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GENERALIZATION OF LIE TRIPLE ALGEBRA

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Abstract. We generalize the concept of Lie triple algebra, introduced as tangent algebra of geodesic homogeneous left Lie loop [19], to some algebraic systems equipped with some more multilinear operations, under an idea based on purely geometric point of view. That is, the operations of Lie triple algebras defined by the parallel torsion and curvature tensors of the canonical connection of homogeneous left Lie loops will be extended to ones defined by some connection whose torsion is not assumed to be parallel. The new algebraic system thus obtained will be called *Lie triple multi-algebra*. We present a method of constructing a Lie triple multi-algebra from double Lie algebras.

1. Introduction

The concept of *Lie triple algebra* has been introduced by Yamaguti [60] under the name of *general Lie triple system*, related with the canonical connection of reductive homogeneous space of Nomizu [51]. It is an algebraic system on a vector space equipped with a bilinear multiplication and a trilinear multiplication, which contains both of the concepts of Lie algebra and Lie triple system as special cases.

In 1975, the author has introduced the concept of *homogeneous Lie loops* (cf. Kikkawa [19] as a non-associative generalization of the concept of Lie groups, that is, a kind of differentiable algebraic binary systems on manifolds. Then, he defined the *tangent Lie triple algebra* of the homogeneous Lie loop, or of homogeneous left Lie loop (cf. Kikkawa [19], [38]) with the bilinear and trilinear multiplications on the tangent space at the unit element, by the values of the torsion tensor and curvature tensor of the canonical connection, respectively. This is a generalization of the Lie algebras of Lie groups. In fact, any Lie group is a homogeneous Lie loop with the $(-)$-connection as the canonical connection, and the bracket operation of the Lie algebra is given by the value of the torsion tensor of this connection. For differentiable homogeneous loops and tangent Lie triple algebras, see also Hofmann-Strambach [10], [11] and Miheev-Sabinin

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[50]. On the same line, the theory of generalization of Lie groups and tangent algebraic systems related with linear connections has been developed by Sagle and others in [12], [55], [56] and [57].

In this paper, we generalize the concept of Lie triple algebra from the viewpoint of algebraic systems whose multiplications are given by the values of torsion and curvature tensors of some linear connection. To do this, in section 2, we recall some identities for the covariant differentiation of linear connections. Then, in section 3, we introduce the new algebraic system called *Lie triple multi-algebra* on any vector space which is a generalized concept of that of Lie triple algebras.

The concept of differentiable local loops was originated in 1936 by S. S. Chern [7] as differentiable webs on manifolds. For the relations between loops and webs, see e.g. Bol [5]. In 1964, the concept of *geodesic local loops* on linearly connected manifolds was introduced by Kikkawa [13] and, independently, by Sabinin [53] in 1972. See also Akivis [3], Miheev-Sabinin [50] and Sabinin [54]. For locally reductive spaces, that is, linearly connected manifolds whose torsion and curvature are parallel, the geodesic local loops are reduced to homogeneous local Lie loops and their tangent algebras are reduced to Lie triple algebras. In [19] - [46] the author has developed the extensive theory of geodesic homogeneous (left) Lie loops and their tangent Lie triple algebras and shown that it is a full generalization of the theory of Lie groups and Lie algebras. The tangent algebraic system of analytic loops in general was considered by Akivis [2], which is called *Akivis Algebra*. The related theory has been developed in Akivis [1] - [3], Akivis-Shelekhov [4] and Hofmann-Strambach [10]. See also Hofmann-Strambach [11], Goldberg [9]. The concept of Akivis algebras was generalized for analytic n-loops in Goldberg [8] and Smith [59].

On the other hand, in [37], the author has introduced the concept of projectivity of geodesic local loops, and found the method to get from a geodesic homogeneous left Lie loops a new one, by means of changing of the canonical connection to the other locally reductive connection. By this method, the theory of projectivity has been developed in Kikkawa [37] - [45] and Sanami-Kikkawa [58]. Especially, the structural theory of projectivity of simple Lie groups has been investigated in [58], where the tangent Lie triple algebras of homogeneous left Lie loops in projective relation with a simple Lie group G are determined by projective double Lie algebras of the Lie algebra of G. In section 4, motivated by this method, we give a method of constructing a Lie triple multi-algebra $\mathfrak{g}$ from two Lie algebras $\mathfrak{s}$ and $\mathfrak{t}$ on the same vector space.

The problem of finding algebraic properties of geodesic local loops on linearly connected manifolds whose tangent algebras form Lie triple multi-algebras is still open.

2. Covariant differentiation of linear connections

In this section, we recall some identities on covariant differentiation of linear connections, that are familiar to differential geometers.

Let M be a differentiable manifold of class C^∞ with a linear connection ∇. In what follows, we assume that all functions, vector fields and tensor fields are of class C^∞. We denote the torsion tensor field and the curvature tensor field of ∇ by S and R, respectively, which are given by the following formulas:

$$S(X,Y) := [X,Y] - \nabla_X Y + \nabla_Y X \tag{2.1}$$

$$R(X,Y)Z := \nabla_{[X,Y]} Z - \nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z, \tag{2.2}$$

for any vector fields X, Y, Z on M. We denote here the Lie bracket of vector fields by the big bracket $[X,Y]$ to distinguish from the usual bracket notation used later for a multilinear algebraic operation.

For any tensor field K of type $(1,p)$, the covariant derivative ∇K of K is a tensor field of type $(1,p+1)$ given by the following equations:(cf. Kobayashi-Nomizu [47] pp.124-125)

$$\begin{aligned}(\nabla K)(X_1, X_2, \ldots, X_p; Y) &:= \nabla_Y(K(X_1, X_2, \ldots, X_p)) \\ &- K(\nabla_Y X_1, X_2, \ldots, X_p) - K(X_1, \nabla_Y X_2, \ldots, X_p) \\ &- \cdots - K(X_1, X_2, \ldots, \nabla_Y X_p),\end{aligned} \tag{2.3}$$

for any vector fields $X_1, X_2, \ldots, X_p$ and Y on M. A tensor field K on M is said to be *parallel* if $\nabla K = 0$. Moreover, for any integer m, the m-th covariant derivative $\nabla^m K$ of any tensor field K is defined inductively to be a tensor field of type $(1, p+m)$ given by the equation

$$\begin{aligned}&(\nabla^m K)(X_1, X_2, \ldots, X_p; Y_1; Y_2; \ldots; Y_m) \\ &\quad := (\nabla(\nabla^{m-1} K))(X_1, X_2, \ldots, X_p; Y_1; Y_2; \ldots; Y_{m-1}; Y_m).\end{aligned} \tag{2.4}$$

In particular, for any vector fields Y_1, Y_2 , we set

$$\begin{aligned}(\nabla_{Y_1} K)(X_1, X_2, \ldots, X_p) &:= (\nabla K)(X_1, X_2, \ldots, X_p; Y_1), \\ (\nabla^2 K)(;Y_1;Y_2)(X_1, X_2, \ldots, X_p)& \\ &:= (\nabla^2 K)(X_1, X_2, \ldots, X_p; Y_1; Y_2).\end{aligned} \tag{2.5}$$

Then, we get tensor fields $\nabla_{Y_1} K = \nabla K(;Y_1)$ and $(\nabla^2 K)(;Y_1;Y_2)$ of type $(1,p)$, which satisfy the following identity:

$$(\nabla^2 K)(;X;Y) = \nabla_Y(\nabla_X K) - \nabla_{\nabla_Y X} K. \tag{2.6}$$

From this equation, we get the following ;

$$(\nabla^2 K)(;X;Y) - (\nabla^2 K)(;Y;X) = R(X,Y)K - \nabla_{S(X,Y)} K \tag{2.7}$$

(Ricci's identity),

where

$$\begin{aligned}(R(X,Y)K)(X_1, X_2, \ldots, X_p) &:= R(X,Y)(K(X_1, X_2, \ldots, X_p)) \\ &- K(R(X,Y)X_1, X_2, \ldots, X_p) - K(X_1, R(X,Y)X_2, \ldots, X_p) \\ &- \cdots - K(X_1, X_2, \ldots, R(X,Y)X_p).\end{aligned} \tag{2.8}$$

In the following we recall some identities for the torsion tensor field S and the curvature tensor field R. Here, we denote by $\mathfrak{S}_{X,Y,Z}$ the cyclic sum with respect to X, Y, Z :

$$\mathfrak{S}_{X,Y,Z}\{R(X,Y)Z + S(S(X,Y),Z) - (\nabla_X S)(Y,Z)\} = 0 \tag{2.9}$$

(Bianchi's 1st identity)

$$\mathfrak{S}_{X,Y,Z}\{R(S(X,Y),Z)W - ((\nabla_X R)(Y,Z)W)\} = 0 \tag{2.10}$$

(Bianchi's 2nd identity)

$$(\nabla^2 S)(;X;Y) - (\nabla^2 S)(;Y;X) = R(X,Y)S - \nabla_{S(X,Y)}S \tag{2.11}$$

(Ricci's identity for S)

$$(\nabla^2 R)(;X;Y) - (\nabla^2 R)(;Y;X) = R(X,Y)R - \nabla_{S(X,Y)}R \tag{2.12}$$

(Ricci's identity for R)

$$\begin{aligned}(\nabla^2(\nabla S))(;X;Y) - (\nabla^2(\nabla S))(;Y;X) \\ = R(X,Y)((\nabla S)) - \nabla_{S(X,Y)}(\nabla S)\end{aligned} \tag{2.13}$$

(Ricci's identity for ∇S)

and so on.

3. Lie triple multi-algebras

In this section, we introduce a new concept of algebraic systems on vector spaces which is induced from the relations of torsion and curvature tensors of linearly connected manifolds mentioned in the last section.

Let M be a manifold with a linear connection ∇. If the torsion tensor S and the curvature tensor R are parallel, we see that the real vector space $\mathfrak{X}(M)$ of all vector fields on M forms a Lie triple algebra $(\mathfrak{X}(M); [X,Y], \langle X,Y,Z\rangle)$, where the bilinear and trilinear operations $[X,Y]$ and $\langle X,Y,Z\rangle$ are given by ;

$$[X,Y] := S(X,Y), \qquad \langle X,Y,Z\rangle := R(X,Y)Z,$$

for any vector fields X, Y, Z. Notice that the bracket $[X,Y]$ does not mean the Lie bracket of vector fields.

Assume that M is a geodesic homogeneous (left) Lie loop and ∇ is its canonical connection. Then, by restricting these operations to the tangent space $T_e(M)$ at the unit element e of M, we get the tangent Lie triple algebra

$$(\mathfrak{g} := (T_e(M); [X,Y], \langle X,Y,Z\rangle)$$

for any elements $X = X_e, Y = Y_e, Z = Z_e \in T_e(M)$ and

$$[X, Y] := S_e(X_e, Y_e), \qquad \langle X, Y, Z\rangle := R_e(X_e, Y_e)Z_e, \tag{3.1}$$

that is, it satisfies the following relations:

$$[X, Y] = -[Y, X] \tag{3.2}$$

$$\langle X, Y, Z\rangle = -\langle Y, X, Z\rangle \tag{3.3}$$

$$\mathfrak{S}_{X,Y,Z}\{\langle X, Y, Z\rangle + [[X, Y], Z]\} = 0 \tag{3.4}$$

$$\mathfrak{S}_{X,Y,Z}\{\langle [X, Y], Z, W\rangle\} = 0 \tag{3.5}$$

$$\langle U, V, [X, Y]\rangle = [\langle U, V, X\rangle, Y] + [X, \langle U, V, Y\rangle] \tag{3.6}$$

$$\begin{aligned}\langle U, V, \langle X, Y, Z\rangle\rangle = {} & \langle\langle U, V, X\rangle, Y, Z\rangle \\ & + \langle X, \langle U, V, Y\rangle, Z\rangle + \langle X, Y, \langle U, V, Z\rangle\rangle\end{aligned} \tag{3.7}$$

for any $X, Y, Z, U, V, W \in \mathfrak{g}$. In fact, under the assumptions $\nabla S = 0$ and $\nabla R = 0$, the formulae (3.4), (3.5), (3.6) and (3.7) are obtained from Bianchi's 1st identity, 2nd identity, Ricci's identity for S and for R in §2, respectively.

Remark. As is well-known, a Lie triple algebra $\mathfrak{g}$ is reduced to a Lie algebra if the ternary operation $\langle X, Y, Z\rangle$ vanishes identically, while it is reduced to a Lie triple system if the binary operation $[X, Y]$ vanishes identically.

Now we introduce the new algebraic system, motivated by the direct correspondence above between the axiom of Lie triple algebra ((3.2) - (3.7)) and the identities of Bianchi and Ricci for the parallel torsion tensor S and the parallel curvature tensor R:

Definition 3.1. A *Lie triple multi-algebra* $\mathfrak{g} = (\mathbf{V}; [X, Y], [X, Y; Z], \langle X, Y, Z\rangle)$ is a vector space $\mathbf{V}$ over a field of characteristic 0 equipped with a family of $\mathbf{V}$-valued multilinear operations on $\mathbf{V}$:

$$[X, Y], \ [X, Y; Z] \ \text{ and } \ \langle X, Y, Z\rangle$$

satisfying the following relations:

$$[X, Y] = -[Y, X] \tag{3.8}$$

$$[X, Y; Z] = -[Y, X; Z] \tag{3.9}$$

$$\langle X, Y, Z\rangle = -\langle Y, X, Z\rangle \tag{3.10}$$

$$\mathfrak{S}_{X,Y,Z}\{\langle X, Y, Z\rangle + [[X, Y], Z] - [X, Y; Z]\} = 0 \tag{3.11}$$

$$\mathfrak{S}_{X,Y,Z}\{\langle [X, Y], Z, W\rangle\} = 0 \tag{3.12}$$

$$\begin{aligned}\langle U, V, [X, Y]\rangle = {} & [\langle U, V, X\rangle, Y] + [X, \langle U, V, Y\rangle] \\ & - [X, Y; [U, V]]\end{aligned} \tag{3.13}$$

$$\langle U,V,[X,Y;Z]\rangle = [\langle U,V,X\rangle,Y;Z] + [X,\langle U,V,Y\rangle;Z] + [X,Y;\langle U,V,Z\rangle] \tag{3.14}$$

$$\langle U,V,\langle X,Y,Z\rangle\rangle = \langle\langle U,V,X\rangle,Y,Z\rangle + \langle X,\langle U,V,Y\rangle,Z\rangle + \langle X,Y,\langle U,V,Z\rangle\rangle \tag{3.15}$$

Indeed, assume that a linear connection ∇ on a manifold M has the torsion tensor S and the curvature tensor R satisfying $\nabla^2 S = 0$ and $\nabla R = 0$. Then the infinite-dimensional vector space $\mathfrak{X}(M)$ of all vector fields on M forms a Lie triple multi-algebra, because (3.8) - (3.10) are evident by anti-symmetricity of S and R, and Bianchi's identities and Ricci's identities for $S, \nabla S$ and R assure the formulas (3.11) - (3.15), respectively.

Remark. If the ternary operation $[X,Y;Z]$ of a Lie triple multi-algebra vanishes identically, it is reduced to a Lie triple algebra. On the other hand, we can define more general concepts of Lie triple multi-algebras of order k for the operation $[X,Y]$ and of order m for the operation $\langle X,Y,Z\rangle$ as $[X,Y;Z_1;Z_2;\dots;Z_k]$ and $\langle X,Y,Z;W_1;W_2;\dots;W_m\rangle$, respectively. In this paper, we do not consider them because we want to investigate the first stage of generalization of the tangent Lie triple multi-algebras of analytic loops .

In what follows, we will consider Lie triple multi-algebras (of order 1 for $[X,Y]$). Let $\mathfrak{g} = (\mathbf{V}; [X,Y], [X,Y;Z], \langle X,Y,Z\rangle)$ be a Lie triple multi-algebra on a vector space $\mathbf{V}$. We can define the concepts of Lie triple multi-subalgebras of $\mathfrak{g}$ and homomorphisms of Lie triple multi-algebras in a natural manner.

Definition 3.2. A Lie triple multi-subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ given on a subspace $\mathbf{H}$ of $\mathbf{V}$ will be called an *ideal* of $\mathfrak{g}$ if it satisfies the following relations:

$$[\mathfrak{g},\mathfrak{h}] \subset \mathfrak{h} \tag{3.16}$$

$$[\mathfrak{g},\mathfrak{h};\mathfrak{g}] \subset \mathfrak{h} \quad \text{and} \quad [\mathfrak{g},\mathfrak{g};\mathfrak{h}] \subset \mathfrak{h} \tag{3.17}$$

$$\langle \mathfrak{g},\mathfrak{h},\mathfrak{g}\rangle \subset \mathfrak{h}. \tag{3.18}$$

A Lie triple multi-algebra $\mathfrak{g}$ is said to be *simple* if it has no non-trivial ideal.

Remark. If the Lie triple multi-algebra $\mathfrak{g}$ is reduced to a Lie triple algebra, the ideals of $\mathfrak{g}$ are reduced to those of the Lie triple algebra (general Lie triple system) which was introduced by Yamaguti in [61].

From this definition, we can easily obtain the following results:

Proposition 3.1. *Let $\mathfrak{h}$ be a Lie triple multi-subalgebra of a Lie triple multi-algebra $\mathfrak{g}$. Then, $\mathfrak{h}$ is an ideal of $\mathfrak{g}$ if and only if it is a kernel of a homomorphism of $\mathfrak{g}$ into some Lie triple multi-algebra.*

Proposition 3.2. *Let $\mathfrak{h}$ be an ideal of a Lie triple multi-algebra $\mathfrak{g}$. Then, the quotient multi-algebra $\mathfrak{g}/\mathfrak{h}$ is defined in a natural manner which forms a Lie triple multi-algebra under the natural multilinear operations $[\tilde{X}, \tilde{Y}]$, $[\tilde{X}, \tilde{Y}; \tilde{Z}]$ and $\langle \tilde{X}, \tilde{Y}, \tilde{Z}\rangle$ defined for $\tilde{X} = X + \mathfrak{h}, \tilde{Y} = Y + \mathfrak{h}$ and $\tilde{Z} = Z + \mathfrak{h}$.*

4. Construction of Lie triple multi-algebras from double Lie algebras

In this section, we construct a Lie triple multi-algebra

$$\mathfrak{g} = (\mathbf{V}; [X, Y], [X, Y; Z], \langle X, Y, Z\rangle)$$

from two Lie algebras (double Lie algebras) given on the same vector space $\mathbf{V}$.

Theorem 4.1. *Let $\mathfrak{s} = (\mathbf{V}; S(X, Y))$ and $\mathfrak{t} = (\mathbf{V}; T(X, Y))$ be two Lie algebras on the same vector space $\mathbf{V}$ whose bracket operations are given by $S(X, Y)$ and $T(X, Y)$, respectively. Consider an algebraic system*

$$\mathfrak{g} = (\mathbf{V}; [X, Y], [X, Y; Z], \langle X, Y, Z\rangle)$$

on $\mathbf{V}$ given by

$$[X, Y] := S(X, Y) + 2T(X, Y) \tag{4.1}$$

$$[X, Y; Z] := T(S(X, Y), Z) - S(T(X, Z), Y) - S(X, T(Y, Z)) \tag{4.2}$$

$$\langle X, Y, Z\rangle := -T(S(X, Y) + T(X, Y), Z) \tag{4.3}$$

for $X, Y, Z \in \mathbf{V}$.

Assume that the two Lie algebras $\mathfrak{s}$ and $\mathfrak{t}$ satisfy the following:

$$T(W, [X, Y; Z]) = 0 \tag{4.4}$$

$$[T(W, X), Y; Z] + [X, T(W, Y); Z] = 0 \tag{4.5}$$

$$[X, Y; S(U, V)] = 0 \tag{4.6}$$

$$[X, Y; T(U, V)] = 0 \tag{4.7}$$

for $X, Y, Z, U, V, W \in \mathbf{V}$. Then $\mathfrak{g}$ forms a Lie triple multi-algebra on $\mathbf{V}$.

Proof. We show that the operations (4.1), (4.2) and (4.3) satisfy the relations (3.8) - (3.15) under the assumptions (4.4) - (4.7). By definition of the operations, the relations (3.8), (3.9) and (3.10) are clear. By a straightforward calculation we get the relation (3.11). We see that the (3.12) is equivalent to

$$\mathfrak{S}_{X,Y,Z} T(W, [X, Y; Z]) = 0,$$

which is valid because (4.4) is assumed. On the other hand, we can show that the relation (3.13) is equivalent to the following condition:

$$[X, Y; 2S(U, V) + T(U, V)]) = 0,$$

which is assured by (4.6) and (4.7). We can also show that the relations (3.14) is deduced by the assumptions (4.5), (4.6) and (4.7). Finally, we can see that the relation (3.15) is equivalent to the following equation:

$$T(W, [X, Y; S(U, V) + T(U, V)]) = 0,$$

which is clear by (4.4). □

Remark. As a special case of the results in the theorem above, we have already known the case of projective double Lie algebras discussed in [58], where the condition

$$T(U, S(X, Y)) = S(T(U, X), Y) + S(X, T(U, Y)) \tag{4.8}$$

is given. In this case, we have

$$[X, Y; Z] = 0 \quad \text{for all} \quad X, Y, Z \in \mathfrak{g},$$

and we see that the Lie triple multi-algebra $\mathfrak{g}$ is reduced to a Lie triple algebra.

As for the Lie triple multi-algebras constructed by ideals of double Lie algebras, we have the following

Theorem 4.2. *Let* $\mathfrak{g} = (\mathbf{V}; [X, Y], [X, Y; Z], \langle X, Y, Z\rangle)$ *be a Lie triple multi-algebra constructed by double Lie algebras* $\mathfrak{s} = (\mathbf{V}; S(X, Y))$ *and* $\mathfrak{t} = (\mathbf{V}; T(X, Y))$ *on a vector space* $\mathbf{V}$*. Assume that a subspace* $\mathbf{H}$ *of* $\mathbf{V}$ *induces ideals of both of the Lie algebras* $\mathfrak{s}$ *and* $\mathfrak{t}$*, that is, the restrictions* $\mathfrak{s}_{\mathbf{H}} = (\mathbf{H}; S_{\mathbf{H}}(X, Y))$ *and* $\mathfrak{t}_{\mathbf{H}} = (\mathbf{H}; T_{\mathbf{H}}(X, Y))$ *to* $\mathbf{H}$ *are ideals of* $\mathfrak{s}$ *and* $\mathfrak{t}$*, respectively. Then, the Lie triple multi-algebra*

$$\mathfrak{h} = (\mathbf{H}; [X, Y]_{\mathbf{H}},\ [X, Y; Z]_{\mathbf{H}},\ \langle X, Y, Z\rangle_{\mathbf{H}})$$

constructed by $\mathfrak{s}_{\mathbf{H}}$ *and* $\mathfrak{t}_{\mathbf{H}}$ *on* $\mathbf{H}$ *is an ideal of* $\mathfrak{g}$*.*

Proof. Since S and T satisfy the relations

$$S(\mathbf{V}, \mathbf{H}) \subset \mathbf{H} \quad \textit{and} \quad T(\mathbf{V}, \mathbf{H}) \subset \mathbf{H}$$

the definitions (4.1), (4.2) and (4.3) of the operations of the constructed Lie triple multi-algebra imply the following relations:

$$[\mathfrak{g}, \mathfrak{h}] \subset \mathfrak{h},\ [\mathfrak{g}, \mathfrak{h}; \mathfrak{g}] \subset \mathfrak{h},\ [\mathfrak{g}, \mathfrak{g}; \mathfrak{h}] \subset \mathfrak{h} \quad \text{and} \quad \langle \mathfrak{g}, \mathfrak{h}, \mathfrak{g}\rangle \subset \mathfrak{h}$$

which show the assertion of the theorem. □

From this theorem we have the following corollary:

Corollary 4.3. *If the Lie triple multi-algebra costructed by double Lie algebras is simple, then the two Lie algebras have no ideals on any non-trivial underlying subspace in common.*

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