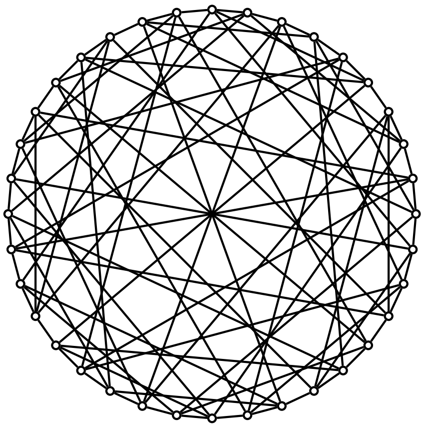

Cubes at Pentagon
and the Sylvester graph

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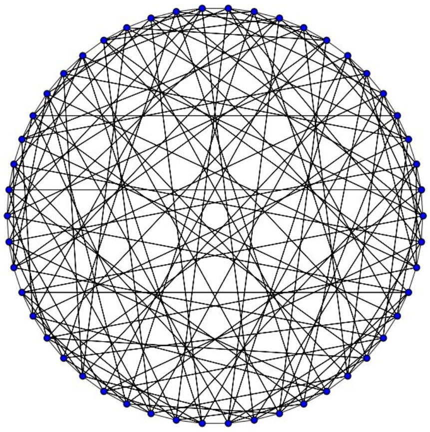

1. INTRODUCTION. The Sylvester graph is a 5-regular, Δ-free and $\square$-free graph with diameter 3, such that for any vertex x of the graph, the second neighbourhood of x induces a union of cycles and the third neighbourhood of x induces a perfect matching. Let Σ be a graph with these properties. We will use purely combinatorial means to show that Σ is actually the Sylvester graph. The standard proof of uniqueness relies on the uniqueness of a bigger graph, namely the Hoffman-Singleton graph (50 vertices), see Brouwer, Cohen & Neumaier [2, Thm. 13.1.2(ii)], whose intersection of the second neighbourhoods from two adjacent vertices induces the Sylvester graph. We will however search for smaller structures. One could say that this is a small case that can be studied with aid of computer(s).[1] Our aim is to reduce the number of cases to such a small number that it will be possible to solve the problem entirely by hand.

2. DISTANCE-REGULARITY. Let d be the diameter of a graph Γ and $\Gamma_i(x)$ $(0 \le i \le d)$ the set of vertices at distance i from a vertex x of Γ, also called the i-th neighbourhood of x. All properties of the Sylvester graph mentioned above are related to the **distance partition** $\{\Gamma_0(x), \Gamma_1(x), \dots, \Gamma_d(x)\}$, see Fig. 1(a). Set $\Gamma_{-1}(x) := \emptyset =: \Gamma_{d+1}(x)$, $\Gamma(x) := \Gamma_1(x)$, let y be vertex in $\Gamma_i(x)$ $(0 \le i \le d)$ and

$$a_i := |\Gamma(y) \cap \Gamma_i(x)|, \quad b_i := |\Gamma(y) \cap \Gamma_{i+1}(x)|, \quad c_i := |\Gamma(y) \cap \Gamma_{i-1}(x)|, \quad k_i := |\Gamma_i(x)|,$$

where $|S|$ denotes the size of a set S. Note that $k_0 = 1 = c_1$ and $c_0 = a_0 = 0 = b_d$. If $y \in \Gamma(x)$, we say that x and y are adjacent or neighbours and often write $x \sim y$.

We can now rewrite the above properties of the graph Σ in the following way:

$$\text{(i) } k_1 = 5, \quad \text{(ii) } a_1 = 0, \quad \text{(iii) } c_2 = 1, \quad \text{(iv) } d = 3, \quad \text{(v) } a_2 = 2, \quad \text{(vi) } a_3 = 1$$

for every vertex x of Σ and every $y \in \Sigma_i(x)$.

We say that a graph Γ with diameter d is **distance-regular** whenever there is an intersection array of constants $\{b_0, \dots, b_{d-1}; 1, c_2, \dots, c_d\}$ so that for every vertex x of Γ and every $y \in \Gamma_i(x)$ $(0 \le i \le d)$ we have $b_i = |\Gamma(y) \cap \Gamma_{i+1}(x)|$ and $c_i = |\Gamma(y) \cap \Gamma_{i-1}(x)|$. Obviously, a distance-regular graph Γ is regular, $k := k_1 = b_0 = a_i + b_i + c_i$ and a two-way counting of edges between the sets $\Gamma_i(x)$ and $\Gamma_{i-1}(x)$ gives us $k_i = k_{i-1}b_{i-1}/c_i$ $(1 \le i \le d)$.

We can write three of the above conditions for Σ also as: (ii) $b_1 = 4$, (iii) $b_2 = 2$, (vi) $c_3 = 4$ and note that the graph Σ satisfying (i)–(vi) is precisely a distance-regular graph with intersection array

$$\{k, b_1, b_2; c_1, c_2, c_3\} = \{5, 4, 2; 1, 1, 4\}. \tag{1}$$

From now on, let Σ be a distance-regular graph with intersection array (1). Then $k_2 = 5 \cdot 4/1 = 20$, $k_3 = 20 \cdot 2/4 = 10$ and $1 + 5 + 20 + 10 = 36$ is the number of all vertices of Σ.

[1] But be careful! If one wants to produce carelessly the list of all possible graphs on 36 points, then some huge numbers are in the way: $2^{\binom{36}{2}} = 2^{630}$. In cryptography it is well known that it is nowadays safe to hide a key in a set with 2^{80} elements.