EXTREMUM PROBLEMS ON A HILBERT NETWORK

MARETSUGU YAMASAKI

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Abstract. As a generalization of a usual infinite network, a Hilbert network is defined as a pair of a graph and a resistance taking values in a Hilbert space. With the sets of nodes and arcs of the graph, we associate variables belonging to a Hilbert space. In this situation, we study several extremum problems related to Hilbert-valued functions on the set of nodes or arcs of the graph and their inverse relations.

1. Introduction with preliminaries

Let $G = \{X, Y, K\}$ be a locally finite infinite graph which is connected and has no self-loop as in [4]. Here $X$ is a countable set of nodes, $Y$ is a countable set of arcs and $K$ is the node-arc incidence matrix.

Let $H$ be a real Hilbert space with an inner product $(\cdot, \cdot)$ and the norm $\|\cdot\|$. Denote by $L(X; H)$ the set of all functions $u$ on $X$ such that $u(x) \in H$ for each $x \in X$ and by $L_0(X; H)$ the set of all $u \in L(X; H)$ such that the support $\{x \in X; u(x) \neq 0\}$ is a finite set. The meaning of the notation $L(Y; H)$ and $L_0(Y; H)$ is similar. Let $\mathcal{L}(H)$ be the set of all bounded, linear, positive and invertible linear operators from $H$ to $H$. Assume that $r \in L(Y; \mathcal{L}(H))$. This is a generalization of the resistance in the usual network theory as in [3] and [4]. We call the pair $N = \{G, r\}$ of the graph $G$ and this generalized resistance $r$ a Hilbert network as in [1], [6] and [7].

For each $y \in Y$, there exists $\rho(y) > 0$ by our assumption (cf. [5]) such that

$$(r(y)h, h) \geq \rho(y)\|h\|^2 \quad \text{for all} \quad h \in H.$$ 

Here $r(y)h$ means the image of $h$ under $r(y)$, i.e., $r(y)(h)$. In this paper, we use this convention unless no confusion occurs from the context. Denote by $r(y)^{-1}$ the inverse operator of $r(y)$. Notice that there exists $\rho^*(y) > 0$ such that

$$(r(y)^{-1}h, h) \geq \rho^*(y)\|h\|^2 \quad \text{for all} \quad h \in H.$$ 

For each $y \in Y$, there exists a unique square root $r(y)^{1/2} \in \mathcal{L}(H)$ of $r(y)$ by [2] i.e.,

$$[r(y)^{1/2}]^2 = r(y).$$

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Before introducing extremum problems on the Hilbert network $N$, we need several preparations.

**Definition 1.1.** Let $e$ be a fixed element of $\mathcal{H}$ such that $\|e\| = 1$.

**Definition 1.2.** For $u \in L(X; \mathcal{H})$, the potential drop $\delta u$ of $u$ and the discrete derivative $du$ of $u$ are defined by

$$\delta u(y) := \sum_{x \in X} K(x, y)u(x),$$

$$du(y) := -r(y)^{-1}(\delta u(y)) = -r(y)^{-1}\delta u(y).$$

The Dirichlet sum of $u$ is defined by

$$D(u) := \sum_{y \in Y} ((r(y)du(y), du(y))) = \sum_{y \in Y} ((r(y)^{-1}\delta u(y), \delta u(y))).$$

**Definition 1.3.** For $w \in L(Y; \mathcal{H})$, the divergence $\partial w(x)$ of $w$ and the energy $H(w)$ of $w$ are defined by

$$\partial w(x) := \sum_{y \in Y} K(x, y)w(y),$$

$$H(w) := \sum_{y \in Y} ((r(y)w(y), w(y))).$$

Notice that $D(u) = H(du)$. Let us put

$$D(N; \mathcal{H}) := \{ u \in L(X; \mathcal{H}) ; D(u) < \infty \},$$

$$L_H(Y; \mathcal{H}) := \{ w \in L(Y; \mathcal{H}) ; H(w) < \infty \}.$$

For every $w_1, w_2 \in L_H(Y; \mathcal{H})$, we define the inner product $H(w_1, w_2)$ by

$$H(w_1, w_2) := \sum_{y \in Y} ((r(y)w_1(y), w_2(y))).$$

For every $u_1, u_2 \in D(N; \mathcal{H})$, we define the mutual Dirichlet sum $D(u_1, u_2)$ by

$$D(u_1, u_2) := H(du_1, du_2) = \sum_{y \in Y} ((r(y)^{-1}\delta u_1(y), \delta u_2(y))).$$

**Lemma 1.1.** Let $h \in \mathcal{H}$. For every $y \in Y$, the following relations hold:

1. $|((r(y)w(y), h))|^2 \leq ((r(y)w(y), w(y)))((r(y)h, h))$.
2. $1 \leq ((r(y)^{-1}h, h))((r(y)h, h))$.

**Proof.** By the Schwarz inequality, we have

$$|((r(y)w(y), h))|^2 = |((r(y)^{1/2}w(y), r(y)^{1/2}h))|^2 \leq \|r(y)^{1/2}w(y)\|^2 \|r(y)^{1/2}h\|^2 = ((r(y)w(y), w(y)))((r(y)h, h)).$$

(2) follows from (1) by taking $w(y) := r(y)^{-1}h$. \qed

**Lemma 1.2.** $|H(w_1, w_2)| \leq H(w_1)^{1/2}H(w_2)^{1/2}$. 

Proof. From the Schwarz inequality, it follows that
\[ |H(w_1, w_2)| \leq \sum_{y \in Y} |(r(y)w_1(y), w_2(y))| \]
\[ = \sum_{y \in Y} |(r(y)^{1/2}w_1(y), r(y)^{1/2}w_2(y))| \]
\[ \leq \sum_{y \in Y} \|r(y)^{1/2}w_1(y)\| \|r(y)^{1/2}w_2(y)\| \]
\[ \leq \left( \sum_{y \in Y} \|r(y)^{1/2}w_1(y)\|^2 \right)^{1/2} \left( \sum_{y \in Y} \|r(y)^{1/2}w_2(y)\|^2 \right)^{1/2} \]
\[ = H(w_1)^{1/2}H(w_2)^{1/2}. \]

Notice that \( L_H(Y; \mathcal{H}) \) is a Hilbert space with this inner product.

Lemma 1.3. If \( w \in L_0(Y; \mathcal{H}) \), then \( \sum_{y \in Y} r(y)w(y) \in \mathcal{H} \) and
\[ \sum_{y \in Y} (r(y)w(y), h) = (\sum_{y \in Y} r(y)w(y), h) \]
for every \( h \in \mathcal{H} \).

Proof. Since \( r(y)w(y) \in \mathcal{H} \) for every \( y \in Y \) and \( w \in L_0(Y; \mathcal{H}) \), our assertion is clear. \( \Box \)

For \( a \in X \), let us put
\[ D(N; \mathcal{H}; a) := \{ u \in D(N; \mathcal{H}); u(a) = 0 \} \].

Lemma 1.4. For any \( x \in X \), there exists a constant \( M_x \) which such that
\[ \|u(x)\| \leq M_x D(u)^{1/2} \]
for all \( u \in D(N; \mathcal{H}; a) \).

Proof. We may assume that \( x \neq a \). There exists a path \( P \) from \( a \) to \( x \). Let \( C_X(P) \) and \( C_Y(P) \) be the sets of nodes and arcs on \( P \) respectively (cf. [4]), i.e.,
\[ C_X(P) := \{ x_0, x_1, \cdots, x_n \} \ (x_0 = a, x_n = x) \],
\[ C_Y(P) := \{ y_1, y_2, \cdots, y_n \} \],
\[ \{ x \in X; K(x, y_i) \neq 0 \} = \{ x_i-1, x_i \} \ (i = 1, 2, \cdots, n) \].

Let \( u \in D(N; \mathcal{H}; a) \). Then we have
\[ D(u) \geq \sum_{y \in C_Y(P)} (r(y)^{-1}\delta u(y), \delta u(y)) \]
\[ = \sum_{i=1}^n (r(y_i)^{-1}\delta u(y_i), \delta u(y_i)) \]
\[ \geq \sum_{i=1}^n \rho^*(y_i) \|u(x_i) - u(x_{i-1})\|^2 \]
\[ \geq \sum_{i=1}^n \rho^*(y_i) \|u(x_i)\| - \|u(x_{i-1})\| \]
so that
\[ \|u(x_i)\| - \|u(x_{i-1})\| \leq D(u)^{1/2}[\rho^*(y_i)]^{-1/2} \]
the closure of quantities:

For each Definition 2.2. $(F.2)$ $\partial w$ is an $(F.1)$ $\partial w$

Let Definition 2.1. in the Hilbert space for all $n, m$

See that $\tilde{k}$ such that $k$ is a Cauchy sequence in

Proof. Let $D(u, m) => 0$ as $n, m \to \infty$. Then $\{D(u_n)\}$ is bounded. It follows from Lemma 1.4 that $\{u_n(x)\}$ is a Cauchy sequence in $H$ for each $x \in X$. Therefore there exists $\tilde{u}(x) \in H$ such that $\|u_n(x) - \tilde{u}(x)\| \to 0$ as $n \to \infty$ for each $x \in X$. Thus $\tilde{u}(a) = 0$ and $\|du_n(y) - d\tilde{u}(y)\| \to 0$ as $n \to \infty$ for each $y \in Y$. Since $\{D(u_n)\}$ is bounded, we see that $\tilde{u} \in D(N; H)$ by Fatou's lemma. For any $\epsilon > 0$, there exists $n_0$ such that $D(u_n - u_m) < \epsilon^2$ for all $n, m \geq n_0$. For any finite subset $Y'$ of $Y$,

$$\sum_{y \in Y'} ((r(y)d(u_n - u_m)(y), d(u_n - u_m)(y))) \leq D(u_n - u_m).$$

Letting $m \to \infty$, we have

$$\sum_{y \in Y'} ((r(y)d(u_n - \tilde{u})(y), d(u_n - \tilde{u})(y))) \leq \epsilon^2$$

for all $n \geq n_0$, so that $D(u_n - \tilde{u}) \leq \epsilon^2$. Hence, $D(u_n - \tilde{u}) \to 0$ as $n \to \infty$. \hfill $\Box$

Denote by $D_0(N; H; a)$ the closure of the set

$$L_0(X; H; a) := \{u \in L_0(X; H); u(a) = 0\}$$

in the Hilbert space $D(N; H; a)$.

2. $H$-flows

Definition 2.1. Let $a$ and $b$ be distinct two nodes. We say that $w \in L(Y; H)$ is an $H$-flow from $a$ to $b$ if the following conditions are fulfilled:

$(F.1)$ $\partial w(x) = 0$ for all $x \in X \setminus \{a, b\}$;

$(F.2)$ $\partial w(a) + \partial w(b) = 0$.

Denote by $F(a, b; H)$ the set of all $H$-flows from $a$ to $b$.

Definition 2.2. For each $w \in F(a, b; H)$, we introduce the following two quantities:

$$I_e(w) := ((\partial w(b), e)) = -((\partial w(a), e)),$$

$$I(w) := \|\partial w(a)\| = \|\partial w(b)\|.$$

Let us put $F_0(a, b; H) := F(a, b; H) \cap L_0(Y; H)$ and denote by $F_H(a, b; H)$ the closure of $F_0(a, b; H)$ in $L_H(Y; H)$. 

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Lemma 2.1. Assume that $N$ is a finite network. If $w \in L(Y; \mathcal{H})$ satisfies (F.1), then it does also (F.2).

**Proof.** Since $N$ is a finite network and
\[
\sum_{x \in X} K(x, y) = 0
\]
for each $y \in Y$, we have by changing the order of summation
\[
\partial \tilde{w}(a) + \partial \tilde{w}(b) = \sum_{x \in X} \partial \tilde{w}(x) = \sum_{y \in Y} \left[ \sum_{x \in X} K(x, y) \right] \tilde{w}(y) = 0. \square
\]

Similarly we have

Lemma 2.2. If $w \in L_0(Y; \mathcal{H})$ satisfies (F.1), then it does (F.2).

Corollary 2.1. (F.1) implies (F.2) for every $w \in F_H(a, b; \mathcal{H})$.

Lemma 2.3. Let $u \in L(X; \mathcal{H})$ and $w \in L_0(Y; \mathcal{H})$. Then
\[
\sum_{y \in Y} ((w(y), \delta u(y))) \leq H(w)^{1/2} D(u)^{1/2}.
\]

**Proof.** We have by Lemma 1.2
\[
\sum_{y \in Y} ((w(y), \delta u(y))) = H(w, du) \leq H(w)^{1/2} H(du)^{1/2} \leq H(w)^{1/2} D(u)^{1/2}.
\]

Corollary 2.2. Let $u \in D(N; \mathcal{H})$ and $w \in F_H(a, b; \mathcal{H})$. Then
\[
\sum_{y \in Y} ((w(y), \delta u(y))) \leq H(w)^{1/2} D(u)^{1/2}.
\]

**Proof.** There exists a sequence $\{w_n\}$ in $F_0(a, b; \mathcal{H})$ such that $H(w_n - w) \to 0$ as $n \to \infty$. We have by Lemma 2.3 $H(w_n, du) \leq H(w_n)^{1/2} D(u)^{1/2}$. Since $du \in L_H(Y; \mathcal{H})$, we see that $H(w_n, du) = H(w, du)$ and $H(w_n) = H(w)$ as $n \to \infty$. \square

Lemma 2.4. Let $u \in D(N; \mathcal{H})$ and $w \in F_H(a, b; \mathcal{H})$. Then
\[
\sum_{x \in X} ((u(x), \partial w(x))) = \sum_{y \in Y} ((\delta u(y), w(y))).
\]

**Proof.** There exists a sequence $\{w_n\}$ in $F_0(a, b; \mathcal{H})$ such that $H(w_n - w) \to 0$ as $n \to \infty$. Since the support of $w_n$ is a finite set, we have
\[
((u(a), \partial w_n(a))) + ((u(b), \partial w_n(b))) = \sum_{x \in X} ((u(x), \partial w_n(x))) = \sum_{y \in Y} ((\delta u(y), w_n(y))) = H(du, w_n).
\]

By letting $n \to \infty$, we obtain the desired inequality, since $du \in L_H(Y; \mathcal{H})$ and $\partial w(x) = 0$ for $x \in X \setminus \{a, b\}$. \square

Denote by $C_0(N)$ the set of all finite cycles on $N$, i.e.,
\[
C_0(N) := \{ \omega \in L_0(Y; \mathcal{H}); \partial \omega(x) = 0 \text{ on } X \}.
\]

Lemma 2.5. Let $\tilde{w} \in F(a, b; \mathcal{H})$ such that $H(\tilde{w}) < \infty$. Suppose that $H(\tilde{w}, \omega) = 0$ for every $\omega \in C_0(N)$. Then there exists $\tilde{\omega} \in D(N; \mathcal{H}; a)$ such that $d\tilde{\omega} = -\tilde{w}$. 
Lemma 2.3

Let \( p_1, p_2 \) be path indices of paths from \( a \) to \( x \) (cf. [4]). First we shall prove

\[
\sum_{y \in Y} p_1(y)r(y)\tilde{w}(y) = \sum_{y \in Y} p_2(y)r(y)\tilde{w}(y).
\]

In fact, for any \( h \in \mathcal{H} \), \( \omega(y) := (p_1(y) - p_2(y))h \) belongs to \( C_0(N) \), so that we have by our assumption

\[
0 = H(\tilde{w}, (p_1 - p_2)h) = \sum_{y \in Y} ((r(y)(p_1(y) - p_2(y))\tilde{w}(y)), h)).
\]

Since \((p_1 - p_2)\tilde{w} \in L_0(Y; \mathcal{H})\), we see by Lemma 1.3.

\[
((\sum_{y \in Y} r(y)((p_1(y) - p_2(y))\tilde{w}(y)), h)) = 0.
\]

Since \( h \in \mathcal{H} \) is arbitrary, our assertion follows. Define \( \tilde{u} \in L(X; \mathcal{H}) \) by \( \tilde{u}(a) = 0 \) and

\[
\tilde{u}(x) := \sum_{y \in Y} p_x(y)\tilde{w}(y) \text{ for } x \neq a,
\]

where \( p_x \) is the path index of a path from \( a \) to \( x \). This function is well-defined by the above observation. Let \( y' \in Y \) and \( \{x \in X; K(x, y') \neq 0\} = \{x_1, x_2\} \). Let \( p_{x_2} \) be the path index of a path \( P_{x_2} \) from \( a \) to \( x_2 \) which passes the arc \( y' \) after the node \( x_1 \). Namely \( P_{x_2} \) consists of a path \( P_{x_1} \) from \( a \) to \( x_1 \) and the single arc \( y' \). We have

\[
\tilde{u}(x_2) = \sum_{y \in Y} p_{x_2}(y)\tilde{w}(y) = \sum_{y \in Y} p_{x_1}(y)\tilde{w} + r(y')K(x_1, y')\tilde{w}(y') = \tilde{u}(x_1) + r(y')K(x_1, y')\tilde{w}(y'),
\]

so that \( \tilde{u}(x_2) = \tilde{u}(x_1) + r(y')K(x_1, y')\tilde{w}(y'), \) or \( \delta\tilde{u}(y') = -r(y')\tilde{w}(y') \).

\[ \square \]

3. INVERSE RELATION I

Now let us consider the following pair of extremum problems on the Hilbert network \( N \) which are related to \( \mathcal{H} \)-valued functions on \( X \) or \( Y \):

\[
d_*(a, b; \mathcal{H}) := \inf\{D(w); u \in L(X; \mathcal{H}), ((u(a), e)) = 0, ((u(b), e)) = 1\},
\]

\[
d^*(a, b; \mathcal{H}; e) := \inf\{H(w); w \in F_H(a, b; \mathcal{H}), \partial w(b) = e\}
\]

First we have

**Theorem 3.1.** \( 1 \leq d_*(a, b; \mathcal{H})d^*(a, b; \mathcal{H}, e) \).

**Proof.** Let \( u \) be a feasible solution for \( d_*(a, b; \mathcal{H}) \) and let \( w \) be a feasible solution for \( d^*(a, b; \mathcal{H}; e) \). It suffices to show that \( 1 \leq H(w^{1/2})D(u^{1/2}) \). There exists a sequence \( \{w_n\} \) in \( F_0(a, b; \mathcal{H}) \) such that \( H(w - w_n) \to 0 \) as \( n \to \infty \). We have by Lemma 2.3

\[
1 = ((u(b), e)) = ((u(b), \partial w(b))) = \lim_{n \to \infty} ((u(b), \partial w_n(b))) = \lim_{n \to \infty} \sum_{x \in X} ((u(x), \partial w_n(x))) = \lim_{n \to \infty} \sum_{y \in Y} ((\delta u(y), w_n(y))) \leq \lim_{n \to \infty} H(w_n^{1/2}D(u)^{1/2}) = H(w)^{1/2}D(u)^{1/2}. \quad \square
\]

To prove the converse inequality, we prepare
Lemma 3.1. There exists a unique optimal solution for $d^*(a, b; \mathcal{H}; e)$.

Proof. Let $\{w_n\}$ be a minimizing sequence for $d^*(a, b; \mathcal{H}; e)$, i.e., $\{w_n\} \subset F_H(a, b; \mathcal{H})$, $\partial w_n(b) = e$ and $H(w_n) \rightarrow d^*(a, b; \mathcal{H}; e)$ as $n \rightarrow \infty$. Since $(w_n + w_m)/2$ is a feasible solution for $d^*(a, b; \mathcal{H}; e)$, we have

$$d^*(a, b; \mathcal{H}; e) \leq H((w_n + w_m)/2) \leq H((w_n + w_m)/2) + H((w_n - w_m)/2) = [H(w_n) + H(w_m)]/2 \rightarrow d^*(a, b; \mathcal{H}; e)$$

as $m, n \rightarrow \infty$. Therefore $H(w_n - w_m) \rightarrow 0$ as $n, m \rightarrow \infty$. It follows that $\{w_n\}$ is a Cauchy sequence in the Hilbert space $L_H(Y; \mathcal{H})$. There exists $\tilde{w} \in L_H(Y; \mathcal{H})$ such that $H(w_n - \tilde{w}) \rightarrow 0$ as $n \rightarrow \infty$. Then $\tilde{w} \in F_H(a, b; \mathcal{H})$, $\partial \tilde{w}(b) = e$ and $d^*(a, b; \mathcal{H}; e) = H(\tilde{w})$. Namely $\tilde{w}$ is an optimal solution for $d^*(a, b; \mathcal{H}; e)$. Since $H(w)$ is a strictly convex function of $w \in L_H(Y; \mathcal{H})$, the uniqueness of the optimal solution follows. \hfill \Box

Lemma 3.2. Let $\tilde{w}$ be the optimal solution for $d^*(a, b; \mathcal{H}; e)$. Then $H(\tilde{w}, \omega) = 0$ for every $\omega \in C_0(N)$.

Proof. For any $\omega \in C_0(N)$ and $t \in \mathbb{R}$, $\tilde{w} + t\omega$ is a feasible solution for $d^*(a, b; \mathcal{H}; e)$. Thus

$$H(\tilde{w}) \leq H(\tilde{w} + t\omega) = H(\tilde{w}) + 2tH(\tilde{w}, \omega) + t^2H(\omega).$$

By the standard variational argument, we obtain $H(\tilde{w}, \omega) = 0$. \hfill \Box

Lemma 3.3. Let $\tilde{w}(y)$ be the same as above. There exists $\hat{u} \in D(N; \mathcal{H})$ such that $\hat{u}(a) = 0$, $((\hat{u}(b), e)) = d^*(a, b; \mathcal{H}; e)$ and $\delta \hat{u} = -\tilde{w}$.

Proof. Let $\hat{u}$ be the function defined by $\tilde{w}$ in Lemma 3.2. Then $\hat{u}(a) = 0$ and $d\hat{u} = -\tilde{w}$. There exists $\{w_n\} \subset F_0(a, b; \mathcal{H})$ such that $H(w_n - \tilde{w}) \rightarrow 0$ as $n \rightarrow \infty$. Let $p_b$ a path index of a path from $a$ to $b$. Since $w_n - p_b\partial w_n(b) \in C_0(N)$, we have $H(w_n, w_n - p_b\partial w_n(b)) = 0$. From $\partial w_n(b) \partial w_n(b) = e$, it follows that $H(\tilde{w}, \tilde{w} - p_b e) = 0$, so that

$$d^*(a, b; \mathcal{H}; e) = H(\tilde{w}) = H(\tilde{w}, p_b e) = ((\hat{u}(b), e)).$$

\hfill \Box

Theorem 3.2. $d_e(a, b; \mathcal{H})d^*(a, b; \mathcal{H}; e) = 1$.

Proof. Let $\hat{w}$ be the optimal solution for $d^*(a, b; \mathcal{H}; e)$ and let $\hat{u}$ be the function defined in Lemma 3.3. Then $v := \hat{u}/d^*(a, b; \mathcal{H}; e)$ is a feasible solution for $d_e(a, b; \mathcal{H})$ and

$$d_e(a, b; \mathcal{H}) \leq D(v) = D(\hat{u})/d^*(a, b; \mathcal{H}; e)^2 = H(\hat{w})/(d^*(a, b; \mathcal{H}; e)^2 = 1/d^*(a, b; \mathcal{H}; e),$$

so that $d_e(a, b; \mathcal{H})d^*(a, b; \mathcal{H}; e) \leq 1$. Thus the equality holds by Theorem 3.1. \hfill \Box
4. INVERSE RELATION II

Let us consider further extremum problems on the Hilbert network $N$:
\[
\begin{align*}
d(a, b; \mathcal{H}; e) &:= \inf\{D(u); u \in L(X; \mathcal{H}), u(a) = 0, u(b) = e\}, \\
d(a, b; \mathcal{H}) &:= \inf\{D(u); u \in L(X; \mathcal{H}), u(a) = 0, \|u(b)\| = 1\}, \\
d^*_e(a, b; \mathcal{H}) &:= \inf\{H(w); w \in F_H(a, b; \mathcal{H}), I_e(w) = 1\}, \\
d^*(a, b; \mathcal{H}) &:= \inf\{H(w); w \in F_H(a, b; \mathcal{H}), I(w) = 1\}.
\end{align*}
\]
Clearly
\[
d_e(a, b; \mathcal{H}) \leq d(a, b; \mathcal{H}; e), \quad d(a, b; \mathcal{H}) \leq d(a, b; \mathcal{H}; e),
\]
\[
d^*_e(a, b; \mathcal{H}) \leq d^*(a, b; \mathcal{H}; e), \quad d^*(a, b; \mathcal{H}) \leq d^*(a, b; \mathcal{H}; e).
\]

We have

**Theorem 4.1.** $1 \leq d(a, b; \mathcal{H}; e)d^*_e(a, b; \mathcal{H})$.

**Proof.** It suffices to show that $1 \leq H(w)^{1/2}D(u)^{1/2}$ holds for any feasible solution $u$ for $d(a, b; \mathcal{H}; e)$ and any feasible solution $w$ for $d^*_e(a, b; \mathcal{H})$. By the corollary of Lemma 2.3 and Lemma 2.4, we have
\[
1 = I_e(w) = ((\partial w(b), e)) = \sum_{x \in X}((\partial w(x), u(x)))
\]
\[
= \sum_{y \in Y}((w(y), \delta u(y)))
\]
\[
\leq H(w)^{1/2}D(u)^{1/2}.
\]

To prove the converse inequality, we prepare

**Lemma 4.1.** There exists a unique optimal solution for $d(a, b; \mathcal{H}; e)$.

**Proof.** Let $\{u_n\}$ be a minimizing sequence for $d(a, b; \mathcal{H}; e)$, i.e., $\{u_n\} \subset D(N; \mathcal{H}; a)$, $u_n(b) = e$ and $D(u_n) \to d(a, b; \mathcal{H}; e)$ as $n \to \infty$. Since $(u_n + u_m)/2$ is a feasible solution for $d(a, b; \mathcal{H}; e)$, we have
\[
d(a, b; \mathcal{H}; e) \leq D((u_n + u_m)/2)
\]
\[
\leq D((u_n + u_m)/2) + D((u_n - u_m)/2)
\]
\[
= [D(u_n) + D(u_m)]/2 \to d(a, b; \mathcal{H}; e)
\]
as $n \to \infty$. Therefore $D(u_n - u_m) \to 0$ as $n, m \to \infty$. It follows from Proposition 1.1 that there exists $\tilde{u} \in D(N; \mathcal{H}; a)$ such that $D(u_n - \tilde{u}) \to 0$ as $n \to \infty$. Clearly $\tilde{u}(b) = e$ and $\alpha = D(\tilde{u})$. Namely $\tilde{u}$ is an optimal solution. The uniqueness of the optimal solution follows from the fact that $D(u)$ is strict convex on $D(N; \mathcal{H}; a)$.

**Lemma 4.2.** Assume that $N$ is a finite network. Let $\tilde{u}$ be the optimal solution for $d(a, b; \mathcal{H}; e)$ and put $\tilde{w}(y) := d\tilde{u}(y)$. Then $\tilde{w} \in F(a, b; \mathcal{H})$ and $I_e(\tilde{w}) = D(\tilde{u})$.

**Proof.** Let $f \in D(N; \mathcal{H})$ satisfy $f(a) = f(b) = 0$. Then, for any $t \in \mathbb{R}$, $\tilde{u} + tf$ is a feasible solution for $d(a, b; \mathcal{H}; e)$, so that
\[
D(\tilde{u}) \leq D(\tilde{u} + tf) = D(\tilde{u}) + 2tD(\tilde{u}, f) + t^2D(f).
\]
By the standard variational argument, we have $D(\hat{u}, f) = 0$. On the other hand, we have
\[
D(\hat{u}, f) = \sum_{y \in Y} \left( (\hat{w}(y), \sum_{z \in X} K(z, y) f(z)) \right)
= \sum_{z \in X} \sum_{y \in Y} \left( (K(z, y) \hat{w}(y), f(z)) \right)
= \sum_{z \in X} \left( (\partial \hat{w}(z), f(z)) \right).
\]

Denote by $\varepsilon_x$ the characteristic function of $\{x\}$, i.e., $\varepsilon_x(x) = 1$ and $\varepsilon_x(z) = 0$ for $z \neq x$. Let $x \neq a, b$. For any $h \in \mathcal{H}$, we may take $\varepsilon_x h$ for $f$, which leads to
\[
((\partial \hat{w}(x), h)) = 0.
\]

Therefore $\partial \hat{w}(x) = 0$ for $x \neq a, b$. Namely $\hat{w}$ satisfies (F.1). Since $N$ is a finite network, we have $\hat{w} \in F(a, b; \mathcal{H})$ by Lemma 2.1. By taking $\hat{u} = \varepsilon_b e$ for $f$, we obtain $D(\hat{u}, \hat{u} - \varepsilon_b e) = 0$, so that
\[
D(\hat{u}) = D(\hat{u}, \varepsilon_b e) = ((\partial \hat{w}(b), e))
\]
Therefore $I_\varepsilon(\hat{w}) = D(\hat{u})$. \qed

**Theorem 4.2.** Assume that $N$ is a finite network. Then the inverse relation $d(a, b; \mathcal{H}; e) d^*_e(a, b; \mathcal{H}) = 1$ holds.

**Proof.** Let $\hat{u}$ be the optimal solution for $d(a, b; \mathcal{H}; e)$ and let $\hat{w} = d \hat{u}$. We see by Lemma 4.2 that $\hat{w}(y)/D(\hat{u})$ is a feasible solution for $d^*_e(a, b; \mathcal{H})$, so that
\[
d^*_e(a, b; \mathcal{H}) \leq H(\hat{w}(y)/D(\hat{u}))
= D(\hat{u})/D(\hat{u})^2
= 1/D(\hat{u}) = 1/d(a, b; \mathcal{H}; e).
\]

Thus $d(a, b; \mathcal{H}; e) d^*_e(a, b; \mathcal{H}) \leq 1$. \qed

In order to establish the equality in Theorem 4.2 in the case where $N$ is an infinite network, we consider an exhaustion $\{G_n\}(G_n := < X_n, y_n >)$ of $G$ (cf. [4]) with $a, b \in X_1$. A Hilbert subnetwork $N_n$ of $N$ is defined as the pair of the
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On each finite subnetwork $N_n$, we define the Dirichlet mutual sum of $u_1, u_2 \in L(X_n; \mathcal{H})$ by
\[
D_n(u_1, u_2) := \sum_{y \in Y_n} ((r(y) du_1(y), du_2(y)))
\]
and put $D_n(u) = D_n(u, u)$. For $w \in L(Y_n; \mathcal{H})$, we define $H_n(w)$ and $\partial_n w$ by
\[
H_n(w) := \sum_{y \in Y_n} ((r(y) w(y), w(y)))
\]
\[
\partial_n w(x) := \sum_{y \in Y_n} K(x, y) w(y).
\]

For large $n$, we have $\partial_n w(a) = \partial w(a)$ and $\partial_n w(b) = \partial w(b)$. Let us consider the following extremum problems on $N_n$:
\[
d_n := d(a, b; N_n; \mathcal{H}; e) := \inf\{ D_n(u); u \in L(X_n; \mathcal{H}), u(a) = 0, u(a) = e \},
\]
\[
d^*_n := d^*_e(a, b; N_n; \mathcal{H}) := \inf\{ H_n(w); w \in F_n(a, b; \mathcal{H}), ((\partial_n w(b), e)) = 1 \},
\]
where $F_n(a, b; \mathcal{H}) := \{ w \in L(Y_n; \mathcal{H}); \partial_n w(x) = 0 \text{ on } X_n \setminus \{a, b\} \}$. 

Corollary 4.1. \( \{d_e^*(a, b; N_n; \mathcal{H})\} \) converges to \( d_e^*(a, b; \mathcal{H}) \) as \( n \to \infty \).

**Proof.** Let \( \tilde{u} \) and \( u_n \) be the optimal solutions of \( d(a, b; \mathcal{H}'; e) \) and \( d_n \) respectively. Then for every \( f \in L(X_n; \mathcal{H}) \) satisfying \( f(a) = f(b) = 0 \), we have \( D_n(u_n, f) = 0 \) as in the proof of Lemma 4.2. For \( n < m \), we have
\[
D_n(\tilde{u} - u_n, u_n) = 0 \quad \text{and} \quad D_n(u_m - u_n, u_n) = 0.
\]
Furthermore
\[
D_n(u_n) \leq D_n(\tilde{u}) \leq D(\tilde{u}) < \infty.
\]
By the relation
\[
0 \leq D_n(u_m - u_n) = D_n(u_m) - D_n(u_n) \leq D_m(u_m) - D_n(u_n),
\]
we see that the limit of \( \{D_n(u_n)\} \) exists, and hence
\[
\lim_{n \to \infty} D_n(u_m - u_n) = 0.
\]
For \( k < n < m \), we have
\[
D_k(u_m - u_n) \leq D_n(u_m - u_n) \to 0 \quad (n \to \infty).
\]
Thus \( \{u_n\} \) is a Cauchy sequence with respect to \( D_k \), and the limit of \( \{u_n(x)\} \)
equals for all \( x \in X_k \) both in the sense of \( D_k \) and in the sense of norm convergence in \( \mathcal{H} \). Let \( v \) be the limit of \( \{u_n\} \). Then \( v(a) = 0 \) and \( v(b) = e \), so that \( D(\tilde{u}) \leq D(v) \). Since \( D_k(u_n) \leq D_n(u_n) \) if \( k \leq n \), we have
\[
D_k(v) = \lim_{n \to \infty} D_k(u_n) \leq \lim_{n \to \infty} D_n(u_n) \leq D(\tilde{u}).
\]
Letting \( k \to \infty \), we obtain \( D(v) \leq D(\tilde{u}) \), and hence \( D(v) = D(\tilde{u}) \). By the uniqueness of the optimal solution, we have \( v = \tilde{u} \) and
\[
\lim_{n \to \infty} D_n(u_n) = D(\tilde{u}). \quad \square
\]

**Theorem 4.3.** \( d(a, b; \mathcal{H}; e) d_e^*(a, b; \mathcal{H}) = 1 \).

**Proof.** It is easily seen that for large \( n \) we have
\[
d_n^* = \inf \{H(w); w \in F(a, b; \mathcal{H}), I_e(w) = 1, w_n = 0 \text{ on } Y \setminus Y_n\}.
\]
Therefore we obtain \( d_n^* \geq d_{n+1}^* \geq d_e^*(a, b; \mathcal{H}) \), so that
\[
d_e^*(a, b; \mathcal{H}) \leq \lim_{n \to \infty} d_n^*.
\]
Since \( d_n \cdot d_n^* = 1 \) by Theorem 4.2, we have by Lemma 4.3
\[
d(a, b; \mathcal{H}; e) d_e^*(a, b; \mathcal{H}) \leq \lim_{n \to \infty} d_n \cdot d_n^* = 1.
\]
Our equality follows from Theorem 4.1. \( \square \)

**Corollary 4.1.** \( \{d_e^*(a, b; N_n; \mathcal{H})\} \) converges to \( d_e^*(a, b; \mathcal{H}) \) as \( n \to \infty \).
5. Extremal length

Let \( a \) and \( b \) be two distinct nodes and let \( P_{a,b} \) be the set of all paths from \( a \) to \( b \). For a path \( P \) and a function \( w \) on \( Y \), we set for simplicity
\[
\sum_P w(y) := \sum_{y \in C_Y(P)} w(y)
\]
The extremal length \( EL(a, b; \mathcal{H}) \) of \( N \) between \( a \) and \( b \) is defined by the inverse of the value of the extremum problem:
\[
EL(a, b; \mathcal{H})^{-1} := \inf \{ H(w); w \in EL(P_{a,b}; \mathcal{H}) \},
\]
where \( EL(P_{a,b}; \mathcal{H}) \) is the set of all \( w \in L(Y; \mathcal{H}) \) satisfying
\[
\sum_P \| r(y)w(y) \| \geq 1 \quad \text{for all} \quad P \in P_{a,b}.
\]
The extremal length \( EL_e(a, b; \mathcal{H}) \) of \( N \) between \( a \) and \( b \) is defined by the inverse of the value of the extremum problem:
\[
EL_e(a, b; \mathcal{H})^{-1} := \inf \{ H(w); w \in EL_e(P_{a,b}; \mathcal{H}) \},
\]
where \( EL_e(P_{a,b}; \mathcal{H}) \) is the set of all \( w \in L(Y; \mathcal{H}) \) satisfying
\[
\sum_P |(r(y)w(y), e)| \geq 1 \quad \text{for all} \quad P \in P_{a,b}.
\]
We have
\[
EL(a, b; \mathcal{H}) \geq EL_e(a, b; \mathcal{H}),
\]
since \( |(r(y)w(y), e)| \leq \| r(y)w(y) \| \| e \| = \| r(y)w(y) \| \).

**Lemma 5.1.** \( EL_e(a, b; \mathcal{H})^{-1} \leq d_e(a, b; \mathcal{H}) \).

**Proof.** Let \( u \) be any feasible solution for \( d_e(a, b; \mathcal{H}) \) and put \( w(y) := du(y) \). Then \( w(y) \in \mathcal{H} \) for each \( y \in Y \). Let \( P \in P_{a,b} \) with \( C_X(P) := \{x_0, x_1, \cdots, x_n\} \) \((x_0 = a, x_n = b)\), \( C_Y(P) := \{y_1, y_2, \cdots, y_m\} \) and \( \{x \in X; K(x, y_i) \neq 0\} = \{x_{i-1}, x_i\} \) for \( i = 1, 2, \cdots, n \) as in the proof of Lemma 1.4. Then we have
\[
\sum_P |((r(y)w(y), e))| = \sum_{i=1}^n |((\delta u(y_i), e))|\]
\[
\geq \sum_{i=1}^n |((u(x_i) - u(x_{i-1}), e))|\]
\[
\geq ((u(b), e)) - ((u(a), e)) = 1.
\]
Therefore
\[
EL_e(a, b; \mathcal{H})^{-1} \leq H(w) = D(u),
\]
and hence \( EL_e(a, b; \mathcal{H})^{-1} \leq d_e(a, b; \mathcal{H}) \). \( \square \)

**Lemma 5.2.** Let \( w \) be a feasible solution for \( EL_e(a, b; \mathcal{H}) \). Then
\[
d_e(a, b; \mathcal{H}) \leq \sum_{y \in Y} ((r(y)w(y), w(y)))((r(y)e, e))((r(y)^{-1}e, e)).
\]
Proof. Put \( V(y) := |(r(y)w(y), e)| \). Then
\[
\sum_P V(y) \geq 1 \quad \text{for all } P \in \mathbf{P}_{a,b}.
\]
By the duality between the max-potential problem and the min-work problem (cf. [4]), we can find \( \beta \in L(X; \mathbb{R}) \) such that \( \beta(a) = 0, \beta(b) = 1 \) and \( |\delta \beta(y)| \leq V(y) \) on \( Y \). Let \( u(x) := \beta(x)e \). Then \( u \in L(X; \mathcal{H}) \), \( u(a) = 0 \) and \( u(b) = e \), so that by Lemma 1.1
\[
d_e(a, b; \mathcal{H}) \leq D(u) = \sum_{y \in Y} (r(y)^{-1} \delta u(y), \delta u(y))
\]
\[
= \sum_{y \in Y} (\delta \beta(y))^2 ((r(y)^{-1} e, e))
\]
\[
\leq \sum_{y \in Y} V(y)^2 ((r(y)^{-1} e, e))
\]
\[
\leq \sum_{y \in Y} ((r(y)w(y), w(y)))((r(y)e, e))((r(y)^{-1} e, e))
\]
\( \square \)

Theorem 5.1. Let \( M(r) := \sup\{(r(y)e, e)|(r(y)^{-1} e, e)); y \in Y\} \). Then
\[
EL_e(a, b; \mathcal{H})^{-1} \leq d_e(a, b; \mathcal{H}) \leq M(r)EL_e(a, b; \mathcal{H})^{-1}.
\]

Corollary 5.1. Assume that \((r(y)e, e)|(r(y)^{-1} e, e)) = 1 \) for all \( y \in Y \). Then
\[
d_e(a, b; \mathcal{H}) = EL_e(a, b; \mathcal{H})^{-1}.
\]

Remark 1. Let \( I \) be the identity map of \( \mathcal{H} \) and let \( \gamma \in L(Y; \mathbb{R}) \) be positive. Then \( r(y) = \gamma(y)I \) is positive and invertible. Clearly, we have \((r(y)e, e) = \gamma(y)\) and \((r(y)^{-1} e, e)) = 1/\gamma(y)\), so that the condition in the above theorem holds in this case.

We shall prove

Theorem 5.2. Assume that the graph \( G = \{X, Y, K\} \) is a tree. Then
\[
d_e(a, b; \mathcal{H}) = EL_e(a, b; \mathcal{H})^{-1} = H(pe)^{-1} = \sum_P ((r(y)e, e)),
\]
where \( p \) is the path index of the path \( P \) from \( a \) to \( b \).

Proof. Since the graph is a tree, there exists a unique path \( P \) from \( a \) to \( b \). Let \( p \) be the path index of \( P \). Then
\[
F_H(a, b; \mathcal{H}) = \{tph; h \in \mathcal{H}, t \in \mathbb{R}\}.
\]
If \( w \) is a feasible solution for \( d^*(a, b; \mathcal{H}; e) \), then \( w = pe \) and
\[
d^*(a, b; \mathcal{H}; e) = H(pe) = \sum_{y \in Y} |p(y)|((r(y)e, e))
\]
\[
= \sum_P ((r(y)e, e)).
\]
Let \( w \) be a feasible solution for \( EL_e(a, b; \mathcal{H})^{-1} \). Then we have by Lemma 1.2
\[
1 \leq \sum_P |((r(y)w(y), e))| = \sum_{y \in Y} |((r(y)w(y), p(y)e))|)
\]
\[
\leq H(w)^{1/2} H(pe)^{1/2},
\]
so that \( H(pe)^{-1} \leq H(w) \). Therefore by Theorem 3.2
\[
d_e(a, b; \mathcal{H}) = H(pe)^{-1} \leq EL_e(a, b; \mathcal{H})^{-1}.
\]
Our equality follows from Lemma 5.1. □

We show by an example that the equality $d_e(a, b; \mathcal{H}) = EL_e(a, b; \mathcal{H})^{-1}$ does not hold in general.

**Example.** Let $X = \{x_0, x_1, x_2\}$ and $Y = \{y_1, y_2, y_3\}$ and define $K$ by

\[
K(x_0, y_1) = K(x_0, y_2) = K(x_1, y_3) = -1,
K(x_1, y_2) = K(x_2, y_1) = K(x_2, y_3) = 1
\]

and $K(x, y) = 0$ for any other pair. Then $G = \{X, Y, K\}$ is a finite graph. Take $\mathcal{H}$ as $\mathbb{R}^2$ with the usual inner product and define $r(y)$ by

\[
r(y_i) := \begin{pmatrix} 1 & 0 & t_i \\ 0 & 1 & 0 \end{pmatrix}
\]

with $t_i > 0$ for $i = 1, 2, 3$. Then

\[
r(y_i)^{-1} = \begin{pmatrix} 1 & 0 & 1/t_i \\ 0 & 1 & 0 \end{pmatrix}.
\]

Let $a = x_0$, $b = x_2$ in the above setting and let $e = (e_1, e_2)^T \in \mathbb{R}^2$. For $w \in L(Y; \mathbb{R}^2)$, set $w(y_i) = (\xi_i, \eta_i)^T$ for $i = 1, 2, 3$. Then

\[
H(w) = \sum_{i=1}^{3}(\xi_i^2 + t_i\eta_i^2).
\]

Let $w$ be a feasible solution for $d^*(a, b; \mathbb{R}^2; e)$. Then $w(y_2) = w(y_3)$ or $\xi_2 = \xi_3$, $\eta_2 = \eta_3$ and

\[
\xi_1 + \xi_2 = e_1, \quad \eta_1 + \eta_2 = e_2.
\]

Minimizing $H(w)$ subject to this constraints, we obtain

\[
d^*(a, b; \mathbb{R}^2; e) = \frac{2}{3}e_2^2 + \frac{t_1(t_2 + t_3)}{t_1 + t_2 + t_3}e_2^2,
\]

so that by Theorem 3.2

\[
d_e(a, b; \mathbb{R}^2) = \frac{3(t_1 + t_2 + t_3)}{2(t_1 + t_2 + t_3)e_1^2 + 3t_1(t_2 + t_3)e_2^2}.
\]

On the other hand, the feasibility of $w \in L(Y; \mathbb{R}^2)$ for $EL_e(a, b; \mathbb{R}^2)$ implies

\[
\xi_1 e_1 + t_1 \eta_1 e_2 \geq 1,
(\xi_2 + \xi_3)e_1 + (t_2 \eta_2 + t_3 \eta_3)e_2 \geq 1.
\]

Minimizing $H(w)$ subject to this constraints, we obtain

\[
EL_e(a, b; \mathbb{R}^2)^{-1} = \frac{3e_2^2 + (t_1 + t_2 + t_3)e_2^2}{(e_1^2 + t_1e_2^2)[2e_1^2 + (t_2 + t_3)e_2^2]}.
\]

We have

\[
d_e(a, b; \mathbb{R}^2) - EL_e(a, b; \mathbb{R}^2)^{-1} = \frac{(t_2 + t_3 - 2t_1)^2e_1^2e_2^2}{\alpha} \geq 0,
\]
\[
\alpha = (e_1^2 + t_1e_3^2)[2e_1^2 + (t_2 + t_3)e_3^2][2(t_1 + t_2 + t_3)e_1^2 + 3t_1(t_2 + t_3)e_2^2].
\]
The equality holds in case \(e_1 = 0\), or \(e_2 = 0\) or \(t_2 + t_3 = 2t_1\).

6. Extremal width

Let \(a\) and \(b\) be distinct two nodes and let \(Q_{a,b}\) be the set of all cuts between \(a\) and \(b\) (cf. [4]).

The extremal width \(EW(a, b; \mathcal{H})\) of \(N\) between \(a\) and \(b\) is defined by the inverse of the value of the extremum problem:

\[
EW(a, b; \mathcal{H})^{-1} := \inf \{ H(w); w \in EW(Q_{a,b}; \mathcal{H}) \},
\]
where \(EW(Q_{a,b}; \mathcal{H})\) is the set of all \(w \in L(Y; \mathcal{H})\) satisfying

\[
\sum_{y \in Q} \|w(y)\| \geq 1 \quad \text{for all} \quad Q \in Q_{a,b}.
\]

The extremal width \(EW_e(a, b; \mathcal{H})\) of \(N\) between \(a\) and \(b\) is defined by the inverse of the value of the extremum problem:

\[
EW_e(a, b; \mathcal{H})^{-1} := \inf \{ H(w); w \in EW_e(Q_{a,b}; \mathcal{H}) \},
\]
where \(EW_e(Q_{a,b}; \mathcal{H})\) is the set of all \(w \in L(Y; \mathcal{H})\) satisfying

\[
\sum_{y \in Q} |((w(y), e))| \geq 1 \quad \text{for all} \quad Q \in Q_{a,b}.
\]

We have

\[
EW(a, b; \mathcal{H}) \geq EW_e(a, b; \mathcal{H}),
\]
since \(||((w(y), e))|| \leq \|w(y)\||e\| = \|w(y)\||.

**Lemma 6.1.** \(EW_e(a, b; \mathcal{H})^{-1} \leq d^*_e(a, b; \mathcal{H})\).

**Proof.** Let \(Q \in Q_{a,b}\). Then there exist two disjoint subsets \(Q(a)\) and \(Q(b)\) of \(X\) such that

\[
a \in Q(a), \quad b \in Q(b), \quad X = Q(a) \cup Q(b) \quad \text{and} \quad Q = Q(a) \cap Q(b).
\]

For a subset \(A\) of \(X\), denote by \(\varepsilon_A \in L(X; \mathbb{R})\) the characteristic function of \(A\). Then \(|\delta \varepsilon_{Q(y)}(y)| = 1\) for \(y \in Q\) and \(|\delta \varepsilon_{Q(y)}(y)| = 0\) for \(y \not\in Q\). Let \(w\) be a feasible solution for \(d^*_e(a, b; \mathcal{H})\). There exists a sequence \(\{w_n\} \subset F_0(a, b; \mathcal{H})\) such that \(H(w - w_n) \rightarrow 0\) as \(n \rightarrow \infty\). We have

\[
I_e(w_n) = ((\partial w_n(b), e)) = \sum_{x \in X} ((\partial w_n(x), \varepsilon_Q(x)e))
\]

\[
= \sum_{y \in Y} ((w_n(y), \delta \varepsilon_Q(y)e))
\]

\[\leq \sum_{y \in Q} |((w_n(y), e))|.
\]

Namely \(w_n/I_e(w_n)\) is a feasible solution for \(EW_e(a, b; \mathcal{H})\), so that

\[
EW_e(a, b; \mathcal{H})^{-1} \leq H(w_n/I_e(w_n)) = H(w_n)/(I_e(w_n))^2.
\]

Letting \(n \rightarrow \infty\), we obtain \(EW_e(a, b; \mathcal{H})^{-1} \leq H(w)\), so that \(EW_e(a, b; \mathcal{H})^{-1} \leq d^*_e(a, b; \mathcal{H})\). \(\square\)
Lemma 6.2. Let \( w \) be a feasible solution for \( EW_e(a, b; \mathcal{H}) \). Then
\[
d^*_e(a, b; \mathcal{H}) \leq \sum_{y \in Y} ((r(y)w(y), w(y))((r(y)e, e)(r(y)^{-1}e, e)).
\]

Proof. Put \( V(y) := ||(w(y), e)|| \). Then
\[
\sum_{y \in Q} V(y) \geq 1 \quad \text{for all} \quad Q \in Q_{a,b}.
\]

By the duality between the max-flow problem and the min-cut problem (cf. [4]), we can find \( \varphi \in L(Y; \mathbb{R}) \) such that \( |\varphi(y)| \leq V(y) \) on \( Y \),
\[
\partial \varphi(x) = 0 \quad \text{for} \quad x \in X \setminus \{a, b\} \quad \text{and} \quad -\partial \varphi(a) = \partial \varphi(b) = 1.
\]
Let \( w(y) := \varphi(y)e \). Then \( w \in F(a, b; \mathcal{H}) \) and \( I_e(w) = 1 \). Thus we have
\[
d^*_e(a, b; \mathcal{H}) \leq H(w) = \sum_{y \in Y} ((r(y)\varphi(y)e, \varphi(y)e)) = \sum_{y \in Y} [\varphi(y)]^2((r(y)e, e)) \leq \sum_{y \in Y} ||(w(y), e)||^2((r(y)e, e)) \leq \sum_{y \in Y} ((r(y)w(y), w(y)))((r(y)^{-1}e, e))(r(y)e, e)).
\]

Theorem 6.1. Let \( M(r) := \sup\{(r(y)e, e)((r(y)^{-1}e, e)); y \in Y\} \). Then
\[
EW_e(a, b; \mathcal{H})^{-1} \leq d^*_e(a, b; \mathcal{H}) \leq M(r)EW_e(a, b; \mathcal{H})^{-1}.
\]

Corollary 6.1. Assume that \( ((r(y)e, e)((r(y)^{-1}e, e)) = 1 \) for all \( y \in Y \). Then \( d^*_e(a, b; \mathcal{H}) = EW_e(a, b; \mathcal{H})^{-1} \).

We show by an example that the equality \( d^*_e(a, b; \mathcal{H}) = EW_e(a, b; \mathcal{H})^{-1} \) does not hold in general.

Example. Let \( X = \{x_0, x_1, x_2\} \) and \( Y = \{y_1, y_2\} \) and define \( K \) by
\[
K(x_i, y_i) = 1, \quad K(x_{i-1}, y_i) = -1 \quad (i = 1, 2)
\]
and \( K(x, y) = 0 \) for any other pair. Then \( G = \{X, Y, K\} \) is a finite graph. Notice that \( G \) is a tree. Take \( \mathcal{H} \) as \( \mathbb{R}^2 \) and define \( r(y) \) by
\[
r(y_i) := \begin{pmatrix} 1 & 0 \\ 0 & t_i \end{pmatrix}
\]
where \( t_i > 0 \) for \( i = 1, 2 \). Then
\[
r(y_i)^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1/t_i \end{pmatrix}.
\]

Let \( a = x_0, \ b = x_2 \) in the above setting and let \( e = (e_1, e_2)^T \in \mathbb{R}^2 \). For \( w(y_i) = (\xi_i, \eta_i) \in L(Y; \mathbb{R}^2) \), we have
\[
H(w) = \sum_{i=1}^2 (\xi_i^2 + t_i \eta_i^2).
If \( w \) is a feasible solution for \( d^*_e(a, b; \mathbb{R}^2) \), then \( \xi_1 = \xi_2, \eta_1 = \eta_2 \) and \( I_e(w) = 1 \) implies \( \xi_1 e_1 + \eta_1 e_2 = 1 \). Minimizing \( H(w) \) subject to this constraints, we obtain

\[
d^*_e(a, b; \mathbb{R}^2) = \frac{1}{e_1^2/2 + e_2^2/(t_1 + t_2)}.
\]

On the other hand, if \( w \) is feasible for \( EW_e(a, b; \mathbb{R}^2)^{-1} \), then we have

\[
\xi_1 e_1 + \eta_1 e_1 \geq 1, \quad \xi_2 e_1 + \eta_2 e_2 \geq 1.
\]

Minimizing \( H(w) \) subject to this constraints, we obtain

\[
EW_e(a, b; \mathbb{R}^2)^{-1} = \frac{t_1}{t_1 e_1^2 + e_2^2} + \frac{t_2}{t_2 e_1^2 + e_2^2}.
\]

Therefore

\[
d^*_e(a, b; \mathbb{R}^2) - EW_e(a, b; \mathbb{R}^2)^{-1} = \frac{(t_1 - t_2)^2 e_1^2 e_2^2}{(t_1 + t_2)(t_1 e_1^2 + e_2^2)(t_2 e_1^2 + e_2^2)} \geq 0,
\]

and the equality holds if \( t_1 = t_2 \) or \( e_1 = 0 \) or \( e_2 = 0 \).

**References**