SOME REMARKS ON REPRESENTATIONS OF GENERALIZED INVERSE $*$-SEMIGROUPS

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Abstract. The Munn representation of an inverse semigroup $S$, in which the semigroup is represented by isomorphisms between principal ideals of the semilattice $E(S)$, is not always faithful. By introducing a concept of a pre-semilattice, Reilly considered of enlarging the carrier set $E(S)$ of the Munn representation in order to obtain a faithful representation of $S$ as an inverse subsemigroup of a structure resembling the Munn semigroup $T_{E(S)}$.

The purpose of this paper is to obtain a generalization of the Reilly’s results for generalized inverse $*$-semigroups.

1. Introduction

A semigroup $S$ with a unary operation $*: S \to S$ is called a regular $*$-semigroup if it satisfies

(i) $(x^*)^* = x$,
(ii) $(xy)^* = y^*x^*$,
(iii) $xx^*x = x$.

Let $S$ be a regular $*$-semigroup. An idempotent $e$ in $S$ is called a projection if it satisfies $e^* = e$. For any subset $A$ of $S$, denote the sets of idempotents and projections of $A$ by $E(A)$ and $P(A)$, respectively.

Let $S$ be a regular $*$-semigroup. It is called a locally inverse $*$-semigroup if, for any $e \in E(S)$, $eSe$ is an inverse subsemigroup of $S$. If $E(S)$ is a normal band, then $S$ is called a generalized inverse $*$-semigroup.

Let $S$ and $T$ be regular $*$-semigroups. A homomorphism $\phi: S \to T$ is called a $*$-homomorphism if $(a\phi)^* = a^*\phi$. A congruence $\sigma$ on $S$ is called a $*$-congruence if $(a\sigma)^* = a^*\sigma$. A $*$-congruence $\sigma$ on $S$ is said to be idempotent-separating if $\sigma \subseteq \mathcal{H}$, where $\mathcal{H}$ is one of the Green’s relations. Denote the maximum idempotent-separating $*$-congruence on $S$ by $\mu_S$ or simply by $\mu$. If $\mu_S$ is the identity relation

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on $S$, $S$ is called fundamental. The following results are well-known, and we use them frequently throughout this paper.

**Result 1.1.** [2] Let $S$ be a regular $\ast$-semigroup. Then we have the following:

1. $E(S) = P(S)^2$;
2. for any $a \in S$ and $e \in P(S)$, $a^* ea \in P(S)$;
3. each $L$-class and each $R$-class have one and only one projection;
4. $\mu_S = \{(a, b) \in S \times S : a^* ea = b^* eb$ and $aea^* = eb^* e$ for all $e \in P(S)\}$.

For a mapping $\alpha : A \rightarrow B$, denote the domain and the range of $\alpha$ by $d(\alpha)$ and $r(\alpha)$, respectively. For a subset $C$ of $A$, $\alpha|_C$ means the restriction of $\alpha$ to $C$.

As a generalization of the Preston-Vagner representations, one of the authors gave two types of representations of locally [generalized] inverse $\ast$-semigroups in [4], [6] and [7]. In this paper, we follow [7]. A non-empty set $X$ with a reflexive and symmetric relation $\sigma$ is called an $\iota$-set, and denoted by $(X; \sigma)$. If $\sigma$ is transitive, that is, if $\sigma$ is an equivalence relation on $X$, $(X; \sigma)$ is called a transitive $\iota$-set.

Let $(X; \sigma)$ be an $\iota$-set. A subset $A$ of $X$ is called an $\iota$-single subset of $(X; \sigma)$ if it satisfies the following condition:

for any $x \in X$, there is at most one element $y \in A$ such that $(x, y) \in \sigma$.

We consider the empty set to be an $\iota$-single subset. We remark that if $(X; \sigma)$ is a transitive $\iota$-set, a subset $A$ of $X$ is an $\iota$-single subset if and only if, for $x, y \in A$, $(x, y) \in \sigma$ implies $x = y$. A mapping $\alpha$ in $I_X$, the symmetric inverse semigroup on $X$, is called a partial one-to-one $\iota$-mapping on $(X; \sigma)$ if $d(\alpha), r(\alpha)$ are both $\iota$-single subsets of $(X; \sigma)$, where $d(\alpha)$ and $r(\alpha)$ are the domain and the range of $\alpha$, respectively. Denote the set of all partial one-to-one $\iota$-mappings of $(X; \sigma)$ by $LI_{(X; \sigma)}$.

For any $\iota$-single subsets $A$ and $B$ of $(X; \sigma)$, define $\theta_{A,B}$ by

$$\theta_{A,B} = \{(a, b) \in A \times B : (a, b) \in \sigma\} = (A \times B) \cap \sigma.$$ 

Since a subset of an $\iota$-single subset is also an $\iota$-single subset, $\theta_{A,B} \in LI_{(X; \sigma)}$.

For any $\alpha, \beta \in LI_{(X; \sigma)}$, define $\theta_{\alpha|,\beta}$ by $\theta_{\alpha|,\beta} = \theta_{r(\alpha),d(\beta)}$, and let $\mathcal{M} = \{\theta_{\alpha|,\beta} : \alpha, \beta \in LI_{(X; \sigma)}\}$, an indexed set of one-to-one partial functions. Now, define a multiplication $\circ$ and a unary operation $\ast$ on $LI_{(X; \sigma)}$ as follows:

$$\alpha \circ \beta = \alpha\theta_{\alpha|,\beta} \beta \quad \text{and} \quad \alpha^* = \alpha^{-1},$$

where the multiplication of the right side of the first equality is that of $I_X$. Denote $(LI_{(X; \sigma)}, \circ, \ast)$ by $LI_{(X; \sigma)}(\mathcal{M})$ or simply by $LI_{(X; \sigma)}$. In this paper, we use $LI_{(X; \sigma)}$ rather than $LI_{(X; \sigma)}(\mathcal{M})$. 
RESULT 1.2. [7] For any $t$-set $(X; \sigma)$, $\mathcal{LI}(X, \sigma)$, defined above, is a locally inverse $*$-semigroup. If $(X; \sigma)$ is a transitive $t$-set, then $\mathcal{LI}(X, \sigma)$ is a generalized inverse $*$-semigroup. In this case, we denote it by $\mathcal{GI}(X, \sigma)$ instead of $\mathcal{LI}(X, \sigma)$.

Moreover, if $\sigma$ is the identity relation on $X$, then $\mathcal{LI}(X, \sigma)$ is the symmetric inverse semigroup $\mathcal{I}_X$ on $X$.

We call $\mathcal{LI}(X, \sigma) [\mathcal{GI}(X, \sigma)]$ the $t$-symmetric locally [generalized] inverse $*$-semigroup on the $t$-set (the transitive $t$-set) $(X; \sigma)$ with the structure sandwich set $\mathcal{M}$.

Let $S$ be a regular $*$-semigroup, and define a relation $\Omega$ on $S$ as follows:

$$(x, y) \in \Omega \iff \text{there exists } e \in E(S) \text{ such that } x \rho_e = y,$$

where $\rho_a (a \in S)$ is the mapping of $Sa^*$ onto $Sa$ defined by $x \rho_a = xa$.

RESULT 1.3. [7] Let $S$ be a locally inverse $*$-semigroup. For each $a \in S$, let

$$\rho_a : x \mapsto xa \quad (x \in d(\rho_a) = Sa^*).$$

Then a mapping $\rho : a \mapsto \rho_a$ is a $*$-monomorphism of $S$ into $\mathcal{LI}(S, \Omega)(\mathcal{M})$.

For a partial groupoid $X$, if there exist a semilattice $Y$, a partition $\pi : X \sim \sum \{X_e : e \in Y\}$ of $X$ and mappings $\varphi_{e,f} : X_e \rightarrow X_f (e \geq f \in Y)$ such that

1. for any $e \in Y$, $\varphi_{e,e} = 1_{X_e}$,
2. if $e \geq f \geq g$, then $\varphi_{e,f} \varphi_{f,g} = \varphi_{e,g}$,
3. for $x \in X_e$, $y \in X_f$, $xy$ is defined in $X$ if and only if $x \varphi_{e,e,f} = y \varphi_{f,e,f}$, and in this case $xy = x \varphi_{e,e,f}$,

then $X$ is called a strong $\pi$-groupoid with mappings $\{\varphi_{e,f} : e, f \in Y, e \geq f\}$, and it is denoted by $X(\pi; Y; \{\varphi_{e,f}\})$ or simply by $X(\pi)$.

Let $X(\pi; Y; \{\varphi_{e,f}\})$ be a strong $\pi$-groupoid. A subset $A$ of $X$ is called a $\pi$-singleton subset of $X(\pi; Y; \{\varphi_{e,f}\})$, if there exists $e \in Y$ such that

$$|A \cap X_f| = \begin{cases} 1 & \text{if } f \in \langle e \rangle, \\ 0 & \text{otherwise}, \end{cases}$$

$$(A \cap X_f) \varphi_{f,g} = A \cap X_g \quad \text{for any } f, g \in \langle e \rangle \text{ such that } f \geq g,$$

where $\langle e \rangle$ is the principal ideal of $Y$ generated by $e$. In this case, we sometimes denote the $\pi$-singleton subset $A$ by $A(e)$. If $A(e)$ is a $\pi$-singleton subset, then $|A \cap X_f| = 1$ for any $f \in \langle e \rangle$. We denote the only one element of $A \cap X_f$ by $a_f$. We remark that, for any $\pi$-singleton subset $A(e)$, $A(e) = \{a_e \varphi_{e,f} : f \in \langle e \rangle\}$. Denote the set of all $\pi$-singleton subsets of $X(\pi; Y; \{\varphi_{e,f}\})$ by $\mathcal{X}$.

Two $\pi$-singleton subsets $A(e)$ and $B(f)$ are said to be $\pi$-isomorphic to each other, if there exists an isomorphism $\pi : \langle e \rangle \rightarrow \langle f \rangle$ as semilattices. In this case, the mapping $\alpha : A(e) \rightarrow B(f)$ defined by $a_e \alpha = b_g \pi (g \in \langle e \rangle)$ is called a $\pi$-isomorphism of $A(e)$ to $B(f)$. It is obvious that $\alpha$ is a bijection of $A(e)$ onto $B(f)$, and hence $\alpha \in \mathcal{I}_X$. 
Let $X(\pi; Y; \{\varphi_{e,f}\})$ be a strong $\pi$-groupoid. Define an equivalence relation $\mathcal{U}$ on $X$ by

$$\mathcal{U} = \{(A(e), B(f)) \in X \times X : \langle e \rangle \cong \langle f \rangle \text{ (as semilattices)}\}.$$  

For $(A(e), B(f)) \in \mathcal{U}$, let $T_{A(e), B(f)}$ be the set of all $\pi$-isomorphisms of $A(e)$ onto $B(f)$, and let

$$T_{X(\pi)} = \bigcup_{(A(e), B(f)) \in \mathcal{U}} T_{A(e), B(f)}.$$  

For any $\alpha, \beta \in T_{X(\pi)}$, define a mapping $\theta_{\alpha, \beta}$ as follows:

$$d(\theta_{\alpha, \beta}) = \{a \in r(\alpha) : \text{there exist } e \in Y \text{ and } b \in d(\beta) \text{ such that } a, b \in X_e\},$$

$$r(\theta_{\alpha, \beta}) = \{b \in d(\beta) : \text{there exist } e \in Y \text{ and } a \in r(\alpha) \text{ such that } a, b \in X_e\},$$

$$a\theta_{\alpha, \beta} = b \text{ if } r(\alpha) \cap X_e = \{a\} \text{ and } d(\beta) \cap X_e = \{b\}.$$  

Then $\theta_{\alpha, \beta} \in T_{X(\pi)}$. Let $\mathcal{M} = \{\theta_{\alpha, \beta} : \alpha, \beta \in T_{X(\pi)}\}$, and define a multiplication $\circ$ and a unary operation $\ast$ on $T_{X(\pi)}$ by

$$\alpha \circ \beta = \alpha \theta_{\alpha, \beta} \beta,$$

$$\alpha^\ast = \alpha^{-1}.$$  

Then $T_{X(\pi)}(\circ, \ast)$ is a regular $\ast$-semigroup. We denote it by $T_{X(\pi)}(\mathcal{M})$.

**Result 1.4.** [5] A regular $\ast$-semigroup $T_{X(\pi)}(\mathcal{M})$ is a generalized inverse $\ast$-semigroup whose set of projections is partially isomorphic to $X$.

Let $S$ be a generalized inverse $\ast$-semigroup. Hereafter, denote $E(S)$ and $P(S)$ simply by $E$ and $P$, respectively. Let $E \sim \sum\{E_i : i \in I\}$ be the structure decomposition of $E$, and let $P_i = P(E_i)$. Then $\pi : P \sim \sum\{P_i : i \in I\}$ is a partition of $P$. For any $i, j \in I (i \geq j)$, define a mapping $\varphi_{i,j} : P_i \rightarrow P_j$ by

$$e\varphi_{i,j} = e f e \text{ for some (any) } f \in P_j.$$  

Then $P(\pi; I; \{\varphi_{i,j}\})$ is a strong $\pi$-groupoid.

**Result 1.5.** [5] Let $S$ be a generalized inverse $\ast$-semigroup. For each $a \in S$, let

$$\tau_a : e \mapsto a^\ast e a \quad (e \in d(\tau_a) = P(Sa^\ast)).$$

Then a mapping $\tau : a \mapsto \tau_a$ is a $\ast$-homomorphism of $S$ into $T_{P(\pi)}(\mathcal{M})$ such that $\tau \circ \tau^{-1} = \mu$.

A regular $\ast$-subsemigroup $T$ of a regular $\ast$-semigroup $S$ is said to be $P$-full if $P(T) = P(S)$. 

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Result 1.6. [5] A generalized inverse \( * \)-semigroup \( S \) is fundamental if and only if it is \( * \)-isomorphic to a \( P \)-full generalized inverse \( * \)-subsemigroup of \( T_X(\pi)(M) \) on a strong \( \pi \)-groupoid \( X(\pi; \{\varphi_i\}) \) such that \( P(T_X(\pi)(M)) \) is partially isomorphic to \( P(S) \).

In \( \S\ 2 \), by introducing the concept of partially ordered \( \varrho \)-set \( (X(\leq); \{\phi_x\}) \), we construct a fundamental generalized inverse \( * \)-semigroup \( T_X(\leq)(M) \). Also, we shall see that \( T_X(\leq)(M) \) has similar properties with \( T_X(\pi)(M) \), where \( T_X(\pi)(M) \) has been given by T. Imaoka, I. Inata and H. Yokoyama [5]. And we shall show that two concepts, strong \( \pi \)-groupoids and partially ordered \( \varrho \)-sets, are equivalent.

In \( \S\ 3 \), we shall introduce the notion of \( \omega \)-set \( (X(\ll); \sigma) \), and construct a generalized inverse \( * \)-semigroup \( T_{(X(\ll);\sigma)}(M) \). Furthermore, let \( S \) be a generalized inverse \( * \)-semigroup with the set of projections \( P \), we shall make two generalized inverse \( * \)-semigroups \( T_{P(\leq)}(M) \) and \( T_{(S(\leq);\Omega)}(M) \), where the former is obtained in \( \S\ 2 \), and the latter is constructed in this section. Then we shall show that these three semigroups make a commutative diagram.

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2. Fundamental generalized inverse \( * \)-semigroups

2.1. \( T_X(\leq)(M) \). Let \( X(\leq) \) be a partially ordered set and, for each \( x \in X \), consider an order-preserving mapping \( \phi_x : X \to X \). If a relation \( \varrho = \{ (x, y) \in X \times X : y \phi_x = x, x \phi_y = y \} \) is an equivalence relation on \( X \) such that

(P1) \( x \leq y \implies \) for each \( y' \in y \varrho \), there exists \( x' \in x \varrho \) such that \( x' \leq y' \),

(P2) a relation \( \leq = \{ (x \varrho, y \varrho) \in X/\varrho \times X/\varrho : \) there exists \( x' \in x \varrho \) such that \( x' \leq y' \} \) is a partial order and \( X/\varrho(\leq) \) is a semilattice,

(P3) \( x_1 \leq y_1, x_2 \leq y_2 \) and \( x_1 \varrho \leq x_2 \varrho \implies x_1 \leq x_2 \),

then \( (X(\leq); \{\phi_x\}) \) is called a partially ordered \( \varrho \)-set.

Let \( (X(\leq); \{\phi_x\}) \) be a partially ordered \( \varrho \)-set. Define an equivalence relation \( \mathcal{U} \) on \( \mathcal{X} \) by

\[ \mathcal{U} = \{ \langle a \rangle, \langle b \rangle \in \mathcal{X} \times \mathcal{X} : \langle a \rangle \simeq \langle b \rangle \text{ (order isomorphic)} \}, \]

where \( \mathcal{X} \) is the set of all principal ideals of \( (X(\leq); \{\phi_x\}) \). For \( \langle a \rangle, \langle b \rangle \in \mathcal{U} \), let \( T_{(a), (b)} \) be the set of all (order) isomorphisms of \( \langle a \rangle \) onto \( \langle b \rangle \), and let

\[ T_X(\leq) = \bigcup_{(a), (b) \in \mathcal{U}} T_{(a), (b)}. \]
For any \( \alpha, \beta \in T_X(\leq) \), define a mapping \( \theta_{\alpha, \beta} \) as follows:
\[
\theta_{\alpha, \beta} = \{(x, y) \in r(\alpha) \times d(\beta) : (x, y) \in \varrho\},
\]
where \( \varrho \) is the equivalence relation on \( X \) induced by \( \{\phi_x\} \), as defined above.

To show that \( \theta_{\alpha, \beta} \in T_X(\leq) \), assume that \( r(\alpha) = \langle a \rangle \), \( d(\beta) = \langle b \rangle \) and \( a\varrho \land b\varrho = c\varrho \) (\( c \in X \)). Since \( c\varrho \leq a\varrho \) and \( c\varrho \leq b\varrho \), there exist \( c_1, c_2 \in c\varrho \) such that \( c_1 \leq a \) and \( c_2 \leq b \). For any \( x \in d(\theta_{\alpha, \beta}) \), there exists \( y \in \langle b \rangle \) such that \( (x, y) \in \varrho \). Since \( x \leq a, c_1 \leq a \) and \( x\varrho \leq c_1\varrho \), we have \( x \leq c_1 \) and so \( x \in \langle c_1 \rangle \). Thus \( d(\theta_{\alpha, \beta}) \subseteq \langle c_1 \rangle \).

Conversely, let \( x \) be any element of \( \langle c_1 \rangle \). Since \( x\varrho \leq c_1\varrho = c_2\varrho \), there exists \( y \in x\varrho \) such that \( y \leq c_2 \). Therefore, \( x \in \langle c_1 \rangle \subseteq \langle a \rangle \), \( y \in \langle c_2 \rangle \subseteq \langle b \rangle \) and \( (x, y) \in \varrho \), and so \( x \in d(\theta_{\alpha, \beta}) \). Thus \( \langle c_1 \rangle \subseteq d(\theta_{\alpha, \beta}) \), and hence \( d(\theta_{\alpha, \beta}) = \langle c_1 \rangle \). Similarly, \( r(\theta_{\alpha, \beta}) = \langle c_2 \rangle \). Since it is obvious that \( \theta_{\alpha, \beta} \) is a bijection, we have \( \theta_{\alpha, \beta} \in T_X(\leq) \).

Let \( \mathcal{M} = \{\theta_{\alpha, \beta} : \alpha, \beta \in T_X(\leq)\} \), and define a multiplication \( \circ \) and a unary operation \( \ast \) on \( T_X(\leq) \) by
\[
\alpha \circ \beta = \alpha \theta_{\alpha, \beta} \beta, \quad \alpha^\ast = \alpha^{-1}.
\]

Then it is clear that \( T_X(\leq)(\circ, \ast) \) is a regular \( \ast \)-subsemigroup of the \( \tau \)-symmetric generalized inverse \( \ast \)-semigroup \( GI_{X, \varrho}(\mathcal{M}) \). Hence it is a generalized inverse \( \ast \)-semigroup and denoted by \( T_X(\leq)(\mathcal{M}) \).

Let \( S \) be a generalized inverse \( \ast \)-semigroup and \( P = P(S) \). We consider \( P \) as a partially ordered set with respect to the natural order. Now, we have the following results.

**Theorem 2.1.** A regular \( \ast \)-semigroup \( T_X(\leq)(\mathcal{M}) \) is a generalized inverse \( \ast \)-semigroup whose set of projections is order isomorphic to \( X(\leq) \).

**Proof.** It remains to show that \( T_X(\leq)(\mathcal{M}) \) is order isomorphic to \( X(\leq) \). It is clear that \( P(T_X(\leq)(\mathcal{M})) = \{1_{(a)} : a \in X\} \). Define a mapping \( \psi : X \to P(T_X(\leq)(\mathcal{M})) \) by \( a\psi = 1_{(a)} \) for \( a \in X \). It is obvious that \( \psi \) is onto. For \( a, b \in X \),
\[
1_{(a)} = 1_{(b)} \quad \implies \quad \langle a \rangle = \langle b \rangle \\
\implies \quad a \leq b \quad \text{and} \quad b \leq a \\
\implies \quad a = b.
\]
Thus \( \psi \) is one-to-one, and hence it is bijection.

Suppose that \( a \leq b \). Then \( \langle a \rangle \subseteq \langle b \rangle \). Thus \( 1_{(a)} \circ 1_{(b)} = \theta_{(a)\ast(\ast)} = \theta_{(a)\ast(a)} = 1_{(a)} \), and so \( 1_{(a)} \leq 1_{(b)} \). Conversely, let \( 1_{(a)} \leq 1_{(b)} \). Then \( 1_{(a)} = 1_{(a)} \circ 1_{(b)} \), and so \( \langle a \rangle = r(1_{(a)}) = r(1_{(a)} \circ 1_{(b)}) \subseteq \langle b \rangle \). Thus \( a \leq b \), and hence \( \psi \) is an isomorphism. \( \square \)

**Corollary 2.2.** A partially ordered set \( X \) is order isomorphic to the set of projections of a generalized inverse \( \ast \)-semigroup if and only if it is a partially ordered \( \varrho \)-set.
2.2. Representations. Let \( S \) be a generalized inverse \( * \)-semigroup. Hereafter, denote \( E(S) \) and \( P(S) \) simply by \( E \) and \( P \), respectively. Let \( E \approx \sum_{i \in I} E_i \) be the structure decomposition of \( E \), and let \( P_i = P(E_i) \). For any \( e \in P \), define a mapping \( \phi_e : P \to P \) by
\[
\phi_e f = efe.
\]
Let \( \leq \) be the natural order on \( S \), that is,
\[
a \leq b \iff a = cb = bf \quad \text{for some} \quad e, f \in P.
\]
Since \( S \) is a generalized inverse \( * \)-semigroup, it follows from [3] that \( \leq \) is compatible. Let \( \leq \) be the restriction of \( \leq \) to \( P \). It is obvious that for \( e, f \in P \),
\[
e \leq f \iff e = fef.
\]
Lemma 2.3. The set \( (P(\leq); \{\phi_e\}) \), defined above, is a partially ordered \( g \)-set.

Proof. Let \( e, f \) and \( g \) be any elements of \( P \) such that \( f \leq g \). Since \( \leq \) is compatible, \( \phi_e = efe \leq ege = g\phi_e \). Thus \( \phi_e \) is order preserving.

For \( e \in P_i \) and \( f \in P_j \),
\[
eg \iff f\phi_e = e \quad \text{and} \quad e\phi_f = f
\]
\[
eg \iff efe = e \quad \text{and} \quad fef = f
\]
\[
eg \iff e\mathcal{J}^E f
\]
\[
eg \iff i = j.
\]
Then \( g = \mathcal{J}^E|_P \), and so \( P/g = \{P_i : i \in I\} \). It is easily to see that \( g \) satisfies the conditions (P1), (P2) and (P3), and we have the lemma. \( \blacksquare \)

Now, we can consider the generalized inverse \( * \)-semigroup \( T_{P(\leq)}(\mathcal{M}) \), where \( \mathcal{M} = \{\theta_{\alpha, \beta} : \alpha \text{ and } \beta \text{ are order isomorphisms among principal ideals of } (P(\leq); \{\phi_e\})\} \).

Lemma 2.4. For any \( a \in S \), \( P(\mathcal{J}a) \) (\( = P(\mathcal{J}a^*a) \)) is a principal ideal of \( (P(\leq); \{\phi_e\}) \).

Proof. We shall show that \( P(\mathcal{J}a) = \langle a^*a \rangle \). Let \( xa \) be any element of \( P(\mathcal{J}a) \). Since \( xa \) is a projection, \( xa = (xa)^*xa \), and so \( xa \leq a^*a \). Thus \( P(\mathcal{J}a) \subseteq \langle a^*a \rangle \). Conversely, let \( e \in P \) such that \( e \leq a^*a \). Then \( a^*aea^*a \), and so \( e \in P(\mathcal{J}a) \). Therefore, we have \( P(\mathcal{J}a) = \langle a^*a \rangle \). \( \blacksquare \)

For any \( a \in S \), define a mapping \( \tau_a : \langle aa^* \rangle \to \langle a^*a \rangle \) by
\[
\tau_a \mapsto a^*ea.
\]
It follows from [5] that \( \tau_a \in T_{S(\leq)} \) and \( \tau_a^* = \tau_a^* \). Moreover, for any \( a, b \in S \), \( \theta_{\tau_a, \tau_b} = \tau_{aabb} \). And we have the following theorems.

Theorem 2.5. Let \( S \) be a generalized inverse \( * \)-semigroup such that \( E(S) = E \) and \( P(S) = P \). Let \( E \approx \sum_{i \in I} E_i \) be the structure decomposition of \( E \) and \( P_i = P(E_i) \). Denote the restriction of the natural order on \( S \) to \( P \) by \( \leq \). For any \( e \in P \), define a mapping \( \phi_e : P \to P \) by \( \phi_e = efe \). Then \( (P(\leq); \{\phi_e\}) \) is a partially ordered \( g \)-set and \( T_{P(\leq)}(\mathcal{M}) \) is a generalized inverse \( * \)-semigroup.
Moreover, for any \(a \in S\), define a mapping \(\tau_a : \langle aa^* \rangle \to \langle a^*a \rangle\) by \(e\tau_a = a^*ea\). Then a mapping \(\tau : S \to T_{P(\preceq)}(M) (a \mapsto \tau_a)\) is a \(*\)-homomorphism and the kernel of \(\tau\) is the maximum idempotent-separating \(*\)-congruence on \(S\).

**Theorem 2.6.** A generalized inverse \(*\)-semigroup \(S\) is fundamental if and only if it is \(*\)-isomorphic to a \(P\)-full generalized inverse \(*\)-subsemigroup of \(T_{X(\preceq)}(M)\) on a partially ordered \(\rho\)-set \((X(\preceq); \{\phi_x\})\) such that \(P(T_{X(\preceq)}(M))\) is order isomorphic to \(P(S)\).

Denote the sets of all partially ordered \(\rho\)-sets and the set of all strong \(\pi\)-groupoids by \(P\) and \(S\), respectively.

**Remark 2.7.** Let \((X(\preceq); \{\phi_x\})\) be any element of \(P\). For any \(x \rho, y \rho \in X/\rho (x \rho \geq y \rho)\), define a mapping \(\overline{\varphi}_{xy} : X/\rho \to X/\rho\) by

\[
x' \overline{\varphi}_{xy} = y', \text{ where } y' \in y \rho \text{ such that } y' \preceq x'.
\]

Moreover, we define a partial product on \(X\) as follows:

\[
xy = \begin{cases} 
  x \overline{\varphi}_{xy}(x \rho)(y \rho) & \text{if } x \overline{\varphi}_{xy}(x \rho)(y \rho) = y \overline{\varphi}_{xy}(x \rho)(y \rho) \\
  \text{undefined} & \text{otherwise.}
\end{cases}
\]

Then \((X(\preceq); \{\phi_x\})\) is a \(\pi\)-groupoid, where \(\pi_{\rho}\) is the partition of \(X\) induced by \(\rho\).

Conversely, let \((X(\preceq); \{\varphi_{e,f}\})\) be any element of \(S\). For any \(x \in X\), define a mapping \(\tilde{\phi}_x : X \to X\) by

\[
y \tilde{\phi}_x = x \varphi_{e,ef},
\]

where \(x \in X_e\) and \(y \in X_f\). If we define \(\triangleright = \{(x, y) \in X \times X : x \varphi_y = x\}\), then \(X(\preceq); \{\varphi_{e,f}\})\) is a partially ordered \(\rho\)-set.

Hence the mappings \(\lambda, \mu\) from \(P\) to \(S\) and from \(S\) to \(P\), respectively, are well-defined. Moreover \(\lambda \mu = 1_S\), and for any \((X(\preceq); \{\phi_x\}) \in P\), if \((X(\preceq); \{\phi_x\})\lambda \mu = (X(\preceq); \{\tilde{\phi}_x\})\) then \(\preceq = \triangleright\).

By the above argument, for any \((X(\preceq); \{\phi_x\})\) in \(P\), without loss of generality, we can consider \((X(\preceq); \{\phi_x\})\) as a member of \(P\lambda \mu\).

Now, let \((X(\preceq); \{\varphi_{e,f}\})\) be any element of \(S\). If \((X(\preceq); \{\varphi_{e,f}\})\mu = (X(\preceq); \{\phi_x\})\), then we can construct two generalized inverse \(*\)-semigroups \(T_{X(\preceq)}(M)\) and \(T_{X(\preceq)}(M)\). In this case, these two generalized inverse \(*\)-semigroups are \(*\)-isomorphic.

### 3. Extensions of \(T_{X(\preceq)}(M)\)

#### 3.1. \(T_{X(\preceq);\rho}(M)\)

By a **pre-order** on a set \(X\) we shall mean a reflexive and transitive relation. Let \((X(\preceq); \{\phi_x\})\) be a pre-ordered set and let \(\nu = \{(a, b) \in X \times X : a \preceq b\}\). Then \(\nu\) is an equivalence relation on \(X\) and \(X/\nu\) is a partially ordered set with respect to the following induced relation.
Now, we define a mapping $P$. Clearly, $\nu$ is the smallest equivalence relation on $X$ for which (C1) defines a partial order on $X/\nu$. We call $\nu$ the minimum partial order congruence (mpo-congruence) on $X$ from $\preceq$.

A subset $A$ of $X$ is an ideal of $X$ provided that $x \preceq y$ and $y \in A$ implies $x \in A$. For $a \in X$, we call $\{x \in X : x \preceq a\}$ the principal ideal generated by $a$ and denote it by $\langle a \rangle$.

A bijection $\alpha$ of one pre-ordered set $X$ onto another $Y$ will be called an isomorphism provided that, for $a, b \in X$, $a \preceq b$ if and only if $a \alpha \preceq b \alpha$. In particular, if $\nu_X$ and $\nu_Y$ denote the respective mpo-congruences then $(a, b) \in \nu_X$ if and only if $(a \alpha, b \alpha) \in \nu_Y$.

Let $X(\preceq)$ be a pre-ordered set and $\nu$ the mpo-congruence from $\preceq$. Then $X$ is a partially pre-ordered $\varrho$-set if and only if $X/\nu$ is a partially ordered $\varrho$-set with respect to the naturally induced order $\preceq$ from $\preceq$.

Let $X(\preceq)$ be a partially pre-ordered $\varrho$-set and $\sigma$ an equivalence relation on $X$ such that

- (O1) for any $x$ in $X$, $\langle x \rangle$ is an $\iota$-single subset with respect to $\sigma$.
- (O2) for $x, y$ in $X$, if $(x, y) \in \sigma$ then $(x \varrho, y \varrho) \in \varrho$.
- (O3) for $x, y, z$ in $X$, if $(x \varrho \varrho, y \varrho \varrho) = (z \varrho \varrho, z \varrho \varrho)$, $z_1 \nu \leq x \nu$ and $z_2 \nu \leq y \nu$ if and only if $\nu, \varrho \varrho \in (z \nu \varrho)$, then for any $a \in \langle z_i \rangle$, there exists $b \in \langle z_j \rangle$ such that $(a, b) \in \sigma$, where $1 \leq i, j \leq 2$.

Then $(X(\preceq); \sigma)$ is called an $\omega$-set.

Let $(X(\preceq); \sigma)$ be an $\omega$-set and let $T_{X(\preceq); \sigma}$ denote the set of all isomorphisms from a principal ideal onto another one.

For any $\alpha, \beta \in T_{X(\preceq); \sigma}$, define a mapping $\theta_{\alpha, \beta}$ as follows:

$$\theta_{\alpha, \beta} = \{(a, b) \in r(\alpha) \times d(\beta) : (a, b) \in \sigma\}.$$ 

Then $\theta_{\alpha, \beta} \in T_{X(\preceq); \sigma}$. Let $\mathcal{M} = \{\theta_{\alpha, \beta} : \alpha, \beta \in T_{X(\preceq); \sigma}\}$, and denote a multiplication $\circ$ and a unary operation $*$ on $T_{X(\preceq); \sigma}$ by

$$\alpha \circ \beta = \alpha \theta_{\alpha, \beta} \beta,$$

$$\alpha^* = \alpha^{-1}.$$ 

Clearly, $\alpha \circ \beta$ is an isomorphism from $\langle z_1 \alpha^{-1} \rangle$ onto $\langle z_2 \beta \rangle$. It is obvious that $T_{X(\preceq); \sigma}(\circ, *)$ is a regular $\ast$-semigroup. Hence it is a generalized inverse $\ast$-semigroup and denoted by $T_{X(\preceq); \sigma}(\mathcal{M})$.

**Theorem 3.1.** A regular $\ast$-semigroup $T_{X(\preceq); \sigma}(\mathcal{M})$ is a generalized inverse $\ast$-subsemi-group of $\mathcal{G}\mathcal{I}_{X; \sigma}(\mathcal{M})$ whose set of projections is order isomorphic to $X/\nu$.

**Proof.** Clearly, $T_{X(\preceq); \sigma}(\mathcal{M})$ is a generalized inverse $\ast$-semigroup of $\mathcal{G}\mathcal{I}_{X; \sigma}(\mathcal{M})$. It remains to show that $P(T_{X(\preceq); \sigma}(\mathcal{M}))$ is order isomorphic to $X/\nu$. Hereafter, denote $P(T_{X(\preceq); \sigma}(\mathcal{M}))$ simply by $P$. It is easy to see that $P = \{1_{\langle x \rangle} : x \in X\}$.

Now, we define a mapping $\psi$ of $P$ to $X/\nu$ as follows: for any $1_{\langle x \rangle} \in P$, 

$$av \leq b \nu \text{ if and only if } a \preceq b.$$

We call $\preceq$ the naturally induced order on $X/\nu$ from $\preceq$. Clearly, $\nu$ is the smallest equivalence relation on $X$ for which (C1) defines a partial order on $X/\nu$. We call $\nu$ the minimum partial order congruence (mpo-congruence) on $X$ from $\preceq$.
\[1_{(x)} \psi = x \nu\]

Let \(1_{(x)}, 1_{(y)}\) be elements of \(P\), then
\[
1_{(x)} = 1_{(y)} \iff \langle x \rangle = \langle y \rangle \\
\iff x \in \langle y \rangle \text{ and } y \in \langle x \rangle \\
\iff x \preceq y \text{ and } y \preceq x \\
\iff x \nu = y \nu,
\]
thus \(\psi\) is well-defined and one-to-one, and we can easily see that it is a bijection. For \(x, y \in X\),
\[
1_{(x)} \leq 1_{(y)} \iff \langle x \rangle = \langle x \rangle \circ \langle y \rangle \\
\iff \langle x \rangle \subseteq \langle y \rangle \\
\iff x \preceq y \\
\iff x \nu \leq y \nu.
\]

Then \(\psi\) is an order isomorphism. \(\square\)

**Remark 3.2.** In \(T_{(X; \preceq; \sigma)}(M)\), if \(\preceq = \triangleq \) and \(\sigma = \emptyset\) then \(T_{(X; \preceq; \sigma)}(M) = T_{(X; \preceq)}(M)\).

Let \((X(\preceq); \sigma)\) be an \(\omega\)-set and let \(Y = X/\nu\), where \(\nu\) is the mpo-congruence from \(\preceq\). For any element \(\alpha\) in \(T_{(X(\preceq); \sigma)}\), assume that \(d(\alpha) = \langle a \rangle\). Then we can define a new mapping \(\alpha' \in T_{Y(\preceq)}\) as follows:
\[
d(\alpha') = \{x \nu : x \in d(\alpha)\}, \\
(x \nu)\alpha' = (x \alpha)\nu.
\]
Since \(\alpha\) is an isomorphism, \(\alpha'\) is a bijection of \(\langle a \nu \rangle\) onto \(\langle (a \alpha)\nu \rangle\). For \(x \nu, y \nu \in \langle a \nu \rangle\), we have
\[
x \nu = y \nu \iff x \preceq y \\
\iff x \alpha \preceq y \alpha \\
\iff (x \alpha) \nu \leq (y \alpha) \nu \\
\iff (x \nu) \alpha' \leq (y \nu) \alpha'.
\]
Then \(\alpha' \in T_{Y(\preceq)}\).

**Proposition 3.3.** The mapping \(\xi : \alpha \mapsto \alpha'\) of \(T_{(X(\preceq); \sigma)}(M)\) into \(T_{Y(\preceq)}(M)\) is a \(\ast\)-homomorphism of \(T_{(X(\preceq); \sigma)}(M)\) onto a \(\mathcal{P}\)-full generalized inverse \(\ast\)-subsemigroup of \(T_{Y(\preceq)}(M)\) such that \(\xi \circ \xi^{-1} = \mu\), where \(\mu\) is the maximum idempotent separating \(\ast\)-congruence on \(T_{(X(\preceq); \sigma)}(M)\).

**Proof.** First we shall show that \(\xi\) is a \(\ast\)-homomorphism. It is obvious that \((\alpha^{-1})' = (\alpha')^{-1}\) for any \(\alpha \in T_{(X(\preceq); \sigma)}(M)\). Let \(\alpha, \beta \in T_{(X(\preceq); \sigma)}(M)\) such that \(r(\alpha) = \langle x \rangle\) and \(d(\beta) = \langle y \rangle\). There exist \(z_1 \nu, z_2 \nu \in (x \nu) \emptyset \land (y \nu) \emptyset\) such that \(z_1 \nu \leq x \nu\) and \(z_2 \nu \leq y \nu\). Then \(d(\theta_{\alpha, \beta}) = \langle z_1 \rangle\) and \(r(\theta_{\alpha, \beta}) = \langle z_2 \rangle\). Thus \(d(\alpha \circ \beta) = \langle z_1 \alpha^{-1} \rangle\) and so \(d((\alpha \circ \beta)') = \langle (z_1 \alpha^{-1}) \nu \rangle\). On the other hand, Since \(r(\alpha') = \langle x \nu \rangle\) and \(d(\beta') = \langle y \nu \rangle\), we have
\[
d(\alpha' \circ \beta') = d(\alpha' \theta_{\alpha', \beta'} \beta') = (z_1 \nu) (\alpha')^{-1} = \langle (z_1 \alpha^{-1}) \nu \rangle.
\]
Then \( d((\alpha \circ \beta)\xi) = d((\alpha\xi) \circ (\beta\xi)) \).

To show that \( \xi \) is a \(*\)-homomorphism, it is sufficient to show that \( \theta_{\alpha',\beta'} = (\theta_{\alpha,\beta})' \). It is clear that \( d(\theta_{\alpha',\beta'}) = d((\theta_{\alpha,\beta})') = \langle z_1 \nu \rangle \). For any \( av \in \langle z_1 \nu \rangle \), set \( av(\theta_{\alpha,\beta})' = (a\theta_{\alpha,\beta})'v = bv \) and \( av\theta_{\alpha',\beta'} = cv \). Since \( (a,b) \in \sigma \), \( (av,bv) \in \mathcal{P} \). On the other hand, \( (av,cv) \in \mathcal{Q} \). Since \( \langle z_1 \nu \rangle \) is an \( \nu \)-set, \( bv = cv \), and we have \( \theta_{\alpha',\beta'} = (\theta_{\alpha,\beta})' \).

It is clear that \( (T(\langle \xi \rangle;\sigma)(\mathcal{M})) \xi \) is \( \mathcal{P} \)-full and fundamental. To show that \( \xi \circ \xi^{-1} = \mu \), it is sufficient to prove \( \xi \) separates projections. For \( 1_{(x)}, 1_{(y)} \in P(T(\langle \xi \rangle;\sigma)(\mathcal{M})) \),

\[
1_{(x)}\xi = 1_{(y)}\xi \implies 1_{(xv)} = 1_{(yv)} \implies xv \in \langle yv \rangle \text{ and } yv \in \langle xv \rangle \implies xv \leq yv \text{ and } yv \leq xv \implies x \preceq y \text{ and } y \preceq x \implies \langle x \rangle \subseteq \langle y \rangle \text{ and } \langle y \rangle \subseteq \langle x \rangle \implies 1_{(x)} = 1_{(y)}.
\]

Thus we have the proposition. \( \square \)

Hereafter, we shall refer to \( \xi \) as the natural projection of \( T(\langle \xi \rangle;\sigma)(\mathcal{M}) \) to \( T_{(\subseteq)}(\mathcal{M}) \).

### 3.2. Inflated representations

Let \( S \) be a generalized inverse \(*\)-semigroup. Hereafter, denote \( E(S) \) and \( P(S) \) simply by \( E \) and \( P \), respectively. Define a relation \( \preceq \) on \( S \) by:

\[
(3.2) \quad a \preceq b \text{ if and only if } a^*a \leq b^*b,
\]

for \( a, b \in S \). Then clearly \( \preceq \) is a pre-order on \( S \) for which the mpo-congruence from \( \preceq \) is \( \nu = \mathcal{L} \). Hence \( S/\mathcal{L} = S/\nu \), under the naturally induced order \( \leq \) from \( \preceq \), is just the set of \( \mathcal{L} \)-classes of \( S \) under the usual partial ordering of the \( \mathcal{L} \)-classes of a generalized inverse \(*\)-semigroup and so is order isomorphic to the partially ordered \( \nu \)-set \( P \) of \( S \). Hence \( S \) is a partially pre-ordered \( \nu \)-set under \( \preceq \). Then \( \varrho = J^E|_\nu \) and hence \( (av)\varrho(bv) \iff a^*aJ^Eb^*b \). Hereafter, for any \( a \in S \), we think \( av = L_{a^*a} \) as \( a^*a \).

For any \( a \in S \), define a mapping \( \rho_a : Sa^* \to Sa \) as follows:

\[
d(\rho_a) = Sa^* (= Saa^*), \quad x\rho_a = xa.
\]

Let \( \rho : S \to \mathcal{GI}(S;\Omega)(\mathcal{M}) \) by \( a\rho = \rho_a \), where the relation \( \Omega \) defined by: for \( x, y \in S \),

\[
(3.3) \quad (x, y) \in \Omega \iff x\rho_e = y \text{ for some } e \in E.
\]

Since \( S \) is a regular \(*\)-semigroup, the representation \( \rho \) is faithful. Moreover, it follows from \cite[Lemma 3.3]{6} that it is a \(*\)-monomorphism.

**Lemma 3.4.** The set \( (S(\preceq);\Omega) \), defined above, is an \( \omega \)-set.
Proof. Let $a$ be any element of $S$. Then
\[
 x \in Sa \iff x^* x = a^* a x a \leq a^* a \\
 \iff x \leq a \\
 \iff a \in \langle a \rangle.
\]

Thus we have $Sa = \langle a \rangle$. By Lemma 3.2 [7], $\langle a \rangle$ is an $\iota$-single subset.

Next, let $(a, b)$ be any element of $\Omega$. It follows from Lemma 3.1 [7] that $b = ab^* b$ and $a^* a R \mathcal{E} \mathcal{L} b^* b$ for some $e \in E$. Thus $a^* a \mathcal{F} b^* b$, and hence $(av) \mathcal{Q} (bv)$.

Assume that $J_{a^* a} \cap J_{b^* b} = J_{c^* c}$. Then $J_{c^* c} = J_{a^* ab^* b} = J_{b^* ba^* a}$. Also, we have $a^* ab^* ba^* a \leq a^* a$, $b^* ba^* ab^* b \leq b^* b$, and hence $b^* ba^* a \leq a$ and $a^* ab^* b \leq b$. Let $x = (xb^* ba^* a)$ be any element of $\langle b^* ba^* a \rangle$ and let $y = x^* ab^* b$. Then it is clear that $x = yx^* x$ and $y = y^* y$. It follows from Lemma 3.1 [7] that $y \in \langle a^* ab^* b \rangle$ and $(x, y) \in \Omega$. Similarly, for any $y \in \langle a^* ab^* b \rangle$, we have $x = yb^* ba^* a \in \langle b^* ba^* a \rangle$ and $(x, y) \in \Omega$. Hence $(S(\zeta); \Omega)$ is an $\omega$-set. \hfill $\Box$

Again, we consider $\rho_a : Sa^* \to Sa$. By Lemma 3.4, $d(\rho_a) = \langle a^* \rangle$ and $r(\rho_a) = \langle a \rangle$. For $x, y \in d(\rho_a)$, $x^* x, y^* y \leq a^* a$. Now $x \leq y$ if and only if $x^* x \leq y^* y$ while $xa \leq ya$ if and only if $a^* x xa = (xa)^* (xa) \leq (ya)^* (ya) = a^* y^* ya$. But, since $x^* x, y^* y \leq a^* a$ it follows that $x^* x \leq y^* y$ if and only if $a^* x xa \leq a^* y^* ya$. Therefore $x \leq y$ if and only if $xa \leq ya$. Thus $\rho_a$ is an isomorphism of $\langle a^* \rangle$ onto $\langle a \rangle$, and hence $Sp \subseteq T(S(\zeta); \Omega)(M)$.

Now, we have the following theorem.

**Theorem 3.5.** Let $S$ be a generalized inverse $*$-semigroup and let $\zeta$ be the relation on $S$ defined in (3.2). Then $\zeta$ is a pre-order on $S$ with respect to which $S$ is a partially pre-ordered $\omega$-set. Moreover, if $\Omega$ is the relation defined in (3.3), then $(S(\zeta); \Omega)$ is an $\omega$-set. The faithful representation $\rho$, defined above, embeds $S$ as a $\mathcal{P}$-full generalized inverse $*$-subsemigroup of $T(S(\zeta); \Omega)(M)$.

If $\nu$ is the mpo-congruence on $S$ from $\zeta$, then $\nu = \mathcal{L}$ and $S/\nu$ is order isomorphic to the partially ordered set $P$ of $S$. Moreover, $\rho \xi = \tau$, where $\xi$ is the natural projection and $\tau$ is the representation which is defined in Theorem 2.5.

**Proof.** It remains to show that $Sp$ is a $\mathcal{P}$-full generalized inverse $*$-subsemigroup of $T(S(\zeta); \Omega)(M)$ and that $\rho \xi = \tau$. Let $1_{\langle a \rangle} (a \in S)$ be any projection of $T(S(\zeta); \Omega)(M)$ and let $e = a^* a$. Then $1_{\langle a \rangle}$ and $\rho_e$ are both identity mappings on $\langle a \rangle$. Thus $1_{\langle a \rangle} = \rho_e$ and $Sp$ is a $\mathcal{P}$-full generalized inverse $*$-subsemigroup of $T(S(\zeta); \Omega)(M)$.

Next, let $\rho_a (a \in S)$ be an element of $Sp$. Then
\[
 d(\rho') = \{x^* x : x \in Sa^* \} \\
 = \{x^* x : x^* x \in Sa^* \cap P \} \\
 = Sa^* \cap P,
\]
and hence $d(\rho') = d(\tau_a)$. Moreover, for any $x^* x \in d(\rho')$,
\[
 (x^* x)\rho'_a = (xa)^* (xa) = a^* x^* xa = (x^* x)\tau_a.
\]

Thus $\rho'_a = \tau_a$, and hence $\rho_a \xi = \tau_a$. Therefore, $\rho \xi = \tau$, as required. \hfill $\Box$
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