infty$, so that $\lambda_m(n) \to 0$ as $n \to \infty$.

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