The principal inverse of the gamma function

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The Euler form of the gamma function $\Gamma(x)$ is given by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$$

for $x > 0$. The Weierstrass form

$$\frac{1}{\Gamma(x)} = x e^{\gamma x} \prod_{n=1}^{\infty} (1 + \frac{x}{n}) e^{-\frac{x}{n}} \tag{1}$$

extends it to $\mathbf{R} \setminus \{0, -1, -2, \cdots\}$, where γ is the Euler constant defined by

$$\gamma = \lim_{n\to\infty} (1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n) = 0.57721 \cdots .$$

From this it follows that

$$\log \Gamma(x) = -\log x - \gamma x + \sum_{n=1}^{\infty} \left(\frac{x}{n} - \log(1 + \frac{x}{n}) \right), \tag{2}$$

$$\frac{\Gamma'(x)}{\Gamma(x)} = -\gamma + \sum_{n=0}^{\infty} (\frac{1}{n+1} - \frac{1}{n+x}). \tag{3}$$

This is called the *psi function* or *digamma function*.

It is clear that $\Gamma(1) = \Gamma(2) = 1,\ \Gamma'(1) = -\gamma,\ \Gamma'(2) = -\gamma + 1$.

Denote the unique zero in $(0, \infty)$ of $\Gamma'(x)$ by α. It is known that $\alpha = 1.4616 \cdots$ and $\Gamma(\alpha) = 0.8856 \cdots$.

We call the inverse function of the restriction of $\Gamma(x)$ to (α, ∞) *the principal inverse function* and denote it by Γ^{-1}.
$\Gamma^{-1}(x)$ is an increasing and concave function defined on $(\Gamma(\alpha), \infty)$.

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(1) guarantees that $\Gamma(x)$ has the holomorphic extension which is a meromorphic function with poles at non-positive integers and (3) holds there.

Let Π_+ and Π_- be respectively the open upper half plane and the open lower half plane. A holomorphic function defined on Π_+ is called a *Pick function* or *Nevanlinna function* if it maps Π_+ into itself.

From (3) it follows that $\Gamma'(z)$ does not vanish on $\mathbf{C} \setminus (-\infty, \alpha]$; in fact, $\Gamma'(z)/\Gamma(z)$ is a Pick function. Hence for each point $\omega_0 \in \Gamma(\mathbf{C} \setminus (-\infty, \alpha])$ there is a local inverse of $\Gamma(z)$ in a neighborhood of ω_0.
The main objective of this paper is to show

Theorem 1 *The principal inverse $\Gamma^{-1}(x)$ of $\Gamma(x)$ has the holomorphic extension $\Gamma^{-1}(z)$ to $\mathbf{C} \setminus (-\infty, \Gamma(\alpha)]$, which satisfies*

(i) $\Gamma^{-1}(\Pi_+) \subset \Pi_+$ *and* $\Gamma^{-1}(\Pi_-) \subset \Pi_-$,

(ii) $\Gamma^{-1}(z)$ *is univalent,*

(iii) $\Gamma(\Gamma^{-1}(z)) = z$ *for* $z \in \mathbf{C} \setminus (-\infty, \Gamma(\alpha)]$.

References

[1] Mitsuru Uchiyama, The principal inverse of the gamma function, PAMS 140(2012) 1343–1348.