

FUNCTIONAL FREENESS FOR THE BERMAN CLASS $K_{m,n}$ OF OCKHAM ALGEBRAS

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ABSTRACT. In this paper we show that an algebra $\Omega(m, n)$ is functionally free for the Berman class $K_{m,n}$ of Ockham algebras, that is, for any two polynomials f and g , they are identically equal in $K_{m,n}$ if and only if $f = g$ holds in $\Omega(m, n)$. This result can be applied to the well-known algebras, e.g., Boolean, de Morgan, Kleene, Stone, Bunge algebras, and so on.

1. INTRODUCTION

It is well known that, in order to show whether two polynomials f and g are identically equal in the class of Boolean algebras, we only calculate the values of polynomials in the typical Boolean algebra $2 = \{0, 1\}$. If their values are always identical then they are equal as polynomials otherwise not. The property is called a functional freeness of Boolean algebras. There are results about the properties of other algebras, e.g., de Morgan, Kleene algebras ([1],[2]). The classes of these algebras are subvarieties of the Berman class $K_{m,n}$ of Ockham algebras. In this paper we think about the functional freeness of the algebras in the Berman class $K_{m,n}$ and show that the algebra $\Omega(m, n)$ defined below is functional free for the Berman class $K_{m,n}$. From the result we can deduce the properties of the other algebras (e.g., Boolean, de Morgan, Kleene, Stone, Bunge algebras) without difficulty.

2. ALGEBRAS IN $K_{m,n}$

We shall define algebras in the Berman class $K_{m,n}$ of Ockham algebras. Let m and n be integers such that $m \geq 1$ and $n \geq 0$. An algebraic structure $L = (L; \wedge, \vee, N, 0, 1)$ of type $(2, 2, 1, 0, 0)$ is called an Ockham algebra when

- (1) $(L; \wedge, \vee, N, 0, 1)$ is a bounded distributive lattice;
- (2) $N : L \rightarrow L$ is a map satisfying the following conditions

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$$(c1) \quad N0 = 1, N1 = 0$$

$$(c2) \quad N(x \wedge y) = Nx \vee Ny, \quad N(x \vee y) = Nx \wedge Ny$$

An algebra in the Berman class $K_{m,n}$ is the Ockham algebra satisfying the condition

$$(c3) \quad N^{2m+n}x = N^n x,$$

where $N^n x$ is defined recursively as $N^0 x = x, N^{n+1} x = N(N^n x)$.

We have many examples of the algebras in the Berman class. We list some familiar algebras which form the subvarieties of the Berman class $K_{m,n}$.

- (a) $K_{1,0}$: It is the class of de Morgan algebras.
- (b) $K_{1,0}$ with the condition $x \wedge Nx = 0$: This is the class of Boolean algebras.
- (c) $K_{1,0}$ with the condition $x \wedge Nx \leq y \vee Ny$: The class of Kleene algebras [2].
- (d) $K_{1,1}$ with the condition $x \wedge Nx = 0$: The class of Stone algebras [3].
- (e) $K_{1,1}$ with the condition $x \vee Nx = 1$: The class of Bunge algebras [9].

Now we define an algebraic structure $\Omega(m, n)$ which is in the Berman class $K_{m,n}$. The algebra plays an important role to prove the functional freeness for the class of algebras.

For brevity we put $k = 2m + n$. Let $\Omega(m, n) = \{(x_1, x_2, \dots, x_k) | x_i \in \{0, 1\}\}$. We denote an element of $\Omega(m, n)$ by x and the i -th factor by $(x)_i = x_i$. For $x, y \in \Omega(m, n)$, we define $x = y$ if every factor of the elements is identical, that is, $(x)_i = (y)_i$ hence $x_i = y_i$ for every i ($1 \leq i \leq k$). We introduce the operations \wedge, \vee , and N in the set $\Omega(m, n)$. If no confusion arises then we denote $\Omega(m, n)$ simply by Ω .

For every $x = (x_i)_i, y = (y_i)_i \in \Omega$, we define

$$(x \wedge y)_i = \begin{cases} \min\{x_i, y_i\} = x_i \cdot y_i & \text{if } i \text{ is odd} \\ \max\{x_i, y_i\} = x_i + y_i - x_i \cdot y_i & \text{if } i \text{ is even} \end{cases}$$

$$(x \vee y)_i = \begin{cases} \max\{x_i, y_i\} = x_i + y_i - x_i \cdot y_i & \text{if } i \text{ is odd} \\ \min\{x_i, y_i\} = x_i \cdot y_i & \text{if } i \text{ is even} \end{cases}$$

$Nx = (x_2, x_3, \dots, x_k, x_{n+1})$, that is,

$$(Nx)_i = \begin{cases} x_{i+1} & \text{if } i \neq k \\ x_{n+1} & \text{if } i = k \end{cases}$$

We set the special elements $0 = (0, 1, 0, 1, \dots)$ and $1 = (1, 0, 1, 0, \dots)$.

Clearly the structure $\Omega = (\Omega; \wedge, \vee, N, 0, 1)$ is a bounded lattice.

Lemma 1. $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

$$N(x \wedge y) = Nx \vee Ny$$

$$N(x \vee y) = Nx \wedge Ny$$

Proof. We show only the first case. The other cases can be proved similarly.

If i is odd, then the left-hand side is

$$\begin{aligned} & (x \wedge (y \vee z))_i \\ &= \min\{x_i, \max\{y_i, z_i\}\} \\ &= x_i \cdot (y_i + z_i - y_i \cdot z_i) \\ &= x_i \cdot y_i + x_i \cdot z_i - x_i \cdot y_i \cdot z_i. \end{aligned}$$

On the right-hand side is

$$\begin{aligned} & ((x \wedge y) \vee (x \wedge z))_i \\ &= \max\{\min\{x_i, y_i\}, \min\{x_i, z_i\}\} \\ &= \max\{x_i \cdot y_i, x_i \cdot z_i\} \\ &= x_i \cdot y_i + x_i \cdot z_i - (x_i)^2 \cdot y_i \cdot z_i, \text{ since } (x_i)^2 = x_i, \\ &= x_i \cdot y_i + x_i \cdot z_i - x_i \cdot y_i \cdot z_i. \end{aligned}$$

We can also show the equality in case of i being even. Therefore we have $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

The above indicates that the structure $\Omega = (\Omega; \wedge, \vee, N, 0, 1)$ is the Ockham algebra.

It is necessary to show that the structure Ω is in the Berman class $K_{m,n}$. Before doing so, we think about the i -th factor of $N^p x$ for every integer $p \geq 1$. To see each factor of the element $N^p x$, when x is denoted by (x_1, x_2, \dots, x_k) , we consider an infinite sequence of factors of x :

$$\begin{aligned} & x_1, x_2, \dots, x_k, x_{k+1}(=x_{n+1}), x_{k+2}(=x_{n+2}), \dots, x_{k+2m}(=x_k), x_{k+2m+1}(=x_{n+1}), \\ & x_{k+2m+2}(=x_{n+2}), \dots, x_{k+2m+2m}(=x_k), x_{k+2m+2m+1}(=x_{n+1}), \dots \end{aligned}$$

It follows that $(N^p x)_i = x_{i+p}$ for every i . In general the j -th term of the sequence is obtained as follows. When j is denoted as $2m \cdot \alpha + \beta$ ($0 \leq \alpha, 0 \leq \beta < 2m$), the j -th term x_j is x_β . Using the fact, we can show the next lemma.

Lemma 2. *For every $x \in \Omega$ and $i \leq k$, we have $N^k x = N^n x$*

Proof. It is sufficient to show that $(N^k x)_i = (N^n x)_i$ for every $i \leq k$. By the argument above, we have that $(N^k x)_i = x_{i+k}$ and $(N^n x)_i = x_{i+n}$. If we denote $i + n = 2m \cdot a + b$ ($0 \leq a, 0 \leq b < 2m$), since $i + k = 2m \cdot (a + 1) + b$, then we have $(N^k x)_i = x_{i+k} = x_b = x_{i+n} = (N^n x)_i$. This means that $(N^k x)_i = (N^n x)_i$ and hence that $N^k x = N^n x$.

Consequently we can conclude that the structure $\Omega = (\Omega; \wedge, \vee, N, 0, 1)$ is in the Berman class $K_{m,n}$.

3. FUNCTIONAL FREENESS

In this section we show that the algebra Ω is functionally free for the Berman class $K_{m,n}$ of Ockham algebras. In general, an algebra A is said to be functionally free for a non-empty class C of algebras provided that the following condition is satisfied: any two polynomials are identically equal in A iff they are identically

equal in each algebra in C . For example, (1) two-element Boolean algebra $2 = \{0, 1\}$ is functionally free for the class B of all Boolean algebras, (2) three-element Kleene algebra $3 = \{0, a, 1\}$ is so for the class K of all Kleene algebras, and (3) four-element de Morgan algebra $M = \{0, a, b, 1\}$ is so for the class M of all de Morgan algebras.

We define polynomials before proving the functional freeness of Ω .

Let $V = \{p_1, p_2, \dots\}$ be the set of variables. We define polynomials as follows.

1. Every variable $p_n \in V$ is a polynomial;
2. If f and g are polynomials, then so are $f \wedge g$, $f \vee g$, and Nf .

Let L be an arbitrary algebra. The map $v : V \rightarrow L$ is called a valuation on L . The valuation v is extended uniquely to v^* of all the polynomials as follows; For any polynomials f and g ,

- (v1) $v^*(p_n) = v(p_n)$ for every $p_n \in V$
- (v2) $v^*(f \wedge g) = v^*(f) \wedge v^*(g)$
- (v3) $v^*(f \vee g) = v^*(f) \vee v^*(g)$
- (v4) $v^*(Nf) = N(v^*(f))$

Hence the value $v^*(f)$ of the polynomial f is determined by the values of p_n which are components of f . We note that the symbols \wedge , \vee , and N of the right-hand side of the equations are in L . If no confusion arises we denote v^* by v simply.

We say that f and g are identically equal in L (or simply $f = g$ in L) if $v^*(f) = v^*(g)$ for every valuation v on L . We also say that f and g are identically equal in the class C of algebras (or simply $f = g$ in C) when $f = g$ holds in every algebra L in C . In the following, we shall show that $f = g$ in $K_{m,n}$ iff $f = g$ in Ω . Therefore, to investigate whether $f = g$ holds or not in the class $K_{m,n}$ of Ockham algebras, it is sufficient only to calculate the values $v^*(f)$ and $v^*(g)$ for all valuations v on Ω .

Proposition 1. *Let D be any bounded distributive lattice and $a, b \in D$. If $a \neq b$, then there is a prime filter P of D such that $a \in P$ but $b \notin P$.*

Proof. This is a well-known fact about distributive lattices. Hence we omit the proof. See [5].

In general if there is a partition of a set then we can introduce an equivalence relation on it. Let P be a prime filter of $L \in K_{m,n}$. The set L can be divided into 2^k subsets by P as follows:

$$\begin{aligned} L_{111\dots 1} &= \{x \mid x \in P, Nx \in P, N^2x \in P, \dots, N^{k-1}x \in P\} \\ L_{101\dots 1} &= \{x \mid x \in P, Nx \notin P, N^2x \in P, \dots, N^{k-1}x \in P\} \\ &\dots \\ L_{000\dots 0} &= \{x \mid x \notin P, Nx \notin P, N^2x \notin P, \dots, N^{k-1}x \notin P\} \end{aligned}$$

Thus we can define an equivalence relation \sim_P on L as

$$\sim_P \ni (x, y) \iff \exists L_{s_1 s_2 \dots s_k} (x, y \in L_{s_1 s_2 \dots s_k}), \text{ where } s_i \in \{0, 1\}.$$

This means that

$$(x, y) \in \sim_P \text{ iff } \forall i (N^i x \in P \Leftrightarrow N^i y \in P).$$

We say \sim_P an induced equivalence relation by P . For that relation we can show the next lemma.

Lemma 3. *If P is a prime filter of L , then the induced relation \sim_P by P is a congruent relation on L .*

Proof. We have to prove that for any $(x, y), (a, b) \in \sim_P$

$$(1) (x \wedge a, y \wedge b) \in \sim_P ;$$

$$(2) (x \vee a, y \vee b) \in \sim_P ;$$

$$(3) (Nx, Ny) \in \sim_P.$$

From the fact $N^k x = N^n x$, it is clear that the condition (3) holds. We only show the case of (1).

We simply denote an element $x \in L$ as a sequence of 0 and 1 as follows:

$x = x_1 x_2 x_3 \dots x_k$, where x_i is deined by

$$x_k = \begin{cases} 1 & \text{if } N^i x \in P \\ 0 & \text{if } N^i x \notin P \end{cases}$$

By definition of \sim_P , we have

$$(x, y) \in \sim_P \text{ iff } \forall i (x_i = y_i).$$

Hence it is sufficient to show that $\forall i ((x \wedge a)_i = (y \wedge b)_i)$ when $x_i = y_i$ and $a_i = b_i$ for all i .

Since P is the prime filter, we have that

$$(x \wedge y)_i = \min\{x_i, y_i\} \text{ if } i \text{ is even}$$

$$(x \wedge y)_i = \max\{x_i, y_i\} \text{ if } i \text{ is odd.}$$

Thus if i is even then it follows that

$$(x \wedge a)_i = \min\{x_i, a_i\} = \min\{y_i, b_i\} = (y \wedge b)_i.$$

In case of i odd, we also obtain that

$$(x \wedge a)_i = \max\{x_i, a_i\} = \max\{y_i, b_i\} = (y \wedge b)_i.$$

Therefore in either case we can conclude that $\forall i ((x \wedge a)_i = (y \wedge b)_i)$, that is, $(x \wedge a, y \wedge b) \in \sim_P$.

The other case (2) can be proved similarly. This means that \sim_P is the congruence relation on L .

When P is the prime filter of L , we define $L/\sim_P = \{[x] \mid x \in L\}$ and $[x] = \{y \in L \mid x \sim_P y\}$. Since the relation \sim_P is congruent on L , we can consistently define the operations \wedge, \vee , and N on L/\sim_P :

$$[x] \wedge [y] = [x \wedge y]$$

$$[x] \vee [y] = [x \vee y]$$

$$N[x] = [Nx]$$

It is easy to show the next theorem.

Theorem 1. (1) The structure $L/\sim_P = (L/\sim_P; \wedge, \vee, N, [0], [1])$ is in the Berman class.

(2) The map $\eta : L \rightarrow L/\sim_P$ defined by $\eta(x) = [x]$ is a homomorphism.

Lemma 4. The map $\xi : L/\sim_P \rightarrow \Omega$ is an embedding, where ξ is defined by $\xi([x]) = (s_1, s_2, \dots, s_k)$ if $x \in L_{s_1 s_2 \dots s_k}$.

Proof. It is clear that ξ is well-defined and an injection. We only show that ξ is a homomorphism, that is,

$$\begin{aligned}\xi([x] \wedge [y]) &= \xi([x]) \wedge \xi([y]) \\ \xi([x] \vee [y]) &= \xi([x]) \vee \xi([y]) \\ \xi(N[x]) &= N(\xi([x])).\end{aligned}$$

Since P is the prime filter, it follows that

$x \wedge y \in P$ iff $x \in P$ and $y \in P$,

$N(x \wedge y) \in P$ iff $Nx \in P$ or $Ny \in P$,

Hence we have $\xi([x] \wedge [y]) = \xi([x]) \wedge \xi([y])$. The other cases are proved similarly.

Theorem 2. Ω is functionally free for the Berman class $K_{m,n}$ of Ockham algebras, that is, $f = g$ in $K_{m,n}$ if and only if $f = g$ in Ω .

Proof. It is clear that a equation $f = g$ holds for polynomials f and g in $K_{m,n}$ then it holds in Ω . To prove the converse we suppose that $f = g$ does not hold in $K_{m,n}$. It is sufficient to indicate the existence of some algebra in $K_{m,n}$ and a valuation τ on it such that $\tau(f) \neq \tau(g)$.

By definition there are algebra $L \in K_{m,n}$ and a valuation $v : V \rightarrow L$ such that $v^*(f) \neq v^*(g)$. By Proposition 1, there is a prime filter P of L such that $v^*(f) \in P$ but $v^*(g) \notin P$. We divide L into 2^k subsets by use of P and take the congruent relation \sim_P induced by P . That is, for every $x, y \in L$,

$$x \sim_P y \iff \exists L_{s_1 s_2 \dots s_k} \text{ such that } x, y \in L_{s_1 s_2 \dots s_k}.$$

We define a valuation $\tau : V \rightarrow \Omega$ by $\tau = \xi \circ \eta \circ v$, that is, $\tau(p_n) = \xi([v(p_n)])$.

It is clear from definition that for each polynomial h ,

$$\tau^*(h) = \xi([v^*(h)]).$$

Since $v^*(f) \in P$ but $v^*(g) \notin P$, we have $[v^*(f)] \neq [v^*(g)]$. Since ξ is injective, it follows that $\xi([v^*(f)]) \neq \xi([v^*(g)])$. This means that $\tau^*(f) \neq \tau^*(g)$.

Thus the theorem can be proved completely.

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