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SOME REMARKS ON REPRESENTATIONS OF GENERALIZED INVERSE $*$-SEMIGROUPS

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Abstract. The Munn representation of an inverse semigroup S, in which the semigroup is represented by isomorphisms between principal ideals of the semilattice $E(S)$, is not always faithful. By introducing a concept of a *pre-semilattice*, Reilly considered of enlarging the carrier set $E(S)$ of the Munn representation in order to obtain a faithful representation of S as an inverse subsemigroup of a structure resembling the Munn semigroup $T_{E(S)}$.

The purpose of this paper is to obtain a generalization of the Reilly's results for generalized inverse $*$-semigroups.

1. Introduction

A semigroup S with a unary operation $* : S \to S$ is called a *regular $*$-semigroup* if it satisfies

(i) $(x^*)^* = x$,
(ii) $(xy)^* = y^*x^*$,
(iii) $xx^*x = x$.

Let S be a regular $*$-semigroup. An idempotent e in S is called a *projection* if it satisfies $e^* = e$. For any subset A of S, denote the sets of idempotents and projections of A by $E(A)$ and $P(A)$, respectively.

Let S be a regular $*$-semigroup. It is called a *locally inverse $*$-semigroup* if, for any $e \in E(S)$, eSe is an inverse subsemigroup of S. If $E(S)$ is a normal band, then S is called a *generalized inverse $*$-semigroup*.

Let S and T be regular $*$-semigroups. A homomorphism $\phi : S \to T$ is called a *$*$-homomorphism* if $(a\phi)^* = a^*\phi$. A congruence σ on S is called a *$*$-congruence* if $(a\sigma)^* = a^*\sigma$. A $*$-congruence σ on S is said to be *idempotent-separating* if $\sigma \subseteq \mathcal{H}$, where $\mathcal{H}$ is one of the Green's relations. Denote the maximum idempotent-separating $*$-congruence on S by μ_S or simply by μ. If μ_S is the identity relation

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on S, S is called *fundamental*. The following results are well-known, and we use them frequently throughout this paper.

Result 1.1. [2] *Let S be a regular $*$-semigroup. Then we have the following:*

(1) $E(S) = P(S)^2$;
(2) *for any* $a \in S$ *and* $e \in P(S), a^*ea \in P(S)$;
(3) *each* $\mathcal{L}$*-class and each* $\mathcal{R}$*-class have one and only one projection;*
(4) $\mu_S = \{(a,b) \in S \times S : a^*ea = b^*eb$ *and* $aea^* = beb^*$ *for all* $e \in P(S)\}$.

For a mapping $\alpha : A \to B$, denote the domain and the range of α by $d(\alpha)$ and $r(\alpha)$, respectively. For a subset C of A, $\alpha|_C$ means the restriction of α to C.

As a generalization of the Preston-Vagner representations, one of the authors gave two types of representations of locally [generalized] inverse $*$-semigroups in [4], [6] and [7]. In this paper, we follow [7]. A non-empty set X with a reflexive and symmetric relation σ is called an *ι-set*, and denoted by $(X;\sigma)$. If σ is transitive, that is, if σ is an equivalence relation on X, $(X;\sigma)$ is called a *transitive ι-set*.

Let $(X;\sigma)$ be an ι-set. A subset A of X is called an *ι-single subset* of $(X;\sigma)$ if it satisfies the following condition:

for any $x \in X$, there is at most one element $y \in A$ such that $(x,y) \in \sigma$.

We consider the empty set to be an ι-single subset. We remark that if $(X;\sigma)$ is a transitive ι-set, a subset A of X is an ι-single subset if and only if, for $x, y \in A$, $(x,y) \in \sigma$ implies $x = y$. A mapping α in $\mathcal{I}_X$, the symmetric inverse semigroup on X, is called a *partial one-to-one ι-mapping* on $(X;\sigma)$ if $d(\alpha), r(\alpha)$ are both ι-single subsets of $(X;\sigma)$, where $d(\alpha)$ and $r(\alpha)$ are the domain and the range of α, respectively. Denote the set of all partial one-to-one ι-mappings of $(X;\sigma)$ by $\mathcal{LI}_{(X;\sigma)}$.

For any ι-single subsets A and B of $(X;\sigma)$, define $\theta_{A,B}$ by

$$\theta_{A,B} = \{(a,b) \in A \times B : (a,b) \in \sigma\} = (A \times B) \cap \sigma.$$

Since a subset of an ι-single subset is also an ι-single subset, $\theta_{A,B} \in \mathcal{LI}_{(X;\sigma)}$. For any $\alpha, \beta \in \mathcal{LI}_{(X;\sigma)}$, define $\theta_{\alpha,\beta}$ by $\theta_{\alpha,\beta} = \theta_{r(\alpha),d(\beta)}$, and let $\mathcal{M} = \{\theta_{\alpha,\beta} : \alpha, \beta \in \mathcal{LI}_{(X;\sigma)}\}$, an indexed set of one-to-one partial functions. Now, define a multiplication $\circ$ and a unary operation $*$ on $\mathcal{LI}_{(X;\sigma)}$ as follows:

$$\alpha \circ \beta = \alpha\theta_{\alpha,\beta}\beta \quad \text{and} \quad \alpha^* = \alpha^{-1},$$

where the multiplication of the right side of the first equality is that of $\mathcal{I}_X$. Denote $(\mathcal{LI}_{(X;\sigma)}, \circ, *)$ by $\mathcal{LI}_{(X;\sigma)}(\mathcal{M})$ or simply by $\mathcal{LI}_{(X;\sigma)}$. In this paper, we use $\mathcal{LI}_{(X;\sigma)}$ rather than $\mathcal{LI}_{(X;\sigma)}(\mathcal{M})$.

Result 1.2. [7] *For any ι-set $(X;\sigma)$, $\mathcal{LI}_{(X;\sigma)}$, defined above, is a locally inverse $*$-semigroup. If $(X;\sigma)$ is a transitive ι-set, then $\mathcal{LI}_{(X;\sigma)}$ is a generalized inverse $*$-semigroup. In this case, we denote it by $\mathcal{GI}_{(X;\sigma)}$ instead of $\mathcal{LI}_{(X;\sigma)}$.*

Moreover, if σ is the identity relation on X, then $\mathcal{LI}_{(X;\sigma)}$ is the symmetric inverse semigroup $\mathcal{I}_X$ on X.

We call $\mathcal{LI}_{(X;\sigma)}$ [$\mathcal{GI}_{(X;\sigma)}$] the *$\iota$-symmetric locally* [*generalized*] *inverse $*$-semigroup* on the ι-set [the transitive ι-set] $(X;\sigma)$ with the *structure sandwich set* $\mathcal{M}$.

Let S be a regular $*$-semigroup, and define a relation Ω on S as follows:

$$(x,y)\in\Omega \iff \text{there exists } e\in E(S) \text{ such that } x\rho_e = y,$$

where $\rho_a(a\in S)$ is the mapping of Sa^* onto Sa defined by $x\rho_a = xa$.

Result 1.3. [7] *Let S be a locally inverse $*$-semigroup. For each $a\in S$, let*

$$\rho_a : x\mapsto xa \qquad (x\in d(\rho_a)=Sa^*).$$

Then a mapping $\rho : a\mapsto\rho_a$ is a $*$-monomorphism of S into $\mathcal{LI}_{(S;\Omega)}(\mathcal{M})$.

For a partial groupoid X, if there exist a semilattice Y, a partition $\pi : X\sim\sum\{X_e : e\in Y\}$ of X and mappings $\varphi_{e,f} : X_e\to X_f$ ($e\geq f$ in Y) such that

(1) for any $e\in Y$, $\varphi_{e,e}=1_{X_e}$,
(2) if $e\geq f\geq g$, then $\varphi_{e,f}\varphi_{f,g}=\varphi_{e,g}$,
(3) for $x\in X_e$, $y\in X_f$, xy is defined in X if and only if $x\varphi_{e,ef}=y\varphi_{f,ef}$, and in this case $xy=x\varphi_{e,ef}$,

then X is called a *strong π-groupoid* with mappings $\{\varphi_{e,f} : e,f\in Y, e\geq f\}$, and it is denoted by $X(\pi;Y;\{\varphi_{e,f}\})$ or simply by $X(\pi)$.

Let $X(\pi;Y;\{\varphi_{e,f}\})$ be a strong π-groupoid. A subset A of X is called a *π-singleton subset* of $X(\pi;Y;\{\varphi_{e,f}\})$, if there exists $e\in Y$ such that

$$|A\cap X_f| = \begin{cases} 1 & \text{if } f\in\langle e\rangle,\\ 0 & \text{otherwise,}\end{cases}$$

$$(A\cap X_f)\varphi_{f,g} = A\cap X_g \qquad \text{for any } f,g\in\langle e\rangle \text{ such that } f\geq g,$$

where $\langle e\rangle$ is the principal ideal of Y generated by e. In this case, we sometimes denote the π-singleton subset A by $A(e)$. If $A(e)$ is a π-singleton subset, then $|A\cap X_f|=1$ for any $f\in\langle e\rangle$. We denote the only one element of $A\cap X_f$ by a_f. We remark that, for any π-singleton subset $A(e)$, $A(e)=\{a_e\varphi_{e,f} : f\in\langle e\rangle\}$. Denote the set of all π-singleton subsets of $X(\pi;Y;\{\varphi_{e,f}\})$ by $\mathcal{X}$.

Two π-singleton subsets $A(e)$ and $B(f)$ are said to be *π-isomorphic* to each other, if there exists an isomorphism $\overline{\alpha} : \langle e\rangle\to\langle f\rangle$ as semilattices. In this case, the mapping $\alpha : A(e)\to B(f)$ defined by $a_g\alpha = b_{g\overline{\alpha}}$ ($g\in\langle e\rangle$) is called a *π-isomorphism* of $A(e)$ to $B(f)$. It is obvious that α is a bijection of $A(e)$ onto $B(f)$, and hence $\alpha\in\mathcal{I}_X$.

Let $X(\pi; Y; \{\varphi_{e,f}\})$ be a strong π-groupoid. Define an equivalence relation $\mathcal{U}$ on $\mathcal{X}$ by

$$\mathcal{U} = \{(A(e), B(f)) \in \mathcal{X} \times \mathcal{X} : \langle e \rangle \cong \langle f \rangle \text{ (as semilattices)}\}.$$

For $(A(e), B(f)) \in \mathcal{U}$, let $T_{A(e),B(f)}$ be the set of all π-isomorphisms of $A(e)$ onto $B(f)$, and let

$$T_{X(\pi)} = \bigcup_{(A(e),B(f)) \in \mathcal{U}} T_{A(e),B(f)}.$$

For any $\alpha, \beta \in T_{X(\pi)}$, define a mapping $\theta_{\alpha,\beta}$ as follows:

$$d(\theta_{\alpha,\beta}) = \{a \in r(\alpha) : \text{there exist } e \in Y \text{ and } b \in d(\beta) \text{ such that } a, b \in X_e\},$$
$$r(\theta_{\alpha,\beta}) = \{b \in d(\beta) : \text{there exist } e \in Y \text{ and } a \in r(\alpha) \text{ such that } a, b \in X_e\},$$
$$a\theta_{\alpha,\beta} = b \quad \text{if } r(\alpha) \cap X_e = \{a\} \text{ and } d(\beta) \cap X_e = \{b\}.$$

Then $\theta_{\alpha,\beta} \in T_{X(\pi)}$. Let $\mathcal{M} = \{\theta_{\alpha,\beta} : \alpha, \beta \in T_{X(\pi)}\}$, and define a multiplication $\circ$ and a unary operation $*$ on $T_{X(\pi)}$ by

$$\alpha \circ \beta = \alpha\theta_{\alpha,\beta}\beta,$$
$$\alpha^* = \alpha^{-1}.$$

Then $T_{X(\pi)}(\circ, *)$ is a regular $*$-semigroup. We denote it by $T_{X(\pi)}(\mathcal{M})$.

Result 1.4. [5] *A regular $*$-semigroup $T_{X(\pi)}(\mathcal{M})$ is a generalized inverse $*$-semigroup whose set of projections is partially isomorphic to X.*

Let S be a generalized inverse $*$-semigroup. Hereafter, denote $E(S)$ and $P(S)$ simply by E and P, respectively. Let $E \sim \sum\{E_i : i \in I\}$ be the structure decomposition of E, and let $P_i = P(E_i)$. Then $\pi : P \sim \sum\{P_i : i \in I\}$ is a partition of P. For any $i, j \in I$ $(i \geq j)$, define a mappig $\varphi_{i,j} : P_i \to P_j$ by

$$e\varphi_{i,j} = efe \quad \text{for some (any) } f \in P_j.$$

Then $P(\pi; I; \{\varphi_{i,j}\})$ is a strong π-groupoid.

Result 1.5. [5] *Let S be a generalized inverse $*$-semigroup. For each $a \in S$, let*

$$\tau_a : e \mapsto a^*ea \quad (e \in d(\tau_a) = P(Sa^*)).$$

Then a mapping $\tau : a \mapsto \tau_a$ is a $*$-homomorphism of S into $T_{P(\pi)}(\mathcal{M})$ such that $\tau \circ \tau^{-1} = \mu$.

A regular $*$-subsemigroup T of a regular $*$-semigroup S is said to be *$\mathcal{P}$-full* if $P(T) = P(S)$.

Result 1.6. [5] *A generalized inverse $*$-semigroup S is fundamental if and only if it is $*$-isomorphic to a $\mathcal{P}$-full generalized inverse $*$-subsemigroup of $T_{X(\pi)}(\mathcal{M})$ on a strong π-groupoid $X(\pi; I; \{\varphi_{i,j}\})$ such that $P(T_{X(\pi)}(\mathcal{M}))$ is partially isomorphic to $P(S)$.*

In § 2, by introducing the concept of partially ordered ϱ-set $(X(\trianglelefteq); \{\phi_x\})$, we construct a fundamental generalized inverse $*$-semigroup $T_{X(\trianglelefteq)}(\mathcal{M})$. Also, we shall see that $T_{X(\trianglelefteq)}(\mathcal{M})$ has similar properties with $T_{X(\pi)}(\mathcal{M})$, where $T_{X(\pi)}(\mathcal{M})$ has been given by T. Imaoka, I. Inata and H. Yokoyama [5]. And we shall show that two concepts, strong π-groupoids and partially ordered ϱ-sets, are equivalent.

In § 3, we shall introduce the notion of ω-set $(X(\preccurlyeq); \sigma)$, and construct a generalized inverse $*$-semigroup $T_{(X(\preccurlyeq);\sigma)}(\mathcal{M})$. Furthermore, let S be a generalized inverse $*$-semigroup with the set of projections P, we shall make two generalized inverse $*$-semigroups $T_{P(\trianglelefteq)}(\mathcal{M})$ and $T_{(S(\preccurlyeq);\Omega)}(\mathcal{M})$, where the former is obtained in § 2, and the latter is constructed in this section. Then we shall show that these three semigroups make a commutative diagram.

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2. Fundamental generalized inverse $*$-semigroups

2.1. $T_{X(\trianglelefteq)}(\mathcal{M})$**.** Let $X(\trianglelefteq)$ be a partially ordered set and, for each $x \in X$, consider an order-preserving mapping $\phi_x : X \to X$. If a relation $\varrho = \{(x, y) \in X \times X : y\phi_x = x, x\phi_y = y\}$ is an equivalence relation on X such that

(P1) $x \trianglelefteq y \Longrightarrow$ for each $y' \in y\varrho$, there exists $x' \in x\varrho$ such that $x' \trianglelefteq y'$,

(P2) a relation $\leq = \{(x\varrho, y\varrho) \in X/\varrho \times X/\varrho :$ there exists $x' \in x\varrho$ such that $x' \trianglelefteq y\}$ is a partial order and $X/\varrho(\leq)$ is a semilattice,

(P3) $x_1 \trianglelefteq y, x_2 \trianglelefteq y$ and $x_1\varrho \leq x_2\varrho \Longrightarrow x_1 \trianglelefteq x_2$,

then $(X(\trianglelefteq); \{\phi_x\})$ is called a *partially ordered ϱ-set.*

Let $(X(\trianglelefteq); \{\phi_x\})$ be a partially ordered ϱ-set. Define an equivalence relation $\mathcal{U}$ on $\mathcal{X}$ by

$$\mathcal{U} = \{(\langle a\rangle, \langle b\rangle) \in \mathcal{X} \times \mathcal{X} : \langle a\rangle \simeq \langle b\rangle (\textit{order isomorphic})\},$$

where $\mathcal{X}$ is the set of all principal ideals of $(X(\trianglelefteq); \{\phi_x\})$. For $(\langle a\rangle, \langle b\rangle) \in \mathcal{U}$, let $T_{\langle a\rangle,\langle b\rangle}$ be the set of all (order) isomorphisms of $\langle a\rangle$ onto $\langle b\rangle$, and let

$$T_{X(\trianglelefteq)} = \bigcup_{(\langle a\rangle,\langle b\rangle)\in\mathcal{U}} T_{\langle a\rangle,\langle b\rangle}.$$

For any $\alpha, \beta \in T_{X(\trianglelefteq)}$, define a mapping $\theta_{\alpha,\beta}$ as follows:

$$\theta_{\alpha,\beta} = \{(x, y) \in r(\alpha) \times d(\beta) : (x, y) \in \varrho\},$$

where ϱ is the equivalence relation on X induced by $\{\phi_x\}$, as defined above.

To show that $\theta_{\alpha,\beta} \in T_{X(\trianglelefteq)}$, assume that $r(\alpha) = \langle a \rangle$, $d(\beta) = \langle b \rangle$ and $a\varrho \wedge b\varrho = c\varrho$ $(c \in X)$. Since $c\varrho \leq a\varrho$ and $c\varrho \leq b\varrho$, there exist $c_1, c_2 \in c\varrho$ such that $c_1 \trianglelefteq a$ and $c_2 \trianglelefteq b$. For any $x \in d(\theta_{\alpha,\beta})$, there exists $y \in \langle b \rangle$ such that $(x, y) \in \varrho$. Since $x \trianglelefteq a$, $c_1 \trianglelefteq a$ and $x\varrho \leq c_1\varrho$, we have $x \trianglelefteq c_1$ and so $x \in \langle c_1 \rangle$. Thus $d(\theta_{\alpha,\beta}) \subseteq \langle c_1 \rangle$.

Conversely, let x be any element of $\langle c_1 \rangle$. Since $x\varrho \leq c_1\varrho = c_2\varrho$, there exists $y \in x\varrho$ such that $y \trianglelefteq c_2$. Therefore, $x \in \langle c_1 \rangle \subseteq \langle a \rangle$, $y \in \langle c_2 \rangle \subseteq \langle b \rangle$ and $(x, y) \in \varrho$, and so $x \in d(\theta_{\alpha,\beta})$. Thus $\langle c_1 \rangle \subseteq d(\theta_{\alpha,\beta})$, and hence $d(\theta_{\alpha,\beta}) = \langle c_1 \rangle$. Similarly, $r(\theta_{\alpha,\beta}) = \langle c_2 \rangle$. Since it is obvious that $\theta_{\alpha,\beta}$ is a bijection, we have $\theta_{\alpha,\beta} \in T_{X(\trianglelefteq)}$

Let $\mathcal{M} = \{\theta_{\alpha,\beta} : \alpha, \beta \in T_{X(\trianglelefteq)}\}$, and define a multiplication $\circ$ and a unary operation $*$ on $T_{X(\trianglelefteq)}$ by

$$\alpha \circ \beta = \alpha\theta_{\alpha,\beta}\beta,$$
$$\alpha^* = \alpha^{-1}.$$

Then it is clear that $T_{X(\trianglelefteq)}(\circ, *)$ is a regular $*$-subsemigroup of the ι-symmetric generalized inverse $*$-semigroup $\mathcal{GI}_{(X;\varrho)}(\mathcal{M})$. Hence it is a generalized inverse $*$-semigroup and denoted by $T_{X(\trianglelefteq)}(\mathcal{M})$.

Let S be a generalized inverse $*$-semigroup and $P = P(S)$. We consider P as a partially ordered set with respect to the natural order. Now, we have the following results.

Theorem 2.1. *A regular $*$-semigroup $T_{X(\trianglelefteq)}(\mathcal{M})$ is a generalized inverse $*$-semigroup whose set of projections is order isomorphic to $X(\trianglelefteq)$.*

Proof. It remains to show that $T_{X(\trianglelefteq)}(\mathcal{M})$ is order isomorphic to $X(\trianglelefteq)$. It is clear that $P(T_{X(\trianglelefteq)}(\mathcal{M})) = \{1_{\langle a \rangle} : a \in X\}$. Define a mapping $\psi : X \to P(T_{X(\trianglelefteq)}(\mathcal{M}))$ by $a\psi = 1_{\langle a \rangle}$ for $a \in X$. It is obvious that ψ is onto. For $a, b \in X$,

$$\begin{aligned} 1_{\langle a \rangle} = 1_{\langle b \rangle} &\implies \langle a \rangle = \langle b \rangle \\ &\implies a \trianglelefteq b \text{ and } b \trianglelefteq a \\ &\implies a = b. \end{aligned}$$

Thus ψ is one-to-one, and hence it is bijection.

Suppose that $a \trianglelefteq b$. Then $\langle a \rangle \subseteq \langle b \rangle$. Thus $1_{\langle a \rangle} \circ 1_{\langle b \rangle} = \theta_{\langle a \rangle,\langle b \rangle} = \theta_{\langle a \rangle,\langle a \rangle} = 1_{\langle a \rangle}$, and so $1_{\langle a \rangle} \leq 1_{\langle b \rangle}$. Conversely, let $1_{\langle a \rangle} \leq 1_{\langle b \rangle}$. Then $1_{\langle a \rangle} = 1_{\langle a \rangle} \circ 1_{\langle b \rangle}$, and so $\langle a \rangle = r(1_{\langle a \rangle}) = r(1_{\langle a \rangle} \circ 1_{\langle b \rangle}) \subseteq \langle b \rangle$. Thus $a \trianglelefteq b$, and hence ψ is an isomorphism. □

Corollary 2.2. *A partially ordered set X is order isomorphic to the set of projections of a generalized inverse $*$-semigroup if and only if it is a partially ordered ϱ-set.*

2.2. Representations. Let S be a generalized inverse $*$-semigroup. Hereafter, denote $E(S)$ and $P(S)$ simply by E and P, respectively. Let $E \sim \sum\{E_i : i \in I\}$ be the structure decomposition of E, and let $P_i = P(E_i)$. For any $e \in P$, define a mapping $\phi_e : P \to P$ by

$$f\phi_e = efe.$$

Let $\leq$ be the natural order on S, that is,

$$a \leq b \iff a = eb = bf \quad \text{for some } e, f \in P.$$

Since S is a generalized inerse $*$-semigroup, it follows from [3] that $\leq$ is compatible. Let $\trianglelefteq$ be the restriction of $\trianglelefteq$ to P. It is obvious that for $e, f \in P$,

$$e \trianglelefteq f \iff e = fef.$$

Lemma 2.3. *The set $(P(\trianglelefteq); \{\phi_e\})$, defined above, is a partially ordered ϱ-set.*

Proof. Let e, f and g be any elements of P such that $f \trianglelefteq g$. Since $\leq$ is compatible, $f\phi_e = efe \trianglelefteq ege = g\phi_e$. Thus ϕ_e is order preserving.

For $e \in P_i$ and $f \in P_j$,

$$\begin{aligned} e\varrho f &\iff f\phi_e = e \quad \text{and} \quad e\phi_f = f \\ &\iff efe = e \quad \text{and} \quad fef = f \\ &\iff e\mathcal{J}^E f \\ &\iff i = j. \end{aligned}$$

Then $\varrho = \mathcal{J}^E|_P$, and so $P/\varrho = \{P_i : i \in I\}$. It is easily to see that ϱ satisfies the conditions (P1), (P2) and (P3), and we have the lemma. $\square$

Now, we can consider the generalized inverse $*$-semigroup $T_{P(\trianglelefteq)}(\mathcal{M})$, where $\mathcal{M} = \{\theta_{\alpha,\beta} : \alpha$ and β are order isomorphisms among principal ideals of $(P(\trianglelefteq); \{\phi_e\})\}$.

Lemma 2.4. *For any $a \in S$, $P(Sa)$ $(= P(Sa^*a))$ is a principal ideal of $(P(\trianglelefteq); \{\phi_e\})$.*

Proof. We shall show that $P(Sa) = \langle a^*a \rangle$. Let xa be any element of $P(Sa)$. Since xa is a projection, $xa = (xa)^*xa$, and so $xa \trianglelefteq a^*a$. Thus $P(Sa) \subseteq \langle a^*a \rangle$. Conversely, let $e \in P$ such that $e \trianglelefteq a^*a$. Then a^*aea^*a, and so $e \in P(Sa)$. Therefore, we have $P(Sa) = \langle a^*a \rangle$. $\square$

For any $a \in S$, define a mapping $\tau_a : \langle aa^* \rangle \to \langle a^*a \rangle$ by

$$e\tau_a = a^*ea.$$

It follows from [5] that $\tau_a \in T_{S(\trianglelefteq)}$ and $\tau_a^* = \tau_{a^*}$. Moreover, for any $a, b \in S$, $\theta_{\tau_a,\tau_b} = \tau_{a^*abb^*}$. And we have the following theorems.

Theorem 2.5. *Let S be a generalized inverse $*$-semigroup such that $E(S) = E$ and $P(S) = P$. Let $E \sim \sum\{E_i : i \in I\}$ be the structure decomposition of E and $P_i = P(E_i)$. Denote the restriction of the natural order on S to P by $\trianglelefteq$. For any $e \in P$, define a mapping $\phi_e : P \to P$ by $f\phi_e = efe$. Then $(P(\trianglelefteq); \{\phi_e\})$ is a partially ordered ϱ-set and $T_{P(\trianglelefteq)}(\mathcal{M})$ is a generalized inverse $*$-semigroup.*

Moreover, for any $a \in S$, define a mapping $\tau_a : \langle aa^* \rangle \to \langle a^*a \rangle$ by $e\tau_a = a^*ea$. Then a mapping $\tau : S \to T_{P(\trianglelefteq)}(\mathcal{M})$ $(a \mapsto \tau_a)$ is a $*$-homomorphism and the kernel of τ is the maximum idempotent-separating $*$-congruence on S.

Theorem 2.6. *A generalized inverse $*$-semigroup S is fundamental if and only if it is $*$-isomorphic to a $\mathcal{P}$-full generalized inverse $*$-subsemigroup of $T_{X(\trianglelefteq)}(\mathcal{M})$ on a partially ordered ϱ-set $(X(\trianglelefteq); \{\phi_x\})$ such that $P(T_{X(\trianglelefteq)}(\mathcal{M}))$ is order isomorphic to $P(S)$.*

Denote the sets of all partially ordered ϱ-sets and the set of all strong π-groupoids by $\mathbb{P}$ and $\mathbb{S}$, respectively.

Remark 2.7. *Let $(X(\trianglelefteq); \{\phi_x\})$ be any element of $\mathbb{P}$. For any $x\varrho, y\varrho \in X/\varrho$ $(x\varrho \geq y\varrho)$, define a mapping $\overline{\varphi}_{x\varrho,y\varrho} : X_{x\varrho} \to X_{y\varrho}$ by*

$$x'\overline{\varphi}_{x\varrho,y\varrho} = y', \text{ where } y' \in y\varrho \text{ such that } y' \trianglelefteq x'.$$

Moreover, we define a partial product on X as follows:

$$xy = \begin{cases} x\overline{\varphi}_{x\varrho,(x\varrho)(y\varrho)} & \text{if } x\overline{\varphi}_{x\varrho,(x\varrho)(y\varrho)} = y\overline{\varphi}_{y\varrho,(x\varrho)(y\varrho)} \\ undefined & \text{otherwise.} \end{cases}$$

Then $(X(\trianglelefteq); \{\phi_x\})\lambda = X(\pi_\varrho; X/\varrho; \{\overline{\varphi}_{x\varrho,y\varrho}\})$ is a strong π-groupoid, where π_ϱ is the partition of X induced by ϱ.

Conversely, let $X(\pi; Y; \{\varphi_{e,f}\})$ be any element of $\mathbb{S}$. For any $x \in X$, define a mapping $\widetilde{\phi}_x : X \to X$ by

$$y\widetilde{\phi}_x = x\varphi_{e,ef},$$

where $x \in X_e$ and $y \in X_f$. If we define $\blacktriangleleft = \{(x, y) \in X \times X : x\widetilde{\phi}_y = x\}$, then $X(\pi; Y; \{\varphi_{e,f}\})\mu = (X(\blacktriangleleft); \{\widetilde{\phi}_x\})$ is a partially ordered ϱ-set.

Hence the mappings λ, μ from $\mathbb{P}$ to $\mathbb{S}$ and from $\mathbb{S}$ to $\mathbb{P}$,respectively, are well-defined. Moreover $\mu\lambda = 1_{\mathbb{S}}$, and for any $(X(\trianglelefteq); \{\phi_x\}) \in \mathbb{P}$, if $(X(\trianglelefteq); \{\phi_x\})\lambda\mu = (X(\blacktriangleleft); \{\widetilde{\phi}_x\})$, then $\trianglelefteq = \blacktriangleleft$.

By the above argument, for any $(X(\trianglelefteq); \{\phi_x\})$ in $\mathbb{P}$, without loss of generality, we can consider $(X(\trianglelefteq); \{\phi_x\})$ as a member of $\mathbb{P}\lambda\mu$.

Now, let $X(\pi; Y; \{\varphi_{e,f}\})$ be any element of $\mathbb{S}$. If $X(\pi; Y; \{\varphi_{e,f}\})\mu = (X(\trianglelefteq); \{\phi_x\})$. Then we can construct two generalized inverse $*$-semigroups $T_{X(\pi)}(\mathcal{M})$ and $T_{X(\trianglelefteq)}(\mathcal{M})$. In this case, these two generalized inverse $*$-semigroups are $*$-isomorphic.

3. Extensions of $T_{X(\trianglelefteq)}(\mathcal{M})$

3.1. $T_{(X(\preccurlyeq);\sigma)}(\mathcal{M})$. By a *pre-order* on a set X we shall mean a reflexive and transitive relation. Let $X(\preccurlyeq)$ be a pre-ordered set and let $\nu = \{(a, b) \in X \times X : a \preccurlyeq b \text{ and } b \preccurlyeq a\}$. Then ν is an equivalence relation on X and X/ν is a partially ordered set with respect to the following induced relation

$$a\nu \trianglelefteq b\nu \text{ if and only if } a \preccurlyeq b. \tag{3.1}$$

We call $\trianglelefteq$ the *naturally induced order* on X/ν from $\preccurlyeq$. Clearly ν is the smallest equivalence relation on X for which (C1) defines a partial order on X/ν. We call ν the *minimum partial order congruence* (mpo-congruence) on X from $\preccurlyeq$.

A subset A of X is an *ideal* of X provided that $x \preccurlyeq y$ and $y \in A$ implies $x \in A$. For $a \in X$, we call $\{x \in X\colon\ x \preccurlyeq a\}$ the *principal ideal generated* by a and denote it by $\langle a\rangle$.

A bijection α of one pre-ordered set X onto another Y will be called an *isomorphism* provided that, for $a, b \in X$, $a \preccurlyeq b$ if and only if $a\alpha \preccurlyeq b\alpha$. In particular, if ν_X and ν_Y denote the respective mpo-congruences then $(a, b) \in \nu_X$ if and only if $(a\alpha, b\alpha) \in \nu_Y$.

Let $X(\preccurlyeq)$ be a pre-ordered set and ν the mpo-congruence from $\preccurlyeq$. Then X is a *partially pre-ordered ϱ-set* if and only if X/ν is a partially ordered ϱ-set with respect to the naturally induced order $\trianglelefteq$ from $\preccurlyeq$.

Let $X(\preccurlyeq)$ be a partially pre-ordered ϱ-set and σ an equivalence relation on X such that

(O1) for any x in X, $\langle x\rangle$ is an ι-single subset with respect to σ,

(O2) for x, y in X, if $(x, y) \in \sigma$ then $(x\nu, y\nu) \in \varrho$,

(O3) for x, y, z in X, if $(x\nu)\varrho \wedge (y\nu)\varrho = (z\nu)\varrho$, $z_1\nu \trianglelefteq x\nu$ and $z_2\nu \trianglelefteq y\nu$ ($z_1\nu, z_2\nu \in (z\nu)\varrho$), then for any $a \in \langle z_i\rangle$, there exists $b \in \langle z_j\rangle$ such that $(a, b) \in \sigma$, where $1 \le i, j \le 2$.

Then $(X(\preccurlyeq); \sigma)$ is called an *ω-set.*

Let $(X(\preccurlyeq); \sigma)$ be an ω-set and let $T_{(X(\preccurlyeq);\sigma)}$ denote the set of all isomorphisms from a principal ideal onto another one.

For any α, $\beta \in T_{(X(\preccurlyeq);\sigma)}$, define a mapping $\theta_{\alpha,\beta}$ as follows:

$$\theta_{\alpha,\beta} = \{(a, b) \in r(\alpha) \times d(\beta) : (a, b) \in \sigma\}.$$

Then $\theta_{\alpha,\beta} \in T_{(X(\preccurlyeq);\sigma)}$. Let $\mathcal{M} = \{\theta_{\alpha,\beta} : \alpha, \beta \in T_{(X(\preccurlyeq);\sigma)}\}$, and denote a multiplication $\circ$ and a unary operation $*$ on $T_{(X(\preccurlyeq);\sigma)}$ by

$$\alpha \circ \beta = \alpha\theta_{\alpha,\beta}\beta,$$
$$\alpha^* = \alpha^{-1}.$$

Clearly, $\alpha \circ \beta$ is an isomorphism from $\langle z_1\alpha^{-1}\rangle$ onto $\langle z_2\beta\rangle$. It is obvious that $T_{(X(\preccurlyeq);\sigma)}(\circ, *)$ is a regular $*$-semigroup. Hence it is a generalized inverse $*$-semigroup and denoted by $T_{(X(\preccurlyeq);\sigma)}(\mathcal{M})$.

Theorem 3.1. *A regular $*$-semigroup $T_{(X(\preccurlyeq);\sigma)}(\mathcal{M})$ is a generalized inverse $*$-subsemi-group of $\mathcal{GI}_{(X;\sigma)}(\mathcal{M})$ whose set of projections is order isomorphic to X/ν.*

Proof. Clearly, $T_{(X(\preccurlyeq);\sigma)}(\mathcal{M})$ is a generalized inverse $*$-semigroup of $\mathcal{GI}_{(X;\sigma)}(\mathcal{M})$. It remains to show that $P(T_{(X(\preccurlyeq);\sigma)}(\mathcal{M}))$ is order isomorphic to X/ν. Hereafter, denote $P(T_{(X(\preccurlyeq);\sigma)}(\mathcal{M}))$ simply by P. It is easy to see that $P = \{1_{\langle x\rangle} : x \in X\}$. Now, we define a mapping ψ of P to X/ν as follows: for any $1_{\langle x\rangle} \in P$,

$$1_{\langle x\rangle}\psi = x\nu$$

Let $1_{\langle x\rangle}$, $1_{\langle y\rangle}$ be elements of P, then

$$\begin{aligned} 1_{\langle x\rangle} = 1_{\langle y\rangle} &\iff \langle x\rangle = \langle y\rangle \\ &\iff x \in \langle y\rangle \text{ and } y \in \langle x\rangle \\ &\iff x \preccurlyeq y \text{ and } y \preccurlyeq x \\ &\iff x\nu = y\nu, \end{aligned}$$

thus ψ is well-defined and one-to-one, and we can easily see that it is a bijection. For $x, y \in X$,

$$\begin{aligned} 1_{\langle x\rangle} \leq 1_{\langle y\rangle} &\iff \langle x\rangle = \langle x\rangle \circ \langle y\rangle \\ &\iff \langle x\rangle \subseteq \langle y\rangle \\ &\iff x \preccurlyeq y \\ &\iff x\nu \trianglelefteq y\nu. \end{aligned}$$

Then ψ is an order isomorphism. □

Remark 3.2. *In $T_{(X(\preccurlyeq);\sigma)}(\mathcal{M})$, if $\preccurlyeq\; = \;\trianglelefteq$ and $\sigma = \varrho$ then $T_{(X(\trianglelefteq);\varrho)}(\mathcal{M}) = T_{X(\trianglelefteq)}(\mathcal{M})$.*

Let $(X(\preccurlyeq);\sigma)$ be an ω-set and let $Y = X/\nu$, where ν is the mpo-congruence from $\preccurlyeq$. For any element α in $T_{(X(\preccurlyeq);\sigma)}$, assume that $d(\alpha) = \langle a\rangle$. Then we can define a new mapping $\alpha^{'} \in T_{Y(\trianglelefteq)}$ as follows:

$$d(\alpha^{'}) = \{x\nu : x \in d(\alpha)\},$$

$$(x\nu)\alpha^{'} = (x\alpha)\nu.$$

Since α is an isomorphism, α' is a bijection of $\langle a\nu\rangle$ onto $\langle (a\alpha)\nu\rangle$. For $x\nu, y\nu \in \langle a\nu\rangle$, we have

$$\begin{aligned} x\nu = y\nu &\iff x \preccurlyeq y \\ &\iff x\alpha \preccurlyeq y\alpha \\ &\iff (x\alpha)\nu \trianglelefteq (y\alpha)\nu \\ &\iff (x\nu)\alpha' \trianglelefteq (y\nu)\alpha'. \end{aligned}$$

Then $\alpha^{'} \in T_{Y(\trianglelefteq)}$.

Proposition 3.3. *The mapping $\xi : \alpha \mapsto \alpha^{'}$ of $T_{(X(\preccurlyeq);\sigma)}(\mathcal{M})$ into $T_{Y(\trianglelefteq)}(\mathcal{M})$ is a $*$-homomorphism of $T_{(X(\preccurlyeq);\sigma)}(\mathcal{M})$ onto a $\mathcal{P}$-full generalized inverse $*$-subsemigroup of $T_{Y(\trianglelefteq)}(\mathcal{M})$ such that $\xi \circ \xi^{-1} = \mu$, where μ is the maximum idempotent separating $*$-congruence on $T_{(X(\preccurlyeq);\sigma)}(\mathcal{M})$.*

Proof. First we shal show that ξ is a $*$-homomorphism. It is obvious that $(\alpha^{-1})' = (\alpha')^{-1}$ for any $\alpha \in T_{(X(\preccurlyeq);\sigma)}(\mathcal{M})$. Let $\alpha, \beta \in T_{(X(\preccurlyeq);\sigma)}(\mathcal{M})$ such that $r(\alpha) = \langle x\rangle$ and $d(\beta) = \langle y\rangle$. There exist $z_1\nu, z_2\nu \in (x\nu)\varrho \wedge (y\nu)\varrho$ such that $z_1\nu \trianglelefteq x\nu$ and $z_2\nu \trianglelefteq y\nu$. Then $d(\theta_{\alpha,\beta}) = \langle z_1\rangle$ and $r(\theta_{\alpha,\beta}) = \langle z_2\rangle$. Thus $d(\alpha \circ \beta) = \langle z_1\alpha^{-1}\rangle$ and so $d((\alpha \circ \beta)') = \langle (z_1\alpha^{-1})\nu\rangle$. On the other hand, Since $r(\alpha') = \langle x\nu\rangle$ and $d(\beta') = \langle y\nu\rangle$, we have

$$d(\alpha' \circ \beta') = d(\alpha'\theta_{\alpha',\beta'}\beta') = \langle z_1\nu\rangle(\alpha')^{-1} = \langle (z_1\alpha^{-1})\nu\rangle.$$

Then $d((\alpha \circ \beta)\xi) = d((\alpha\xi) \circ (\beta\xi))$.

To show that ξ is a $*$-homomorphism, it is sufficient to show that $\theta_{\alpha',\beta'} = (\theta_{\alpha,\beta})'$. It is clear that $d(\theta_{\alpha',\beta'}) = d((\theta_{\alpha,\beta})') = \langle z_1\nu\rangle$. For any $a\nu \in \langle z_1\nu\rangle$, set $a\nu(\theta_{\alpha,\beta})' = (a\theta_{\alpha,\beta})\nu = b\nu$ and $a\nu\theta_{\alpha',\beta'} = c\nu$. Since $(a,b) \in \sigma$, $(a\nu, b\nu) \in \varrho$. On the other hand, $(a\nu, c\nu) \in \varrho$. Since $\langle z_1\nu\rangle$ is an ι-set, $b\nu = c\nu$, and we have $\theta_{\alpha',\beta'} = (\theta_{\alpha,\beta})'$.

It is clear that $(T_{(X(\preccurlyeq);\sigma)}(\mathcal{M}))\xi$ is $\mathcal{P}$-full and fundamental. To show that $\xi \circ \xi^{-1} = \mu$, it is sufficient to prove ξ separates projections. For $1_{\langle x\rangle}, 1_{\langle y\rangle} \in P(T_{(X(\preccurlyeq);\sigma)}(\mathcal{M}))$,

$$\begin{aligned} 1_{\langle x\rangle}\xi = 1_{\langle y\rangle}\xi &\implies 1_{\langle x\nu\rangle} = 1_{\langle y\nu\rangle} \\ &\implies x\nu \in \langle y\nu\rangle \text{ and } y\nu \in \langle x\nu\rangle \\ &\implies x\nu \trianglelefteq y\nu \text{ and } y\nu \trianglelefteq x\nu \\ &\implies x \preccurlyeq y \text{ and } y \preccurlyeq x \\ &\implies \langle x\rangle \subseteq \langle y\rangle \text{ and } \langle y\rangle \subseteq \langle x\rangle \\ &\implies 1_{\langle x\rangle} = 1_{\langle y\rangle}. \end{aligned}$$

Thus we have the proposition. □

Hereafter, we shall refer to ξ as the *natural projection* of $T_{(X(\preccurlyeq);\sigma)}(\mathcal{M})$ to $T_{Y(\trianglelefteq)}(\mathcal{M})$.

3.2. Inflated representations. Let S be a generalized inverse $*$-semigroup. Hereafter, denote $E(S)$ and $P(S)$ simply by E and P, respectively. Define a relation $\preccurlyeq$ on S by:

$$a \preccurlyeq b \text{ if and only if } a^*a \leq b^*b, \tag{3.2}$$

for $a, b \in S$. Then clearly $\preccurlyeq$ is a pre-order on S for which the mpo-congruence from $\preccurlyeq$ is $\nu = \mathcal{L}$. Hence $S/\mathcal{L} = S/\nu$, under the naturally induced order $\trianglelefteq$ from $\preccurlyeq$, is just the set of $\mathcal{L}$-classes of S under the usual partial ordering of the $\mathcal{L}$-classes of a generalized inverse $*$-semigroup and so is order isomorphic to the partially ordered ϱ-set P of S. Hence S is a partially pre-ordered ϱ-set under $\preccurlyeq$. Then $\varrho = \mathcal{J}^E|_P$ and hence $(a\nu)\varrho(b\nu) \iff a^*a\mathcal{J}^E b^*b$. Hereafter, for any $a \in S$, we think $a\nu = L_{a^*a}$ as a^*a.

For any $a \in S$, define a mapping $\rho_a : Sa^* \to Sa$ as follows:

$$d(\rho_a) = Sa^*(= Saa^*),$$

$$x\rho_a = xa.$$

Let $\rho : S \to \mathcal{GI}_{(S;\Omega)}(\mathcal{M})$ by $a\rho = \rho_a$, where the relation Ω defined by: for $x, y \in S$,

$$(x,y) \in \Omega \iff x\rho_e = y \text{ for some } e \in E. \tag{3.3}$$

Since S is a regular $*$-semigroup, the representation ρ is faithful. Moreover, it follows from [6, Lemma 3.3] that it is a $*$-monomorphism.

Lemma 3.4. *The set $(S(\preccurlyeq);\Omega)$, defined above, is an ω-set.*

Proof. Let a be any element of S. Then

$$\begin{aligned} x \in Sa &\iff x^*x = a^*ax^*xa^*a \leq a^*a \\ &\iff x \preccurlyeq a \\ &\iff a \in \langle a \rangle. \end{aligned}$$

Thus we have $Sa = \langle a \rangle$. By Lemma 3.2 [7], $\langle a \rangle$ is an ι-single subset.

Next, let (a, b) be any element of Ω. It follows from Lemma 3.1 [7] that $b = ab^*b$ and $a^*a\mathcal{R}e\mathcal{L}b^*b$ for some $e \in E$. Thus $a^*a\mathcal{J}^E b^*b$, and hence $(a\nu)\varrho(b\nu)$.

Assume that $J_{a^*a} \wedge J_{b^*b} = J_{c^*c}$. Then $J_{c^*c} = J_{a^*ab^*b} = J_{b^*ba^*a}$. Also, we have $a^*ab^*ba^*a \leq a^*a$, $b^*ba^*ab^*b \leq b^*b$, and hence $b^*ba^*a \preccurlyeq a$ and $a^*ab^*b \preccurlyeq b$. Let x $(= xb^*ba^*a)$ be any element of $\langle b^*ba^*a \rangle$ and let $y = xa^*ab^*b$. Then it is clear that $x = yx^*x$ and $y = y^*y$. It follows from Lemma 3.1 [7] that $y \in \langle a^*ab^*b \rangle$ ans $(x, y) \in \Omega$. Similarly, for any $y \in \langle a^*ab^*b \rangle$, we have $x = yb^*ba^*a \in \langle b^*ba^*a \rangle$ and $(x, y) \in \Omega$. Hence $(S(\preccurlyeq); \Omega)$ is an ω-set. □

Again, we consider $\rho_a : Sa^* \to Sa$. By Lemma 3.4, $d(\rho_a) = \langle a^* \rangle$ and $r(\rho_a) = \langle a \rangle$. For $x, y \in d(\rho_a)$, $x^*x, y^*y \leq a^*a$. Now $x \preccurlyeq y$ if and only if $x^*x \leq y^*y$ while $xa \preccurlyeq ya$ if and only if $a^*x^*xa = (xa)^*(xa) \leq (ya)^*(ya) = a^*y^*ya$. But, since $x^*x, y^*y \leq a^*a$ it follows that $x^*x \leq y^*y$ if and only if $a^*x^*xa \leq a^*y^*ya$. Therefore $x \preccurlyeq y$ if and only if $xa \preccurlyeq ya$. Thus ρ_a is an isomorphism of $\langle a^* \rangle$ onto $\langle a \rangle$, and hence $S\rho \subseteq T_{(S(\preccurlyeq);\Omega)}(\mathcal{M})$.

Now, we have the following theorem.

Theorem 3.5. *Let S be a generalized inverse $*$-semigroup and let $\preccurlyeq$ be the relation on S defined in* (3.2). *Then $\preccurlyeq$ is a pre-order on S with respect to which S is a partially pre-ordered ϱ-set. Moreover, if Ω is the relation defined in* (3.3), *then $(S(\preccurlyeq); \Omega)$ is an ω-set. The faithful representation ρ, defined above, embeds S as a $\mathcal{P}$-full generalized inverse $*$-subsemigroup of $T_{(S(\preccurlyeq);\Omega)}(\mathcal{M})$.*

If ν is the mpo-congruence on S from $\preccurlyeq$, then $\nu = \mathcal{L}$ and S/ν is order isomorphic to the partially ordered ϱ-set P of S. Moreover, $\rho\xi = \tau$, where ξ is the natural projection and τ is the representation which is defined in Theorem 2.5.

Proof. It remains to show that $S\rho$ is a $\mathcal{P}$-full generalized inverse $*$-subsemigroup of $T_{(S(\preccurlyeq);\Omega)}(\mathcal{M})$ and that $\rho\xi = \tau$. Let $1_{\langle a \rangle}$ $(a \in S)$ be any projection of $T_{(S(\preccurlyeq);\Omega)}(\mathcal{M})$ and let $e = a^*a$. Then $1_{\langle a \rangle}$ and ρ_e are both identity mappings on $\langle a \rangle$. Thus $1_{\langle a \rangle} = \rho_e$ and $S\rho$ is a $\mathcal{P}$-full generalized inverse $*$-subsemigroup of $T_{(S(\preccurlyeq);\Omega)}(\mathcal{M})$.

Next, let ρ_a $(a \in S)$ be an element of $S\rho$. Then

$$\begin{aligned} d(\rho') &= \{x^*x : x \in Sa^*\} \\ &= \{x^*x : x^*x \in Sa^* \cap P\} \\ &= Sa^* \cap P, \end{aligned}$$

and hence $d(\rho') = d(\tau_a)$. Moreover, for any $x^*x \in d(\rho')$,

$$(x^*x)\rho'_a = (xa)^*(xa) = a^*x^*xa = (x^*x)\tau_a.$$

Thus $\rho'_a = \tau_a$, and hence $\rho_a\xi = \tau_a$. Therefore, $\rho\xi = \tau$, as required. □

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