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IS A NET MEASURE AN OUTER MEASURE?

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Abstract. In this short note we prove that the net measure m_α is not an outer measure in case $0 < \alpha \le n-1$.

It is well known that a net measure is an outer measure in R^1. But in general it is not known whether the measure is so or not ([1, p.9]). In this note we prove that in $R^n (n \ge 2)$ the measure m_α is not an outer measure in case $0 < \alpha \le n-1$. For the definition of the net measure and particularly m_α, see [1, p.5].

Example. Assume that $n \ge 2$ and $0 < \alpha \le n-1$. Let F be the set $\{x = (x_1, x_2, \dots, x_n);\ 0 \le x_k \le 1\ (k = 1, 2, \dots, n)\ \}$. Then $m_\alpha(F) = 1$ but

$$\inf_{O \supset F, O\, open} m_\alpha(O) \ge 2,$$

thus m_α is not an outer measure.

To prove this, at first it is easily seen that $m_\alpha(F) \le 1$. (As in the following proof we can obtain $m_\alpha(F) = 1$.) Thus we shall prove that $\inf_{O \supset F, O\, open} m_\alpha(O) \ge 2$. Let O be an open set $\supset F$.Then there exists a positive number a such that $H_1 \cup H_2 \subset O$,
where

$$H_1 = \{x;\ 0 \le x_k \le 1\ (k = 1, 2, \dots, n-1),\ x_n = -a\},$$
$$H_2 = \{x;\ 0 \le x_k \le 1\ (k = 1, 2, \dots, n-1),\ x_n = 1 + a\}.$$

Let $\{Q_\nu\}$ be a closed dyadic covering of O with side length δ_ν. Hence, it is sufficient to show that $\sum {\delta_\nu}^\alpha \ge 2$. Let

$$N_1 = \{\nu;\ Q_\nu \cap H_1 \ne \emptyset\},\ N_2 = \{\nu;\ Q_\nu \cap H_2 \ne \emptyset\},$$

then

$$H_1 \subset \cup_{\nu \in N_1} Q_\nu,\ H_2 \subset \cup_{\nu \in N_2} Q_\nu \textit{ and } N_1 \cap N_2 = \emptyset.$$

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Therefore we have

$$1 = |H_1| \leq \sum_{\nu \in N_1} |H_1 \cap Q_\nu| \leq \sum_{\nu \in N_1} {\delta_\nu}^{n-1},$$

where $|E|$ means the (n - 1) dimensional Lebesgue measure on the hyperplane $\{x; x_n = -a\}$. Using the inequality $(a_1 + a_2 + \dots)^\kappa \leq \sum {a_j}^\kappa$ for $a_j \geq 0$ in case $0 < \kappa \leq 1$, we obtain

$$\sum_{\nu \in N_1} {\delta_\nu}^\alpha \geq 1,$$

because $0 < \alpha \leq (n-1)$. Similarly,

$$\sum_{\nu \in N_2} {\delta_\nu}^\alpha \geq 1.$$

Since $N_1 \cap N_2 = \emptyset$, we obtain $\sum {\delta_\nu}^\alpha \geq 2$ and so $m_\alpha(O) \geq 2$. Hence the proof is complete.

Remark 1. In case $\alpha > n - 1$, it is easily seen that m_α is an outer measure, because any hyperplane perpendicular to an axis has zero m_α measure.

Remark 2. In the above, we used the net measure defined by coverings consisting of closed dyadic cubes(see, [1, p.5]). Even if we replace such coverings with that consisting of half open dyadic cubes, we can prove that the new net measure is also not an outer measure, in case $0 < \alpha \leq (n-1)$.

Remark 3. By a similar argument, we can prove that the net measure is not translation-invariant.

Remark 4. Let $h(t)$ be an increasing continuous function defined on $[0, \infty)$ with $h(0) = 0, h(t) > 0$ for $t > 0$ and $\underline{\lim}_{t \to 0} h(t) t^{1-n} > 0$. Set $g(t) = h(t^{\frac{1}{n-1}})$. Assume that g is subadditive, i.e., $g(t_1 + t_2) \leq g(t_1) + g(t_2)$, for all $t_1, t_2 \geq 0$. Then, by a similar method as above, for the same F we can obtain $m_h(F) \leq h(1)$ and $\inf_{O \supset F, O\, open} m_h(O) \geq 2h(1)$ and so m_h is not an outer measure.

References

[1] H. Aikawa and M. Essén *Potential Theory-Selected Topics* Lecture Notes in Math. vol.1633, Springer, 1996.

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